TSUKUBA J. MATH.
Vol. 16 No. 2 (1992), 363-376

ON THE VANISHING OF HOCHSCHILD COHOMOLOGY $H^1(\Lambda, \Lambda\otimes\Lambda)$ FOR A LOCAL ALGEBRA Λ

By

Qiang ZENG

§ 0. Introduction.

Throughout this paper we assume that Λ is a finite dimensional local algebra over an algebraically closed field K. By considering certain subgroups of the Hochschild cohomology groups of Λ-bimodule $\Lambda\otimes\Lambda$ for a generalized biserial commutative algebra Λ the author proved in [7] that Λ is selfinjective if and only if $H^1(\Lambda, \Lambda\otimes\Lambda)=0$. Here Λ is called to be *generalized biserial* if the both composition lengths of $_\Lambda((\mathrm{rad}\,\Lambda)^i/(\mathrm{rad}\,\Lambda)^{i+1})$ and $((\mathrm{rad}\,\Lambda)^i/(\mathrm{rad}\,\Lambda)^{i+1})_\Lambda$ $\leqq 2$ for all $i=1, 2, \cdots$.

On the other hand for a commutative algebra Λ with cube zero radical using Hoshino's results Asashiba proved in [1] that Λ is selfinjective if and only if $\mathrm{Ext}^1_\Lambda({}_\Lambda\mathrm{Hom}_K(\Lambda_\Lambda, K), {}_\Lambda\Lambda)\cong H^1(\Lambda, \Lambda\otimes\Lambda)=0$.

One of the purposes of this paper is to show in § 1 that Asashiba's results together with Hoshino's can be proved directly by calculating the similar subgroups of the Hochschild cohomology of Λ-bimodule $\Lambda\otimes\Lambda$ with [7].

It was conjectured in [5] that Λ is selfinjective if $H^i(\Lambda, \Lambda\otimes\Lambda)=0$ for $i=1, 2, \cdots$. The above results implies that a commutative algebra Λ is selfinjective if $H^1(\Lambda, \Lambda\otimes\Lambda)=0$ and Λ is either generalized biserial or of cube zero radical. So it is interesting to consider the same problem for an algebra with quartic zero radical which is a homomorphic image of the polynomial ring $K[x, y]$ of variables x and y. In § 2 we shall prove that for such algebras we have also an affirmative answer. However it is to be noted here that for this case it needs to consider the larger subgroups of the Hochschild cohomology of Λ-bimodule $\Lambda\otimes\Lambda$ different than ones for the above stated cases.

As was seen in [6] and [7] for commutative algebra Λ it holds that the both composition lengths of $(\mathrm{rad}\,\Lambda)/(\mathrm{rad}\,\Lambda)^2$ and $(\mathrm{rad}\,\Lambda)^2/(\mathrm{rad}\,\Lambda)^3\leqq 2$ implies that Λ is generalized biserial. In § 3 we shall show that we can generalize the above fact for non-commutative algebras. At the end of this section we shall quote that for a (not necessarily commutative) positively **Z**-gradable algebra Λ

Received November 21, 1991, Revised December 13, 1991.

we can choose a set of homogeneous elements with respect to the grading of Λ as a system of minimal generators of rad Λ.

§1. $H^1(\Lambda, \Lambda\otimes\Lambda)$ for algebra Λ with cube zero radical.

Let Λ be a local algebra over an algebraically closed field K having a cube zero radical N. Then the following results were obtained by Asashiba and Hoshino.

PROPOSITION 1.1. (M. Hoshino, see [1]) *If* $H^1(\Lambda, \Lambda\otimes\Lambda)=0$, *then* $\dim_K N^2 \leqq 2$.

THEOREM 1.2 (H. Asashiba [1]) *If Λ is commutative and $H^1(\Lambda, \Lambda\otimes\Lambda)=0$, then Λ is selfinjective.*

Let $x_1, x_2, \cdots, x_n$ be the elements of Λ such that $x_1+N, x_2+N, \cdots, x_n+N$ are a K-basis of N/N^2 and $w_1, w_2, \cdots, w_m$ a K-basis of N^2.

