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HAUSDORFF APPROXIMATIONS ON HADAMARD MANIFOLDS AND THEIR IDEAL BOUNDARIES

Dedicated to Professor Tsunero Takahashi on his sixtieth birthday

By

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§1. Introduction

The concept of ideal boundary of Hadamard manifolds was first introduced by Eberlein and O'Neill [3], and then their Tits metrics were defined by Gromov [2] in the following manner.

Let M be a Hadamard manifold, that is, a simply connected complete Riemannian manifold of nonpositive curvature. In what follows, geodesics are always assumed to be parametrized by arc length. Two geodesic rays $\gamma_1, \gamma_2 : [0, \infty) \to M$ are said to be *asymptotic* if the distance function $t \to d_M(\gamma_1(t), \gamma_2(t))$ is bounded from above for all $t \geq 0$. Then the *ideal boundary* $M(\infty)$ of M is defined to be the set of all asymptotic classes of geodesic rays in M. For $z_1, z_2 \in M(\infty)$ and $p \in M$, let γ_1, γ_2 be rays from p to z_1, z_2. The function $t \to d(\gamma_1(t), \gamma_2(t))/t$ is then monotone non-decreasing and is bounded from above by 2. Thus we can define a metric l on $M(\infty)$ by

$$l(z_1, z_2) := \lim_{t \to \infty} \frac{d(\gamma_1(t), \gamma_2(t))}{t}.$$

It is easy to see that the definition of l is independent of the choice of p and that l is indeed a metric on $M(\infty)$. The *Tits metric* $Td(\cdot, \cdot)$ is then the interior metric l_i induced from this metric.

Subsequently, the concept of ideal boundary was also defined for other classes of Riemannian manifolds in a similar fashion. Among them, Kasue [5] defined it on asymptotically nonnegatively curved manifolds, and Shioya [8], [9] on complete open surfaces admitting total curvature.

On the other hand, we know the concepts of rough isometry and Hausdorff approximation between two metric spaces, which preserve certain asymptotic properties, in the following way (cf. Kanai [4]).

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Let X and Y be metric spaces. A map $\phi: X \to Y$ (not necessarily continuous) is said to be an (α, Δ)-*rough isometry* for some constants $\alpha \geqq 1$ and $\Delta \geqq 0$ if ϕ satisfies the following two conditions:

$$\text{(1)} \qquad \overline{B_\Delta(\phi(X))} := \{y \in Y \mid d(y, \phi(X)) \leqq \Delta\} = Y,$$

$$\text{(2)} \qquad \frac{1}{\alpha} d_X(x_1, x_2) - \Delta \leqq d_Y(\phi(x_1), \phi(x_2)) \leqq \alpha \cdot d_X(x_1, x_2) + \Delta,$$

for all $x_1, x_2 \in X$. If $\alpha = 1$ in particular, we call ϕ a Δ-*Hausdorff approximation*.

It is then an interesting problem to study relationships between Hausdorff approximations and ideal boundaries. Recently, in this direction, Kubo [6] and the author [7] prove the following result.

Let M, N be either Hadamard manifolds, asymptotically nonnegatively curved manifolds or complete open surfaces admitting total curvature. Assume that their ideal boundaries are compact with respect to the Tits-topology. If there exists a Hausdorff approximation between M and N, then their ideal boundaries are isometric with respect to the Tits metric.

In this paper, we shall be concerned with the same problem in the case where given ideal boundaries are noncompact. Our first object is to prove the following theorem, which gives an extension of Theorem A in [7].

THEOREM 1. *If there exists a Hausdorff approximation between two Hadamard manifolds, then their ideal boundaries are isometric with respect to the Tits metric.*

It should be remarked that recently another definition of ideal boundaries of complete metric spaces is given by Adachi [1], which coincides with $M(\infty)$ when M is a Hadamard manifold. Then he proves the same result as Theorem 1 with respect to a metric d_∞ equivalent to $l : l/2 \leqq d_\infty \leqq l$.

