

A DIFFERENTIAL GEOMETRIC CHARACTERIZATION OF HOMOGENEOUS SELF-DUAL CONES

(Dedicated to Professor K. Murata on his sixtieth birthday)

By

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In this note we give a differential geometric characterization of self-dual cones among affine homogeneous convex domains not containing any full straight line. Let Ω be an affine homogeneous convex domain in an n -dimensional real vector space V^n . Then Ω admits an invariant volume element

$$(1) \quad v = \phi dx^1 \wedge \cdots \wedge dx^n$$

and the canonical bilinear form defined by

$$(2) \quad g = \sum_{i,j} \frac{\partial^2 \log \phi}{\partial x^i \partial x^j} dx^i dx^j$$

is positive definite and so gives an invariant Riemannian metric on Ω , where $\{x^1, \dots, x^n\}$ is an affine coordinate system on V^n [5]. In an affine coordinate system $\{x^1, \dots, x^n\}$ the components of the Riemannian connection Γ and the Riemannian curvature tensor R for g are expressed as follows

$$(3) \quad \Gamma^i_{jk} = \frac{1}{2} \sum_p g^{ip} \frac{\partial^3 \log \phi}{\partial x^j \partial x^k \partial x^p},$$

$$(4) \quad R^i_{jkl} = \sum_p (\Gamma^i_{pk} \Gamma^p_{jl} - \Gamma^i_{pl} \Gamma^p_{jk}),$$

where $g_{ij} = \frac{\partial^2 \log \phi}{\partial x^i \partial x^j}$ and $\sum_p g^{ip} g_{pj} = \delta^i_j$ (Kronecker's delta). Since $\frac{1}{2} \sum_p g^{ip} \frac{\partial^3 \log \phi}{\partial x^j \partial x^k \partial x^p}$ defines a tensor field on Ω , we denote this tensor field by the same letter Γ .

An open convex set Ω in V^n is called a cone with vertex o if $o + \lambda(x - o) \in \Omega$ for all $x \in \Omega$ and $\lambda > 0$. An open convex cone Ω with vertex o is said to be a self-dual cone if V^n admits an inner product $\langle \cdot, \cdot \rangle$ such that

(i) $\langle x - o, y - o \rangle > 0$ for all $x, y \in \Omega$;

(ii) if $x \in V^n$ is a vector such that $\langle x - o, y - o \rangle \geq 0$ for all $y \in \bar{\Omega}$ then $x \in \bar{\Omega}$, where $\bar{\Omega}$ is the closure of Ω in V^n .

THEOREM 1. *A homogeneous convex cone Ω not containing any full straight line is a self-dual cone if and only if R is parallel with respect to Γ .^(*)*

THEOREM 2. *A homogeneous convex domain Ω not containing any full straight line is a self-dual cone if and only if Γ is parallel with respect to Γ .*

The necessary conditions of these theorems have been proved by O. S. Rothaus [2].

If Γ is parallel with respect to Γ , then by (4) R is parallel with respect to Γ . The converse is not true. For example, in the case of the interior of a paraboloid; $x^2+1-\frac{1}{2}(x^1)^2>0$, R is parallel but Γ is not parallel with respect to Γ .

We denote by \mathcal{F}_X and L_X the covariant differentiation for Γ and the Lie differentiation in the direction of a vector field X respectively. We set

$$(5) \quad A_X = L_X - \mathcal{F}_X.$$

Then A_X is a derivation of the algebra of tensor fields and for a vector field Y we have

$$(6) \quad A_X Y = -\mathcal{F}_Y X.$$

If X and Y are Killing vector fields, then we know [1]

$$(7) \quad R(X, Y) = [A_X, A_Y] - A_{[X, Y]}.$$

For vector fields $X = \sum_i \xi^i \frac{\partial}{\partial x^i}$ and $Y = \sum_i \eta^i \frac{\partial}{\partial x^i}$ we define a vector field $X \square Y$ by

$$(8) \quad X \square Y = - \sum_{i, j, k} \Gamma^i_{jk} \xi^j \eta^k \frac{\partial}{\partial x^i},$$

and we put

$$(9) \quad S_X Y = X \square Y.$$

The condition that Γ is parallel with respect to Γ is equivalent to

$$(10) \quad \mathcal{F}_Z(X \square Y) = (\mathcal{F}_Z X) \square Y + X \square (\mathcal{F}_Z Y).$$