Put $x_i x_j=\sum_{k=1}^m a_{ij}^k w_k$ for $a_{ij}^k\in K$ and $1\leqq i, j\leqq n$, and let us denote by A_k the $n\times n$ matrix (a_{ij}^k) and by ${}^tA_k$ the transpose of A_k.

In order to prove the above results we shall introduce the following Theorem 1.3.

THEOREM 1.3. *Let T be the following $nm^2\times 2nm$ matrix*

$$\begin{bmatrix}
A_1 & & & {}^tA_1 & & & & \\
& A_1 & & {}^tA_2 & & & & \\
& & \ddots & \vdots & & & & \\
& & A_1 & {}^tA_m & & & & \\
A_2 & & & & {}^tA_1 & & & \\
& A_2 & & & {}^tA_2 & & & \\
& & \ddots & & \vdots & & & \\
& & A_2 & & {}^tA_m & & & \\
& & \vdots & & & \ddots & & \\
A_m & & & & & & \cdots & {}^tA_1 \\
& A_m & & & & & \cdots & {}^tA_2 \\
& & \ddots & & & & & \vdots \\
& & A_m & & & & \cdots & {}^tA_m
\end{bmatrix}.$$

If $H^1(\Lambda, \Lambda\otimes\Lambda)=0$, *then* rank $T\geqq nm^2$.

PROOF. Let

$$0 \longrightarrow J \longrightarrow \Lambda\otimes\Lambda \xrightarrow{\varepsilon} \Lambda \longrightarrow 0$$

be an exact sequence of Λ^e-modules with a canonical homomorphism $\varepsilon: \Lambda\otimes\Lambda \to\Lambda$ defined by putting $\varepsilon(x\otimes y)=xy$, where $\Lambda^e=\Lambda\otimes\Lambda^o$ and Λ^o is the opposite ring of Λ. Then

$$\begin{aligned} H^1(\Lambda, \Lambda\otimes\Lambda) &\cong \mathrm{Ext}^1_{\Lambda^e}({}_{\Lambda^e}\Lambda, {}_{\Lambda^e}\Lambda\otimes\Lambda) \\ &\cong \mathrm{Hom}_{\Lambda^e}(J, \Lambda\otimes\Lambda)/\{(\Phi\,|\,J)\,|\,\Phi\in\mathrm{Hom}_{\Lambda^e}(\Lambda\otimes\Lambda, \Lambda\otimes\Lambda)\}. \end{aligned}$$

Cf. [2]. Since ${}_{\Lambda^e}J$ is generated by $\{\iota_i=x_i\otimes 1-1\otimes x_i\,|\,i=1, 2, \cdots, n\}$ and $N^2\subset \mathrm{soc}\,{}_\Lambda\Lambda\cap\mathrm{soc}\,\Lambda_\Lambda$ and $\mathrm{soc}\,{}_{\Lambda^e}\Lambda\otimes\Lambda=\mathrm{soc}\,{}_\Lambda\Lambda\otimes\mathrm{soc}\,\Lambda_\Lambda$, we can define a Λ^e-homomorphism $\psi: J\to\Lambda\otimes\Lambda$ by giving n elements of $N^2\otimes N^2$ as the values of $\psi(\iota_i)$'s respectively. Let us denote by H the subgroup of $H^1(\Lambda, \Lambda\otimes\Lambda)$ which is generated by the residue classes of $\mathrm{Hom}_{\Lambda^e}(J, \Lambda\otimes\Lambda)$ whose representatives are such ψ's. Then by $H\subset H^1(\Lambda, \Lambda\otimes\Lambda)=0$ we have an extension $\Psi: \Lambda\otimes\Lambda\to\Lambda\otimes\Lambda$ of ψ and we can put

$$\Psi(1\otimes 1)=1\otimes h_0+\textstyle\sum_{j=1}^n x_j\otimes h_j+\sum_{k=1}^m w_k\otimes g_k\,,$$