It will be also remarkable that it is difficult to construct a map between ideal boundaries from a rough isometry which is not a Hausdorff approximation.

In general the converse is not true; for two Hadamard manifolds whose ideal boundaries are isometric, they need not be roughly isometric. Our second object is to consider the converse problem more precisely. Namely we shall investigate the converse problem under some additional condition (E), which is concerned with the expanding growth rate and defined precisely in Section 3. In fact, we prove the following

THEOREM 2. *Let M, N be Hadamard manifolds satisfying the condition (E).*

If ($M(\infty)$, Td) *is isometric to* ($N(\infty)$, Td), *then for any* $\varepsilon>0$, *there exists a* $(1+\varepsilon, T_\varepsilon)$*-rough isometry between M and N, where* T_ε *is a constant depending on* ε.

We here note that there is an example of a pair of two Hadamard manifolds satisfying the condition (E) but no Hausdorff approximation exists between them (see [7]).

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§2. Proof of Theorem 1

In this section, we shall prove Theorem 1. First the following lemma concerning triangles in a Euclidean plane is proved.

LEMMA 1. *For a triangle* $\Delta(p,q,r)$ *in* $\boldsymbol{R}^2$, *let* α, β *and* γ *be the lengths of the opposite sides of p, q and r, respectively. If there is a constant* $c \geqq 0$ *satisfying* $\alpha+\beta-\gamma \leqq c$, *then the following inequalities hold:*

$$\cos \angle rpq \geqq 1-\frac{c}{\beta}, \tag{1}$$

$$h \leqq \sqrt{2\beta c}, \tag{2}$$

where h denotes the distance between r and the foot of a perpendicular from r to the opposite side.

PROOF. Let $\theta_p = \angle rpq$ and $\theta_q = \angle pqr$. From the assumption, we have

$$\begin{aligned} \alpha+\beta &\leqq \gamma + c \\ &= \beta\cos\theta_p + \alpha\cos\theta_q + c \\ &\leqq \beta\cos\theta_p + \alpha + c, \end{aligned}$$

which gives the first inequality.

Since $h=\beta\sin\theta_p = \beta\sqrt{1-\cos^2\theta_p}$, the first inequality implies the second one in the case that $\beta \geqq c$. If not, it is clear. ■

Now we are going to prove Theorem 1.

PROOF OF THEOREM 1. Assume that a Δ-Hausdorff approximation f is given. First we define a map $\partial f: M(\infty) \to N(\infty)$ induced from f.

Let p be an arbitrarily fixed point in M. For any $z \in M(\infty)$, there is a unique ray γ emanating from p to z. Denote by $\tilde{\gamma}(t)$ the curve $f(\gamma(t))$ and let $\tilde{\gamma}_t$ be a geodesic segment from $\tilde{\gamma}(0) = f(p)$ *to* $\tilde{\gamma}(t)$. Then $\tilde{\gamma}_t$ converges to a ray $\tilde{\gamma}_\infty$ as t tends to ∞.

In fact, look at the geodesic triangle $\Delta(\tilde{\gamma}(0), \tilde{\gamma}(s), \tilde{\gamma}(t))$ for $s > t > 0$. Then, concerning the lengths of the sides, we have

$$\begin{aligned}
&|d(\tilde{\gamma}(0), \tilde{\gamma}(t)) + d(\tilde{\gamma}(t), \tilde{\gamma}(s)) - d(\tilde{\gamma}(0), \tilde{\gamma}(s))| \\
\leqq\ & |d(\tilde{\gamma}(0), \tilde{\gamma}(t)) - t| + |d(\tilde{\gamma}(t), \tilde{\gamma}(s)) - (s-t)| + |d(\tilde{\gamma}(0), \tilde{\gamma}(s)) - s| \\
\leqq\ & 3\Delta.
\end{aligned}$$

Hence, applying the lemma above and Toponogov's comparison Theorem, it holds that for $t > \Delta$

$$\cos \angle(\tilde{\gamma}_s'(0), \tilde{\gamma}_t'(0)) \geqq 1 - \frac{3\Delta}{d(\tilde{\gamma}(0), \tilde{\gamma}(t))} \geqq 1 - \frac{3\Delta}{t - \Delta}.$$

Therefore we have $\lim_{t\to\infty} \angle(\tilde{\gamma}_s'(0), \tilde{\gamma}_t'(0)) = 0$, proving the assertion.