We shall now recall the construction of clans from affine homogeneous convex domains [5]. It is known that a homogeneous convex domain Ω not containing any full straight line admits a simply transitive triangular affine Lie group T . Let \mathfrak{t} denote the Lie algebra of T . We fix a point $e \in \Omega$ and choose an affine coordinate system $\{x^1, \dots, x^n\}$ such that $x^1(e) = \dots = x^n(e) = 0$. Identifying $X \in \mathfrak{t}$ with the vector field induced by the one parameter group of transformations $\exp(-tX)$, X

(*) T. Tsuji obtained the same result independently [7].

has an expression $X = \sum_i (\sum_j a^i_j x^j + a^i) \frac{\partial}{\partial x^i}$, where a^i_j and a^i are constants. Let V denote the tangent space of Ω at e . Since T acts simply transitively on Ω , for each $a \in V$ there exists a unique element $X_a \in \mathfrak{t}$ such that the values of X_a at e is equal to a . For $a, b \in V$ we define a multiplication $a \Delta b$ in V by

$$(11) \quad a \Delta b = \sum_i (\sum_j a^i_j b^j) \left(\frac{\partial}{\partial x^i} \right)_e,$$

where a^i_j and b^j are constants given by

$$X_a = \sum_i (\sum_j a^i_j x^j + a^i) \frac{\partial}{\partial x^i} \quad \text{and} \quad X_b = \sum_i (\sum_j b^i_j x^j + b^i) \frac{\partial}{\partial x^i}.$$

Then we have

$$(12) \quad [X_a, X_b] = X_{b \Delta a - a \Delta b}.$$

Denoting by L_a the left multiplication by $a \in V$;

$$(13) \quad L_a b = a \Delta b,$$

we have

$$(14) \quad [L_a, L_b] = L_{a \Delta b - b \Delta a}.$$

Let \langle , \rangle denote the inner product on V induced by g and we put

$$(15) \quad s(a) = \text{Tr } L_a.$$

Then we know

$$\langle a, b \rangle = s(a \Delta b).$$

The algebra V together with the linear form s is said to be the *clan* of Ω with respect to $e \in \Omega$ and the simply transitive triangular group T and is denoted by $V(\Omega)$.

PROPOSITION 1. For $a, b \in V$ we denote by S_a, A_a , and $R(a, b)$ the values of S_{X_a}, A_{X_a} , and $R(X_a, X_b)$ at e respectively. Then we have

$$(i) \quad S_a = \frac{1}{2} (L_a + {}^t L_a), \quad S_a b = S_b a,$$

$$(ii) \quad A_a = -\frac{1}{2} (L_a - {}^t L_a),$$

$$(iii) \quad R(a, b) = -[S_a, S_b],$$

where ${}^t L_a$ is the transpose of L_a with respect to \langle , \rangle .

PROOF. We may assume $\phi(e) = 1$. Since $v = \phi dx^1 \wedge \cdots \wedge dx^n$ is invariant under

the one parameter group of transformations $\text{Exp } tX_a$ generated by X_a , we have $\phi((\text{Exp } tX_a)e) = \exp(-t \text{Tr } L_a)$ and so

$$(16) \quad \log \phi((\text{Exp } tX_a)e) = -ts(a).$$

Expanding the left side in a power series of t and evaluating the terms of the first, the second and the third orders, we have

$$(17) \quad \sum_i \frac{\partial \log \phi}{\partial x^i} (e) a^i = -s(a),$$

$$(18) \quad \sum_{i,j} \frac{\partial^2 \log \phi}{\partial x^i \partial x^j} (e) a^i a^j = \langle a, a \rangle = s(a \triangle a),$$

$$(19) \quad \sum_{i,j,k} \frac{\partial^3 \log \phi}{\partial x^i \partial x^j \partial x^k} (e) a^i a^j a^k = -2\langle a, a \triangle a \rangle = -2s(a \triangle (a \triangle a)),$$

where $a = \sum_i a^i \left(\frac{\partial}{\partial x^i} \right)_e$. Taking $a+b$ and $a+b+c$ instead of a in the formulae (18) and (19) respectively we obtain

$$(18') \quad \sum_{i,j} \frac{\partial^2 \log \phi}{\partial x^i \partial x^j} (e) a^i b^j = \langle a, b \rangle = s(a \triangle b),$$

$$(19') \quad 3 \sum_{i,j,k} \frac{\partial^3 \log \phi}{\partial x^i \partial x^j \partial x^k} (e) a^i b^j c^k \\ = -(\langle a, b \triangle c \rangle + \langle a, c \triangle b \rangle + \langle b, c \triangle a \rangle + \langle b, a \triangle c \rangle + \langle c, a \triangle b \rangle + \langle c, b \triangle a \rangle).$$