where $h_j=\alpha_{j0}1+\sum_{i=1}^n\alpha_{ji}x_i+\sum_{k=1}^m\beta_{jk}w_k$ with $\alpha_{j0}, \alpha_{ji}, \beta_{jk}\in K$ for $j=0, 1, \cdots, n$ and $g_k=\gamma_{k0}1+\sum_{i=1}^n\gamma_{ki}x_i+\sum_{l=1}^m\delta_{kl}w_k$ with $\gamma_{k0}, \gamma_{ki}, \delta_{kl}\in K$ for $k=1, 2, \cdots, m$. It follows that

$$\begin{aligned} \Psi(\iota_i)=x_i\otimes h_0&+\textstyle\sum_{j=1}^n x_ix_j\otimes h_j-1\otimes h_0x_i \\ &-\textstyle\sum_{j=1}^n x_j\otimes h_jx_i-\sum_{k=1}^m w_k\otimes g_kx_i \end{aligned}$$

for all i. From $x_ix_j=\sum_{l=1}^m a_{ij}^l w_l$ and the assumption that $\Psi(\iota_i)=\psi(\iota_i)\in N^2\otimes N^2$ it follows that

$$\begin{aligned} \Psi(\iota_i)&=\textstyle\sum_{j=1}^n(\sum_{l=1}^m a_{ij}^l w_l)\otimes h_j-\sum_{k=1}^m w_k\otimes g_kx_i \\ &=\textstyle\sum_{l=1}^m w_l\otimes(\sum_{j=1}^n a_{ij}^l h_j-g_lx_i) \end{aligned}$$

and we can put

$$\textstyle\sum_{j=1}^n a_{ij}^l h_j-g_lx_i=\sum_{s=1}^m\xi_{ils}w_s\,, \qquad \xi_{ils}\in K\,.$$

On the other hand

$$\begin{aligned} \textstyle\sum_{j=1}^n a_{ij}^l h_j-g_lx_i&=\textstyle\sum_{j=1}^n a_{ij}^l(\alpha_{j0}1+\sum_{r=1}^n\alpha_{jr}x_r+\sum_{s=1}^m\beta_{js}w_s) \\ &\quad-(\gamma_{l0}1+\textstyle\sum_{p=1}^n\gamma_{lp}x_p+\sum_{q=1}^m\delta_{lq}w_q)x_i \\ &=(\textstyle\sum_{j=1}^n a_{ij}^l)\alpha_{j0}+\sum_{r=1}^n(\sum_{j=1}^n a_{ij}^l\alpha_{jr})x_r-\gamma_{l0}x_i \\ &\quad+\textstyle\sum_{s=1}^m(\sum_{j=1}a_{ij}^l\beta_{js})w_s-\sum_{s=1}^m(\sum_{p=1}\gamma_{lp}a_{pi}^s)w_s\,. \end{aligned}$$

Thus we have the following simultaneous linear equations with unknown β_{js} and γ_{lp}:

$$\sum_{j=1}^{n} a_{ij}^{l}\beta_{js} - \sum_{p=1}^{n} a_{pi}^{s}\gamma_{lp} = \xi_{ils}$$

for $i=1, 2, \cdots, n$ and $l, s=1, 2, \cdots, m$. But for any ξ_{ils} we can define Λ^e-homomorphism $\psi: J \to \Lambda\otimes\Lambda$ and by noting $a_{ij}^{l}=(i, j)$-component of A_l, $a_{pi}^{s}=(i, p)$-component of ${}^tA_s$, we get that $nm^2 \leqq$ the rank of T.

PROOF OF PROPOSITION 1.1. Since the number of the columns is $2mn$ it is necessary to hold $2mn \geqq m^2n$. Thus $m \leqq 2$.

Now we have the following immediately

COROLLARY 1.4. (H. Asashiba [1]) *If* $H^1(\Lambda, \Lambda\otimes\Lambda)=0$ *and* $m=2$, *then*

$$\begin{bmatrix} A_1 & & {}^tA_1 & \\ & A_1 & {}^tA_2 & \\ A_2 & & & {}^tA_1 \\ & A_2 & & {}^tA_2 \end{bmatrix}$$

is regular.

LEMMA 1.5. *Let* Λ *be a commutative algebra with* $N^3=0$. *If* $H^1(\Lambda, \Lambda\otimes\Lambda)=0$, *then* $\operatorname{soc}\Lambda=N^2$.