Now define a map $\partial f : M(\infty) \to N(\infty)$ by

$$\partial f(z) = \tilde{\gamma}_\infty(\infty).$$

Note that this definition is independent of the choice of the reference point p.

We will prove that ∂f is surjective and is an isometry. First we prove the surjectivity. For any $w \in N(\infty)$, let σ be a ray emanating from $q := f(p)$ to w. Then for any $t \geqq 0$ there is a point $x_t \in M$ with $d(f(x_t), \sigma(t)) \leqq \Delta$. Then it is easily checked that for $s > t > 0$

$$|d(p, x_t) + d(x_t, x_s) - d(p, x_s)| \leqq 7\Delta,$$

which implies, similarly to the argument above, that the geodesic segment from p to x_t converges to a ray γ as $t \to \infty$. It suffices to show that the image of the asymptotic class of γ is $w = \sigma(\infty)$.

For any x_t there exists $t' \geqq 0$ satisfying $d(x_t, \gamma(t')) = d(x_t, \gamma)$. Then, applying Lemma 1 and Toponogov's comparison Theorem for a geodesic triangle $\Delta(p, x_t, x_s)(s > t)$, we have that the distance between x_t and the ray emanating from p through x_s is not greater than $\sqrt{14\Delta l_t}$, where $l_t := d(p, x_t)$. Since these rays converge to the ray γ, it follows

$$d(x_t, \gamma(t')) \leqq \sqrt{14\Delta l_t}.$$

Note that $t' \leqq l_t \leqq t + 2\Delta$. Hence

$$d(\sigma(t), f(\gamma(t'))) \leqq d(\sigma(t), f(x_t)) + d(f(x_t), f(\gamma(t'))) \leqq \sqrt{14\Delta(t + 2\Delta)} + 2\Delta.$$

Let $\hat{\sigma}$ be a ray emanating from q determined by $f(\gamma)$, namely $\hat{\sigma}(\infty)=\partial f(\gamma(\infty))$. Then it also holds that

$$d(f(\gamma(t')),\hat{\sigma})\leqq\sqrt{6\Delta m_t},$$

where $m_t:=d(q,f(\gamma(t')))\leqq t'+\Delta\leqq t+3\Delta$. Therefore we have

$$\begin{aligned}d(\sigma(t),\hat{\sigma})&\leqq d(\sigma(t),f(\gamma(t')))+d(f(\gamma(t')),\hat{\sigma})\\&\leqq\sqrt{14\Delta(t+2\Delta)}+\sqrt{6\Delta(t+3\Delta)}+2\Delta.\end{aligned}$$

Hence

$$\lim_{t\to\infty}\frac{d(\sigma(t),\hat{\sigma})}{t}=0,$$

which means that $\sigma\equiv\hat{\sigma}$, that is, $w=\hat{\sigma}(\infty)$.

It remains to show that ∂f is an isometry. Let $z_1,z_2\in M(\infty)$ be arbitrarily fixed points. Denote by γ_i a ray emanating from p to z_i and by σ_i that from q to $\partial f(z_i)(i=1,2)$, respectively. Then, for any $s\geqq 0$,

$$\begin{aligned}&|d_N(\sigma_1(s),\sigma_2(s))-d_M(\gamma_1(s),\gamma_2(s))|\\&\leqq|d_N(\sigma_1(s),\sigma_2(s))-d_N(f(\gamma_1(s),f(\gamma_2(s)))|\\&\qquad+|d_N(f(\gamma_1(s)),f(\gamma_2(s)))-d_M(\gamma_1(s),\gamma_2(s))|\\&\leqq d_N(\sigma_1(s),\ f(\gamma_1(s)))+d_N(\sigma_2(s),f(\gamma_2(s)))+\Delta\\&\leqq\sum_{i=1}^{2}\{d_N(\sigma_i(s),\sigma_i(s_i'))+d_N(\sigma_i(s_i'),f(\gamma_i(s)))\}+\Delta\\&\leqq 3\Delta+4\sqrt{6\Delta(s+\Delta)},\end{aligned}$$