By (14) and (18') we have

$$(20) \quad \langle a \triangle b, c \rangle + \langle b, a \triangle c \rangle = \langle b \triangle a, c \rangle + \langle a, b \triangle c \rangle.$$

Using this we get

$$(19'') \quad \sum_{i,j,k} \frac{\partial^3 \log \phi}{\partial x^i \partial x^j \partial x^k} (e) a^i b^j c^k = -\langle a \triangle b, c \rangle - \langle b, a \triangle c \rangle \\ = -\langle (L_a + {}^t L_a) b, c \rangle.$$

On the other hand it follows from (3) (8) (9) that

$$\sum_{i,j,k} \frac{\partial^3 \log \phi}{\partial x^i \partial x^j \partial x^k} (e) a^i b^j c^k = 2 \sum_{i,j,k,l} (g_{kl} l^l {}^t j_i) (e) a^i b^j c^k \\ = -2\langle S_a b, c \rangle.$$

Thus we have

$$S_a b = \frac{1}{2} (L_a + {}^t L_a) b$$

and (i) is proved.

For $X_a = \sum_i (\sum_j \alpha^i_j x^j + \alpha^i) \frac{\partial}{\partial x^i}$ and $X_b = \sum_i (\sum_j b^i_j x^j + b^i) \frac{\partial}{\partial x^i}$ we have

$$\begin{aligned} \mathcal{F}_{X_a} X_b &= \sum_i \left\{ \sum_p b^i_p (\sum_q \alpha^p_q x^q + \alpha^p) \right. \\ &\quad \left. + \sum_{r,s} I^i_{rs} (\sum_p \alpha^r_p x^p + \alpha^r) (\sum_q b^s_q x^q + b^s) \right\} \frac{\partial}{\partial x^i}. \end{aligned}$$

Since $x^i(e) = 0$, by (8), (11) and (i) the value $(\mathcal{F}_{X_a} X_b)_e$ of $\mathcal{F}_{X_a} X_b$ at e is reduced to

$$\begin{aligned} (\mathcal{F}_{X_a} X_b)_e &= \sum_i \left(\sum_p b^i_p \alpha^p + \sum_{r,s} I^i_{rs} \alpha^r b^s \right) \left(\frac{\partial}{\partial x^i} \right)_e \\ &= b \Delta a - b \square a \\ &= \frac{1}{2} (L_b - {}^t L_b) a. \end{aligned}$$

Therefore by (6) we get

$$A_{ab} = (A_{X_a} X_b)_e = -(\mathcal{F}_{X_b} X_a)_e = -\frac{1}{2} (L_a - {}^t L_a) b,$$

and (ii) is proved.

By (7) we have

$$R(X_a, X_b) = [A_{X_a}, A_{X_b}] - A_{[X_a, X_b]}.$$

Using (12), (14), (i) and (ii) we obtain

$$\begin{aligned} R(a, b) &= [A_a, A_b] - A_{b \Delta a - a \Delta b} \\ &= \left[-\frac{1}{2} (L_a - {}^t L_a), -\frac{1}{2} (L_b - {}^t L_b) \right] + \frac{1}{2} (L_{b \Delta a - a \Delta b} - {}^t L_{b \Delta a - a \Delta b}) \\ &= \frac{1}{4} [L_a - {}^t L_a, L_b - {}^t L_b] - \frac{1}{2} ([L_a, L_b] - {}^t [L_a, L_b]) \\ &= -[S_a, S_b], \end{aligned}$$

and so (iii) is proved.

Q. E. D.

PROPOSITION 2. (i) If Γ is parallel with respect to Γ , then we have

$$[A_a, S_b] = S_{A_a b}.$$

(ii) If R is parallel with respect to Γ , then we have

$$[A_a, [S_b, S_c]] = [S_{A_a b}, S_c] + [S_b, S_{A_a c}].$$

PROOF. Since $X_a = \sum_i \xi^i \frac{\partial}{\partial x^i}$ is an infinitesimal affine transformation with respect to Γ , we have