PROOF. We may assume that $\dim_K(\operatorname{rad}\Lambda/(\operatorname{rad}\Lambda)^2 \geqq 2$ because otherwise Λ is uniserial. Take $x_{i_0} \in (\operatorname{soc}\Lambda)\setminus N^2$ and define Λ^e-homomorphism $\varphi: J \to (\operatorname{soc}\Lambda)\otimes(\operatorname{soc}\Lambda) \subset \Lambda\otimes\Lambda$ by putting $\varphi(\iota_{i_1})=x_{i_0}\otimes x_{i_0}$ for some $i_1\neq i_0$ and $\varphi(\iota_i)=0$ for all $i\neq i_1$. Then by the assumption that $H^1(\Lambda, \Lambda\otimes\Lambda)=0$ we have an extension $\Phi: \Lambda\otimes\Lambda \to \Lambda\otimes\Lambda$ of φ with

$$\Phi(1\otimes 1)=1\otimes h_0+\sum_{j=1}^{n} x_j\otimes h_j+\sum_{k=1}^{m} w_k\otimes g_k .$$

It follows from $\Phi(\iota_{i_1})=x_{i_0}\otimes x_{i_0}$ that $-h_{i_0}x_{i_1}=x_{i_0}$, a contradiction, for x_{i_0} and x_{i_1} are K-linearly independent.

PROOF OF THEOREM 1.2. By Lemma 1.5 it is enough to prove $m=1$. Suppose $m\neq 1$. Then by Proposition 1.1 $m=2$ but then T is non-regular since $A_i={}^tA_i$ for $i=1, 2$. This implies $H^1(\Lambda, \Lambda\otimes\Lambda)\neq 0$ and a contradiction.

§2. The case where Λ is a homomorphic image of $K[x, y]/(x, y)^4$.

As mensioned in the introduction we shall consider for an algebra Λ with quartic zero radical which is a homomorphic image of the polynomial ring

$K[x, y]$ of variables x and y. However Λ is same with a homomorphic image of $K[x, y]/(x, y)^4$, since Λ is an artin ring and K is an algebraically closed field.

At the beginning we shall prove

LEMMA 2.1. *Let Λ be a homomorphic image of $K[x, y]/(x, y)^4$. Then Λ is generalized biserial if Λ is selfinjective.*

PROOF. Suppose that Λ is selfinjective but not generalized biserial. Then $\{x^2+(\mathrm{rad}\,\Lambda)^3,\ xy+(\mathrm{rad}\,\Lambda)^3,\ y^2+(\mathrm{rad}\,\Lambda)^3\}$ is a free K-basis of $(\mathrm{rad}\,\Lambda)^2/(\mathrm{rad}\,)^3$, because otherwise they are K-linearly dependent and hence Λ is a homomorphic image of $K[x, y]/((x, y)^4, f)$, where the polynomial $f\in K[x, y]$ has a non-zero homogeneous term of degree two and then Λ is generalized biserial as was proved in [6]. Hence we can suppose that Λ is one of the following cases:

Case 0: $\Lambda=K[x, y]/(x, y)^4$;

Case I: $\Lambda=K[x, y]/((x, y)^4, f_1)$ where $0\neq f_1=ax^3+bx^2y+cxy^2+dy^3$ with a, b, c and $d\in K$.

Case II: $\Lambda=K[x, y]/((x, y)^4, f_1, f_2)$ where f_1 and f_2 are K-linearly independent and $f_i=a_ix^3+b_ix^2y+c_ixy^2+d_iy^3$ with a_i, b_i, c_i and $d_i\in K$ for $i=1, 2$.

Case III: $\Lambda=K[x, y]/((x, y)^4, f_1, f_2, f_3)$ where f_1, f_2 and f_3 are K-linearly independent and $f_i=a_ix^3+b_ix^2y+c_ixy^2+d_iy^3$ with a_i, b_i, c_i and $d_i\in K$ for $i=1, 2, 3$.