where $\sigma_i(s_i')$ is the foot of a perpendicular from $f(\gamma_i(s))$ to a ray σ_i. Note that $s-\Delta-\sqrt{6\Delta(s+\Delta)}\leqq s_i'\leqq s+\Delta$. Hence

$$\lim_{s\to\infty}\left|\frac{d_N(\sigma_1(s),\sigma_2(s))}{s}-\frac{d_M(\gamma_1(s),\gamma_2(s))}{s}\right|=0,$$

which means that $l(\sigma_1(\infty),\sigma_2(\infty))=l(\gamma_1(\infty),\gamma_2(\infty))$, hence $\mathrm{Td}(\sigma_1(\infty),\sigma_2(\infty))=\mathrm{Td}$ $(\gamma_1(\infty),\gamma_2(\infty))$. ■

§3. Proof of Theorem 2

In this section, we shall introduce the condition (E) and prove Theorem 2.

Let M be a Hadamard manifold and $p\in M$ be an arbitrarily fixed point. From now on, we denote a ray emanating from $p\in M$ to $z\in M(\infty)$ by γ_z. For $z_1,z_2\in M(\infty),z_1\neq z_2$ and $t>0$, we define two continuous maps $a_t(z_1,z_2)$ and a_t respectively by

$$a_t(z_1,z_2):=\frac{d(\gamma_{z_1}(t),\gamma_{z_2}(t))}{t\cdot l(z_1,z_2)},$$
$$a_t:=\inf_{z_1\neq z_2\in M(\infty)} a_t(z_1,z_2).$$

Since M is a Hadamard manifold, it holds that $0<a_t(z_1,z_2)\leqq 1$. Furthermore, $a_t(z_1,z_2)$ is monotone non-decreasing with respect to variable t and converges to 1 as $t\to\infty$. It then follows that $0\leqq a_t\leqq 1$ and a_t is monotone non-decreasing. Hence there is a constant $a:=\lim_{t\to\infty} a_t$ with $0\leqq a\leqq 1$. Note that this constant a is independent of the choice of p, namely, a is a scalar expressing some global property of M and we call a the *E-constant*.

DEFINITION. If the E-constant a is equal to 1, we say that M satisfies *the condition (E)*.

The condition (E) implies that the expanding growth rate of any radial direction is similar each other to some degree. For example, if M is a Euclidean space then a = 1, and if M is a Hyperbolic space then a = 0. In the case that $M=\boldsymbol{R}\times\boldsymbol{H}^n$, it also follows that a = 0, but with respect to the points at infinity $S,N\in\boldsymbol{R}(\infty)$ we have $a_t(S,z)=a_t(N,z)$ = 1 for any $t>0$ and $z\in M(\infty)$. The next proposition is useful to check the property concerning E-constant, which implies that if the sphere topology of an ideal boundary does not coincide with the Tits-topology, then the E-constant is equal to 0.

PROPOSITION. *Let M be a Hadamard manifold with a positive E-constant. Then the ideal boundary* $(M(\infty),\mathrm{Td})$ *of M is compact.*

PROOF. For a fixed point $p\in M$, let S_t be a geodesic sphere around p of radius t. We define the natural bijection $\varphi_t:(S_t,d_M/t)\to(M(\infty),l)$ by $\varphi_t(\gamma(t))$ $:=\gamma(\infty)$, where γ is a ray emanating from p. Let a be the positive E-constant of M. Then there is a large number T such that for any $t>T$ and for any two distinct points $z_1,z_2\in M(\infty)$

$$1\geqq\frac{d_M(\gamma_{z_1}(t),\gamma_{z_2}(t))}{t\cdot l(z_1,z_2)}\geqq\frac{a}{2}.$$