$$\begin{aligned}
0 &= \frac{\partial^2 \xi^i}{\partial x^j \partial x^k} + \sum_p \left(\xi^p \frac{\partial \Gamma^i_{jk}}{\partial x^p} + \Gamma^i_{pk} \frac{\partial \xi^p}{\partial x^j} + \Gamma^i_{jp} \frac{\partial \xi^p}{\partial x^k} - \Gamma^p_{jk} \frac{\partial \xi^i}{\partial x^p} \right) \\
&= \frac{\partial^2 \xi^i}{\partial x^j \partial x^k} + L_{X_a} \Gamma^i_{jk},
\end{aligned}$$

where $L_{X_a} \Gamma^i_{jk}$ is the Lie derivative of the tensor field Γ^i_{jk} by X_a . Since $\xi^i = \sum_j a^i_j x^j + a^i$, we get

$$(21) \quad L_{X_a} \Gamma^i_{jk} = 0.$$

From this we have

$$(22) \quad L_{X_a}(X_b \square X_c) = (L_{X_a} X_b) \square X_c + X_b \square (L_{X_a} X_c).$$

Therefore by (5), (10) and (22) the condition $\nabla_{X_a} \Gamma = 0$ is equivalent to

$$A_{X_a}(X_b \square X_c) = (A_{X_a} X_b) \square X_c + X_b \square (A_{X_a} X_c).$$

This implies (i). By (4) and (21) it follows

$$L_{X_a} R = 0.$$

Thus by (5) the condition $\nabla_{X_a} R = 0$ is equivalent to $A_{X_a} R = 0$. Since A_{X_a} is a derivation of the algebra of tensor fields, we have

$$\begin{aligned}
((A_{X_a} R)(X_b, X_c))X_d &= A_{X_a}(R(X_b, X_c)X_d) - R(A_{X_a} X_b, X_c)X_d \\
&\quad - R(X_b, A_{X_a} X_c)X_d - R(X_b, X_c)A_{X_a} X_d \\
&= ([A_{X_a}, R(X_b, X_c)] - R(A_{X_a} X_b, X_c) - R(X_b, A_{X_a} X_c))X_d
\end{aligned}$$

and so by Proposition 1 (iii)

$$\begin{aligned}
((A_a R)(b, c))d &= ([A_a, R(b, c)] - R(A_a b, c) - R(b, A_a c))d \\
&= ([A_a, -[S_b, S_c]] + [S_{A_a b}, S_c] + [S_b, S_{A_a c}])d
\end{aligned}$$

This proves (ii).

Q. E. D.

LEMMA 1. *If Γ is parallel with respect to Γ , then Ω is a cone.*

PROOF. It is known that if the clan $V(\Omega)$ of Ω has a unit element then Ω is a cone [5]. Therefore by Proposition 2 (i) it suffices to show that if $[A_a, S_b] = S_{A_a b}$ holds for all $a, b \in V(\Omega)$, then $V(\Omega)$ has a unit element. Let u be the principal idempotent of the clan $V(\Omega)$, i. e., u is an element in $V(\Omega)$ determined by $\langle u, a \rangle = s(a)$ for all $a \in V(\Omega)$. Then we get the principal decomposition

$$V(\Omega) = V_0 + N,$$

where $V_0 = \{a \in V(\Omega); u \Delta a = a\}$ and $N = \{a \in V(\Omega); u \Delta a = \frac{1}{2}a\}$. The principal de-

composition $V(\Omega) = V_0 + N$ is orthogonal with respect to the inner product \langle , \rangle and the following relations hold [5]

$$(23) \quad \begin{aligned} V_0 \Delta V_0 &\subset V_0, & V_0 \Delta N &\subset N, \\ N \Delta V_0 &= \{0\}, & N \Delta N &\subset V_0. \end{aligned}$$

Let p be an element in N . By our assumption and Proposition 1 we have

$$\begin{aligned} 0 &= \langle ([A_p, S_p] - S_{A_p p})p, p \rangle = \langle A_p S_p p - 2S_p A_p p, p \rangle \\ &= -3 \langle S_p p, A_p p \rangle = \frac{3}{4} \langle (L_p + {}^t L_p)p, (L_p - {}^t L_p)p \rangle \\ &= \frac{3}{4} (\langle L_p p, L_p p \rangle - \langle {}^t L_p p, {}^t L_p p \rangle). \end{aligned}$$

Therefore by the orthogonality of the principal decomposition and by (23) we get

$$\langle p \Delta p, p \Delta p \rangle = \langle {}^t L_p p, {}^t L_p p \rangle = \langle p, L_p {}^t L_p p \rangle = 0.$$

This means $p \Delta p = 0$, $\langle p, p \rangle = s(p \Delta p) = 0$ and so $p = 0$. Thus we have $V(\Omega) = V_0$ and u is a unit element of the clan $V(\Omega)$. Q. E. D.