It is easy to see that if Λ is one of the Case 0—II, then $\mathrm{soc}\,\Lambda$ is not simple. Hence it is enough to consider the Case III only because Λ is selfinjective and $\mathrm{soc}\,\Lambda$ is simple. Since Λ is not generalized biserial the series of composition lengths of factor modules with respect to the upper Loewy series of Λ is (1, 2, 3, 1), that is, $\dim_K \Lambda/(\mathrm{rad}\,\Lambda)=1$, $\dim_K (\mathrm{rad}\,\Lambda)/(\mathrm{rad}\,\Lambda)^2=2$, $\dim_K (\mathrm{rad}\,\Lambda)^2/(\mathrm{rad}\,\Lambda)^3=3$ and $\dim_K (\mathrm{rad}\,\Lambda)^4=1$.

Now we observe the lower Loewy series of Λ. At first, it does not happen that $\dim_K \mathrm{soc}^2\,\Lambda/\mathrm{soc}\,\Lambda=3$ and $\dim_K \mathrm{soc}\,\Lambda=1$. Because otherwise the series of composition lengths of factor modules with respect to the upper Loewy series of $D\Lambda$, where D denotes the usual selfduality $\mathrm{Hom}_K(-, K)$, is $(1, 3, *, *)$ but $D\Lambda\cong\Lambda$ and this is a contradiction. Hence the series of composition lengths of factor modules with respect to the lower Loewy series of Λ is either $(*, *, 2, 1)$ or $(*, *, 1, 1)$. But they do not happen because x^2, xy and $y^2\in(\mathrm{rad}\,\Lambda)^2\setminus(\mathrm{rad}\,\Lambda)^3$ and $(\mathrm{rad}\,\Lambda)^3$ $(\subset\mathrm{soc}\,\Lambda)$ is simple by the K-linearly independent assumption of f_1, f_2 and f_3. It concludes that Λ is generalized biserial.

COROLLARY 2.2. *Let Λ be isomorphic to $K[x, y]/((x, y)^4, f_1, f_2, f_3)$ where f_1, f_2 and f_3 are K-linearly independent and $f_i=a_ix^3+b_ix^2y+c_ixy^2+d_iy^3$ with a_i, b_i, c_i and $d_i\in K$ for $i=1, 2, 3$, then there exists an element $\alpha\in(\mathrm{rad}\,\Lambda)^2\setminus(\mathrm{rad}\,\Lambda)^3$ such that $\alpha\in\mathrm{soc}\,\Lambda$.*

Now our main purpose of this section is to prove

THEOREM 2.3. *Let Λ be a homomorphic image of $K[x, y]/(x, y)^4$. If $H^1(\Lambda, \Lambda\otimes\Lambda)=0$, then Λ is selfinjective.*

Note that it was proved in [7] that the theorem is true if Λ is generalized biserial. Thus it is enough to prove that if Λ is not generalized biserial, then $H^1(\Lambda, \Lambda\otimes\Lambda)\neq 0$. Throughout this section we assume hereafter that Λ is not generalized biserial.

Let us denote $x\otimes 1-1\otimes x$ and $y\otimes 1-1\otimes y$ by ι_1 and ι_2 respectively.

LEMMA 2.4. *Let Λ be not a generalized biserial. If $\lambda\iota_1=\mu\iota_2$ for λ, $\mu\in\Lambda^e$, then $\lambda=a\iota_2+\lambda_1$ and $\mu=a\iota_1+\mu_1$ with some $a\in K$ and λ_1, $\mu_1\in(\mathrm{rad}\,\Lambda^e)^2$.*