That is, for any points $x,y\in S_t$, it holds that $l(\varphi_t(x),\varphi_t(y))\geqq d_M(x,y)/t\geqq(a/2)l$ $(\varphi_t(x),\varphi_t(y))$. This means that φ_t is a bi-Lipschitz homeomorphism. Hence the compactness of S_t implies that so are $(M(\infty),l)$ and $(M(\infty),\mathrm{Td})$. ■

Now we shall prepare some notations and a lemma, and prove Theorem 2.

For any distinct points $z_1, z_2 \in M(\infty)$ and $s, t \geqq 0$, we define

$$\theta_{z_1 z_2} := 2 \arcsin \frac{l(z_1, z_2)}{2},$$
$$\theta_{z_1 z_2}(s,t) := \arccos\left(\frac{s^2 + t^2 - d(\gamma_{z_1}(s), \gamma_2(t))^2}{2st}\right),$$

with a condition that $0 \leqq \theta \leqq \pi$. It is clear that $\lim_{s,t\to\infty} \theta_{z_1 z_2}(s,t) = \theta_{z_1 z_2}$ and, by Toponogov's comparison theorem, that $\theta_{z_1 z_2}(s_1, t_1) \leqq \theta_{z_1 z_2}(s_2, t_2)$ provided $s_1 \leqq s_2$ and $t_1 \leqq t_2$.

LEMMA 2. *Let M be a Hadamard manifold satisfying the condition (E). Then for any small $\varepsilon > 0$ there exists a large number T_ε such that for $s,t > T_\varepsilon$ and for two distinct points $z_1, z_2 \in M(\infty)$*

$$\frac{d(\gamma_{z_1}(s), \gamma_{z_2}(t))}{d_\infty((s, z_1), (t, z_2))} > 1 - \varepsilon,$$

where $d_\infty((s, z_1), (t, z_2)) := \sqrt{s^2 + t^2 - 2st \cos \theta_{z_1 z_2}}$.

PROOF. Since M satisfies the condition (E), for any $\varepsilon > 0$ there is a large number T_ε such that for any $t > T_\varepsilon$ and for two distinct points $z_1, z_2 \in M(\infty)$

$$\frac{d(\gamma_{z_1}(t), \gamma_{z_2}(t))}{t \cdot l(z_1, z_2)} > 1 - \varepsilon.$$

Furthermore, since M is a Hadamard manifold, for $s \geqq t > T_\varepsilon$ and $z_1 \neq z_2 \in M(\infty)$ we have

$$\begin{aligned}
\frac{1 - \cos \theta_{z_1 z_2}(s,t)}{1 - \cos \theta_{z_1 z_2}} &\geqq \frac{1 - \cos \theta_{z_1 z_2}(t,t)}{1 - \cos \theta_{z_1 z_2}} \\
&= \left(\frac{d(\gamma_{z_1}(t), \gamma_{z_2}(t))}{t \cdot l(z_1, z_2)}\right)^2 \\
&> (1 - \varepsilon)^2.
\end{aligned}$$

Hence it follows that $\cos \theta_{z_1 z_2}(s,t) - (1-\varepsilon)^2 \cos \theta_{z_1 z_2} < 1 - (1-\varepsilon)^2$. Therefore we have

$$\begin{aligned}
&d^2(\gamma_{z_1}(s), \gamma_{z_2}(t)) - (1-\varepsilon)^2 d_\infty^2((s, z_1), (t, z_2)) \\
&= (1 - (1-\varepsilon)^2)(s^2 + t^2) - 2st\{\cos \theta_{z_1 z_2}(s,t) - (1-\varepsilon)^2 \cos \theta_{z_1 z_2}\} \\
&> (1 - (1-\varepsilon)^2)(s - t)^2 \geqq 0,
\end{aligned}$$

which completes the proof. ■

Now we are in a position to prove Theorem 2.