We shall now recall the notion of T -algebras [5] [6].

A *matrix algebra with involution* is an algebra over the real number field \mathbf{R} which is bigraded by the subspaces \mathfrak{A}_{ij} ($i, j = 1, \dots, m$) and provided with an involutive anti-automorphism $*$ in such a way that

$$\begin{aligned} \mathfrak{A}_{ij} \mathfrak{A}_{jk} &\subset \mathfrak{A}_{ik}, \\ \mathfrak{A}_{ij} \mathfrak{A}_{ik} &= \{0\} \quad \text{if } j \neq l, \\ \mathfrak{A}_{ij}^* &= \mathfrak{A}_{ji}. \end{aligned}$$

The general element of \mathfrak{A}_{ij} will be denoted by a_{ij}, b_{ij} , etc..

A matrix algebra with involution is said to be a T -algebra if the following axioms are satisfied:

(T.1) For any i the algebra \mathfrak{A}_{ii} is one-dimensional and admits an isomorphism $\rho: \mathfrak{A}_{ii} \rightarrow \mathbf{R}$ with the following properties.

(T.2) $a_{ii} b_{ij} = \rho(a_{ii}) b_{ij}$;

(T.3) $n_i \rho(a_{ij} b_{ji}) = n_j \rho(b_{ji} a_{ij})$, where $n_i = 1 + \frac{1}{2} \sum_{s \neq i} \dim \mathfrak{A}_{is}$;

(T.4) $\rho(a_{ij} a_{ij}^*) > 0$ if $a_{ij} \neq 0$;

(T.5) $a_{ij} (b_{jk} c_{ki}) = (a_{ij} b_{jk}) c_{ki}$;

(T.6) $a_{ij} (b_{jk} c_{kl}) = (a_{ij} b_{jk}) c_{kl}$ if $i < j < k$ and $j < l$;

(T.7) $a_{ij} (b_{jk} b_{jk}^*) = (a_{ij} b_{jk}) b_{jk}^*$ if $i < j < k$.

Let \mathfrak{K} denote the space of hermitian matrices in the T -algebra \mathfrak{A} ; $\mathfrak{K} = \{\alpha \in \mathfrak{A};$

$a^*=a$). For each $a=\sum_{i,j} a_{ij}\in\mathfrak{X}$ we put

$$\hat{a}=\frac{1}{2}\sum_i a_{ii}+\sum_{i<j} a_{ij},$$

$$a=\frac{1}{2}\sum_i a_{ii}+\sum_{i>j} a_{ij},$$

We define a multiplication $L_a b=a\Delta b$ in \mathfrak{X} by the formula

$$a\Delta b=\hat{a}b+ba,$$

Then \mathfrak{X} is a clan with unit element and we denote this clan by $\mathfrak{X}(\mathfrak{A})$. Let $\Omega(\mathfrak{A})$ be the set of matrices which are expressible in the form tt^* , where t is an upper triangular matrix with positive elements on the diagonal. Then $\Omega(\mathfrak{A})$ is a homogeneous convex cone in $\mathfrak{X}(\mathfrak{A})$. For every homogeneous convex cone Ω there exists a T -algebra \mathfrak{A} such that Ω is isomorphic to $\Omega(\mathfrak{A})$ and a clan $V(\Omega)$ of Ω is isomorphic to $\mathfrak{X}(\mathfrak{A})$.

Now we return to the proof of our theorems. By the above fact we may assume $V(\Omega)=\mathfrak{X}(\mathfrak{A})$. Then it is known that for $a, b\in\mathfrak{X}(\mathfrak{A})$

$$\text{Tr } L_a = \text{Spur } a,$$

$$\langle a, b \rangle = \text{Spur } ab,$$

where $\text{Spur } a = \sum_i n_i \rho(a_{ii})$. It is easy to see

$${}^t L_a b = qb + b\hat{a}.$$

Therefore we get

$$(24) \quad S_a b = \frac{1}{2}(ab + ba),$$

$$(25) \quad A_a b = -\frac{1}{2}\{(\hat{a}-a)b - b(\hat{a}-a)\}.$$

Let n_{ij} denote the dimension of \mathfrak{A}_{ij} . We define inductively an equivalence relation \bar{R} in the set $\{1, \dots, m\}$ of indices:

- (1) $i \equiv i \pmod{\bar{R}}$ for all i ,
- (2) if we have already determined whether the i, j such that $|i-j| < r$ are comparable modulo \bar{R} or not, then for $|i-j|=r$ we define $i \equiv j \pmod{\bar{R}}$ if and only if (i) $n_{ij} \neq 0$, (ii) $n_{ik} = n_{jk}$ for all $k \neq i, j$, and (iii) for all k lying between i and j (except i and j) either $n_{ik} = n_{kj} = 0$ or $i \equiv k \pmod{\bar{R}}$ and $k \equiv j \pmod{\bar{R}}$.

We put

$$\mathfrak{A}_{ij}^c = \begin{cases} \mathfrak{A}_{ij} & \text{if } i \equiv j \pmod{\bar{R}} \\ \{0\} & \text{if } i \not\equiv j \pmod{\bar{R}}. \end{cases}$$

$$\mathfrak{A}^c = \sum_{i,j} \mathfrak{A}_{ij}^c.$$

Then \mathfrak{A}^c is a T -algebra and the homogeneous convex cone $\Omega(\mathfrak{A}^c)$ corresponding to \mathfrak{A}^c is self-dual.

LEMMA 2. *If the clan $\mathfrak{X}(\mathfrak{A})$ corresponding to a T -algebra \mathfrak{A} satisfies the condition*

$$[A_a, [S_b, S_c]] = [S_{A_a b}, S_c] + [S_b, S_{A_a c}],$$

then we have $\mathfrak{A} = \mathfrak{A}^c$.

PROOF. By the condition we have

$$[[A_a, S_b] - S_{A_a b}, S_c] = [[A_a, S_c] - S_{A_a c}, S_b].$$

Let $a_{ij} \in \mathfrak{A}_{ij}$, $b_{jk} \in \mathfrak{A}_{jk}$ and $e_i \in \mathfrak{A}_{ii}$, where $i < j$, $k \neq i, j$ and $\rho(e_i) = 1$. We put $a = a_{ij} + a_{ij}^*$, $b = b_{jk} + b_{jk}^*$ and $c = e_i$ and calculate the following formula

$$[[A_a, S_b] - S_{A_a b}, S_c]b = [[A_a, S_c] - S_{A_a c}, S_b]b$$

Using (24) and (25), the left side is equal to

$$\frac{1}{4} \{a_{ij}(b_{jk}b_{jk}^*) - (a_{ij}b_{jk})b_{jk}^* + (b_{jk}b_{jk}^*)a_{ij}^* - b_{jk}(b_{jk}^*a_{ij}^*)\}$$

and the right side is reduced to 0. Considering \mathfrak{A}_{ij} -component, by (T.2) we get

$$(a_{ij}b_{jk})b_{jk}^* = a_{ij}(b_{jk}b_{jk}^*) = \rho(b_{jk}b_{jk}^*)a_{ij}.$$

Multiplying both sides on the right by a_{ij}^* we obtain

$$((a_{ij}b_{jk})b_{jk}^*)a_{ij}^* = \rho(b_{jk}b_{jk}^*)a_{ij}a_{ij}^*.$$

Therefore, by (T.2) we have

$$(26) \quad \rho((a_{ij}b_{jk})(a_{ij}b_{jk})^*) = \rho((a_{ij}b_{jk})b_{jk}^*)a_{ij}^* = \rho(b_{jk}b_{jk}^*)\rho(a_{ij}a_{ij}^*).$$

Assume $n_{ij} \neq 0$. For $a_{ij} \neq 0 \in \mathfrak{A}_{ij}$, by (26) the linear mapping given by $\mathfrak{A}_{jk} \ni b_{jk} \rightarrow a_{ij}b_{jk} \in \mathfrak{A}_{ik}$ is injective and so $n_{jk} \leq n_{ik}$. In the same way we have $n_{ik} \leq n_{jk}$. Therefore we have $n_{ik} = n_{jk}$ for all $k \neq i, j$. This implies that $i \equiv j \pmod{\bar{R}}$ if $n_{ij} \neq 0$. Thus we have $\mathfrak{A} = \mathfrak{A}^c$. Q. E. D.

Since the homogeneous convex cone $\Omega(\mathfrak{A}^c)$ determined by \mathfrak{A}^c is self-dual, in view of Proposition 2, Lemma 1 and 2 the sufficient conditions of our theorems are proved.

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