PROOF. Put $\lambda=a_{1,1}(1\otimes 1)+a_{x,1}(x\otimes 1)+a_{1,x}(1\otimes x)+a_{y,1}(y\otimes 1)+a_{1,y}(1\otimes y)+\lambda_1$ and $\mu=b_{1,1}(1\otimes 1)+b_{x,1}(x\otimes 1)+b_{1,x}(1\otimes x)+b_{y,1}(y\otimes 1)+b_{1,y}(1\otimes y)+\mu_1$ with λ_1, $\mu_1\in(\mathrm{rad}\,\Lambda^e)^2$. Then it is clear that $\{a_{1,1}(1\otimes 1)+a_{x,1}(x\otimes 1)+a_{1,x}(1\otimes x)+a_{1,y}(1\otimes y)+a_{y,1}(y\otimes 1)\}\times\iota_1\equiv\{b_{1,1}(1\otimes 1)+b_{x,1}(x\otimes 1)+b_{1,x}(1\otimes x)+b_{y,1}(y\otimes 1)+b_{1,y}(1\otimes y)\}\times\iota_2$ $\mathrm{mod}\,(\mathrm{rad}\,\Lambda^e)^3$. That is $a_{1,1}\iota_1+a_{x,1}(x^2\otimes 1)+(a_{1,x}-a_{x,1})(x\otimes x)-a_{1,x}(1\otimes x^2)+a_{y,1}(xy\otimes 1)-a_{y,1}(y\otimes x)+a_{1,y}(x\otimes y)-a_{1,y}(1\otimes xy)\equiv b_{1,1}\iota_2+b_{x,1}(xy\otimes 1)-b_{x,1}(x\otimes y)+b_{1,x}(y\otimes x)-b_{1,x}(1\otimes xy)+b_{y,1}(y^2\otimes 1)+(b_{1,y}-b_{y,1}))y\otimes y)-b_{1,y}(1\otimes y^2)\,\mathrm{mod}\,(\mathrm{rad}(\Lambda^e)^3$ Since x^2, xy and y^2 are K-linearly independent $\mathrm{mod}\,(\mathrm{rad}\,\Lambda)^3$ we have that $a_{1,1}=0=b_{1,1}$, $a_{x,1}=0=a_{1,x}$, $b_{y,1}=0=b_{1,y}$, $a_{y,1}=b_{x,1}=-a_{1,y}=-b_{1,x}$. Now put $a_{y,1}=a$ for some $a\in K$.

Let

$$0\longrightarrow J\longrightarrow\Lambda\otimes\Lambda\xrightarrow{\varepsilon}\Lambda\longrightarrow 0$$

be an exact sequence of the left Λ^e-modules with a canonical homomorphism ε defined by putting $\varepsilon(s\otimes t)=st$ for s, $t\in\Lambda$. Then ${}_{\Lambda^e}J$ is generated by ι_1 and ι_2. And

$$H^1(\Lambda, \Lambda\otimes\Lambda)\cong\mathrm{Hom}_{\Lambda^e}(J, \Lambda\otimes\Lambda)/\{(\Phi\,|\,J)\,|\,\Phi\in\mathrm{Hm}_{\Lambda^e}(\Lambda\otimes\Lambda, \Lambda\otimes\Lambda)\}.$$

Since $\mathrm{soc}_{\Lambda^e}(\Lambda\otimes\Lambda)=(\mathrm{soc}_\Lambda\Lambda)\otimes(\mathrm{soc}\,\Lambda_\Lambda)$, for any elements u_1 and $u_2\in(\mathrm{soc}_\Lambda\Lambda)\otimes(\mathrm{soc}\,\Lambda_\Lambda)$ we can define a Λ^e-homomorphism $f: J\to\Lambda\otimes\Lambda$ by putting $f(\iota_1)=u_1$

and $f(\iota_2)=u_2$. In [7] and in §1 we have consider a subgroup $\underline{H}$ of $H^1(\Lambda, \Lambda\otimes\Lambda)$ which is generated by the residue classes of $\mathrm{Hom}_{\Lambda^e}(J, \Lambda\otimes\Lambda)$ whose representatives are such homomorphisms f's.

However in the proof of Theorem 2.3 it needs to consider some Λ^e-homomorphisms $g: J\to\Lambda\otimes\Lambda$ which do not belong to $\underline{H}$. Then the following lemma is useful to check whether a map from J to $\Lambda\otimes\Lambda$ is a Λ^e-homomorphism.