PROOF OF THEOREM 2. Let ψ be an isometry from $M(\infty)$ to $N(\infty)$. Then we define a map $f: M \to N$ as follows.

Fix two points $p \in M$ and $q \in N$ arbitrarily. For $x \in M (x \neq p)$, let $t = d(p,x)$ and let $z \in M(\infty)$ be the asymptotic class of a ray emanating from p through x. Then we define

$$f(x) := \gamma_{\psi(z)}(t),$$

where $\gamma_{\psi(z)}$ denotes a ray emanating from q to $\psi(z)$, and $f(p) := q$.

Now we shall see that f is a desired rough isometry. More precisely, for any sufficient small $\varepsilon > 0$ there are constants $T_\varepsilon^M > 0$ such that $(1-\varepsilon) < a_t^M(z_1, z_2) \leqq 1$ for any $t > T_\varepsilon^M$ and $z_1 \neq z_2 \in M(\infty)$, and $T_\varepsilon^N > 0$ satisfying the same condition for N. Let $T_\varepsilon := \max\{T_\varepsilon^M, T_\varepsilon^N\}$. Then f is a $((1-\varepsilon)^{-1} 4T_\varepsilon)$-rough isometry.

In fact, since f is surjective, it suffices to check the following inequality:

$$(*) \qquad (1-\varepsilon)d(x,y) - 4T_\varepsilon \leqq d(f(x), f(y)) \leqq \frac{1}{(1-\varepsilon)} d(x,y) + 4T_\varepsilon,$$

for all $x, y \in M$.

We express $x, y \in M$ as $x = \gamma_v(s), y = \gamma_w(t) \in M (v, w \in S_p, s, t \geqq 0)$, where S_p denotes a unit tangent sphere at p. If $v = w$ then $d(x,y) = d(f(x), f(y)) = |s-t|$, namely the inequality $(*)$ holds. So we suppose $v \neq w$. In the case $\max(s,t) \leqq T_\varepsilon$, it holds that

$$d(x,y) - 2T_\varepsilon \leqq 0 \leqq d(f(x), f(y)) \leqq 2T_\varepsilon.$$

Next we consider the case $\min(s,t) \leqq T_\varepsilon < \max(s,t)$. We may suppose $t \leqq T_\varepsilon < s$. Let $x' := \gamma_v(t)$. It is then verified that

$$d(f(x), f(y)) \leqq d(f(x), f(x')) + d(f(x'), f(y)) \leqq d(x,y) + 2T_\varepsilon,$$

and conversely

$$d(f(x), f(y)) \geqq d(f(x), f(x')) - d(f(x'), f(y)) \geqq d(x,y) - 4T_\varepsilon.$$

In the case $T_\varepsilon < \min(s,t)$, let $z_1 = \gamma_v(\infty)$ and $z_2 = \gamma_w(\infty)$. By Lemma 2 it holds

$$\begin{aligned} \frac{d(f(x), f(y))}{d(x,y)} &= \frac{d(\gamma_{\psi(z_1)}(s), \gamma_{\psi(z_2)}(t))}{d(\gamma_{z_1}(s), \gamma_{z_2}(t))} \\ &\leqq \frac{d_\infty((s, \psi(z_1)), (t, \psi(z_2)))}{d(\gamma_{z_1}(s), \gamma_{z_2}(t))} \end{aligned}$$

$$= \frac{d_\infty((s, z_1), (t, z_2))}{d(\gamma_{z_1}(s), \gamma_{z_2}(t))} < \frac{1}{1-\varepsilon},$$

and

$$\frac{d(f(x), f(y))}{d(x, y)} \geqq \frac{d(\gamma_{\psi(z_1)}(s), \gamma_{\psi(z_2)}(t))}{d_\infty((s, z_1), (t, z_2))}$$
$$= \frac{d(\gamma_{\psi(z_1)}(s), \gamma_{\psi(z_2)}(t))}{d_\infty((s, \psi(z_1)), (t, \psi(z_2)))} > 1-\varepsilon,$$

which completes the proof. ∎

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