LEMMA 2.5. *Take two elements γ and δ belonging to* $\mathrm{rad}^5_{\Lambda^e}(\Lambda\otimes\Lambda)$ *and define $f: J\to\Lambda\otimes\Lambda$ by putting $f(\alpha\iota_1+\beta\iota_2)=\alpha\gamma+\beta\delta$ for any $\alpha, \beta\in\Lambda^e$. If $\delta\iota_1=\gamma\iota_2$, then f is a Λ^e-homomorphism.*

PROOF. If $\lambda\iota_1=\mu\iota_2$ for any $\lambda, \mu\in\Lambda^e$, then by Lemma 2.4 $\lambda=a\iota_2+\lambda_1$ and $\mu=a\iota_1+\mu_1$ with $\lambda_1, \mu_1\in(\mathrm{rad}\,\Lambda^e)^2$. Hence $\lambda_1 f(\iota_1)$, $\mu_1 f(\iota_2)\in(\mathrm{rad}\,\Lambda^e)^7=0$. Therefore $\delta\iota_1=\gamma\iota_2$ implies $\lambda f(\iota_1)=\mu f(\iota_2)$. Thus f is well-defined.

PROOF OF THEOREM 2.3. As in the proof of Lemma 2.1, we shall divide the proof of this theorem into the following cases:

Case 0: $\Lambda=K[x, y]/(x, y)^4$;

Case I: $\Lambda=K[x, y]/((x, y)^4, f_1)$ where $0\neq f_1=ax^3+bx^2y+cxy^2+dy^3$ with a, b, c and $d\in K$.

Case II: $\Lambda=K[x, y]/((x, y)^4, f_1, f_2)$ where f_1 and f_2 are K-linearly independent and $f_i=a_ix^3+b_ix^2y+c_ixy^2+d_iy^3$ with a_i, b_i, c_i and $d_i\in K$ for $i=1, 2$.

Case III: $\Lambda=K[x, y]/((x, y)^4, f_1, f_2, f_3)$ where f_1, f_2 and f_3 are K-linearly independent and $f_i=a_ix^3+b_ix^2y+c_ixy^2+d_iy^3$ with a_i, b_i, c_i and $d_i\in K$ for $i=1, 2, 3$.

Now for all cases 0-III we shall denote the defining ideal by I and denote the residue classes $u+I$ for $u\in K[x, y]$, that is an element of Λ, simply by u.

At first we shall prove for the Case 0: $\Lambda=K[x, y]/(x, y)^4$. We define a K-homomorphism $f: J\to\Lambda\otimes\Lambda$ by setting $f(\iota_1)=y^3\otimes y^3$ and $f(\iota_2)=0$. Since $y^3\otimes y^3\in\mathrm{soc}_{\Lambda^e}(\Lambda\otimes\Lambda)$, f is obviously well-defined as a Λ^e-homomorphism. But f cannot be extended to any $\varphi: \Lambda\otimes\Lambda\to\Lambda\otimes\Lambda$. Because if $\varphi: \Lambda\otimes\Lambda\to\Lambda\otimes\Lambda$ is an extension of f, then φ is defined by $\varphi(1\otimes 1)=\sum_{v\in V}v\otimes h_v$ since $\Lambda\otimes\Lambda$ is generated by $1\otimes 1$ as a Λ^e-module, where $h_v=h_v(x, y)$ is a K-linear combination of $V=\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2$ and $y^3\}$ the K-basis of Λ. And

$$\varphi(\iota_1)=y^3\otimes y^3 \qquad \text{(i)}$$

$$\varphi(\iota_2)=0 \qquad \text{(ii)}$$

But $\varphi(\iota_1)=\sum_{v\in V}xv\otimes(h_v-xh_{xv})+\sum_{u\in V\setminus\{xv|v\in V\}}u\otimes(-xh_u)=-1\otimes xh_1+x\otimes(h_1-xh_x)$

$-y\otimes xh_y+x^2\otimes(h_x-xh_{x^2})+xy\otimes(h_y-xh_{xy})-y^2\otimes xh_{y^2}+x^3\otimes(h_{x^2}-xh_{x^3})+x^2y\otimes(h_{xy}-xh_{x^2y})+xy^2\otimes(h_{y^2}-xh_{xy^2})-y^3\otimes xh_{y^3}$, thus from (i) $-xh_{y^3}=y^3$, but this is a contradiction because $xh_{y^3}+y^3$ does not belong to $(x, y)^4$.

Now we shall prove for the Case I: $\Lambda=K[x, y]/((x, y)^4, f_1)$. Here we may assume that $f_1=ax^3+bx^2y+cxy^2+dy^3=s(x-\alpha y)(x-\beta y)(x-\gamma y)$ with $s\neq 0$, α, β and $\gamma\in K$ because K is assumed to be algebraically closed. Therefore according to (1) $\alpha=\beta=\gamma$; (2) $\alpha=\beta\neq\gamma$; (3) $\alpha\neq\beta$, $\alpha\neq\gamma$ and $\beta\neq\gamma$, we can change the variables of the polynomial ring $K[x, y]$ so that f_1 is one of the following three cases: (1) $f_1=x^3$; (2) $f_1=x^2y$; (3) $f_1=xy(x-y)$. But for any case, if we define $f: J\to\Lambda\otimes\Lambda$ by setting $f(\iota_1)=y^3\otimes y^3\in\operatorname{soc}_{\Lambda^e}(\Lambda\otimes\Lambda)$ and $f(\iota_2)=0$, then f is obviously well-defined and f has no any extension to $\Lambda\otimes\Lambda\to\Lambda\otimes\Lambda$ by a similar argument as in Case 0. Namely $xh_{y^3}+y^3$ does not belong to $((x, y)^4, f_1)$.

Next we shall prove for the Case II: $\Lambda=K[x, y]/((x, y)^4, f_1, f_2)$ where f_1, f_2 are K-linearly independent and $f_i=a_ix^3+b_ix^2y+c_ixy^2+d_iy^3$ for $i=1, 2$. As we considered at Case I, Λ has to be one of the following three cases:

(1) $f_1=x^3$, $f_2=ax^2y+bxy^2+cy^3$;

(2) $f_1=x^2y$, $f_2=ax^3+bxy^2+cy^3$;

(3) $f_1=x^2\mathcal{K}-xy^2$, $f_2=ax^3+bx^2y+cy^3$.

For case (1), if $c=0$, we define $f: J\to\Lambda\otimes\Lambda$ by setting $\iota_1\to y^3\otimes y^3$ and $\iota_2\to 0$, then f is clearly well-defined and has no any extension to $\Lambda\otimes\Lambda$ as we prove in the Case 0.

If $c\neq 0$, we may put $c=1$. Then $\operatorname{soc}\Lambda=\langle x^2y, xy^2\rangle$ $(=Kx^2y+Kxy^2)$ and $y^3=-ax^2y-bxy^2\in\operatorname{soc}\Lambda$. So we can take $\{1, x, y, x^2, xy, y^2, x^2y, xy^2\}$ as a K-basis of Λ. Define $f: J\to\Lambda\otimes\Lambda$ by setting $f(\iota_1)=xy^2\otimes y^2+xy\otimes y^3=xy^2\otimes y^2-xy\otimes(ax^2y+bxy^2)$ and $f(\iota_2)=0$. Then $\iota_2f(\iota_1)=(y\otimes 1-1\otimes y)(xy^2\otimes y^2+xy\otimes y^3)=0=\iota_1f(\iota_2)$, hence by Lemma 2.5 f is well-defined. Now if $\varphi: \Lambda\otimes\Lambda\to\Lambda\otimes\Lambda$ is an extension of f with $\varphi(1\otimes 1)=1\otimes h_1+x\otimes h_x+y\otimes h_y+x^2\otimes h_{x^2}+xy\otimes h_{xy}+y^2\otimes h_{y^2}+x^2y\otimes h_{x^2y}+xy^2\otimes h_{xy^2}$. Then

$$\varphi(\iota_1)=xy^2\otimes y^2-xy\otimes(ax^2y+bxy^2) \qquad \text{(i)}$$

$$\varphi(\iota_2)=0 \qquad \text{(ii)}$$

From (i) we get that $-xh_{y^2}=0$ and $h_{y^2}-xh_{xy^2}=y^2$. By the second equation, the polynomial h_{y^2} has nonzero term of y^2, but it contradicts to the first equation.

For case (2), if $a=0$, we define $f: J\to\Lambda\otimes\Lambda$ by setting $f(\iota_1)=0$ and $f(\iota_2)=x^3\otimes x^3$, then f is clearly well-defined and has no extension to $\Lambda\otimes\Lambda$ as we proved in the Case 0.

If $c=0$, similarly defined $f: J\to\Lambda\otimes\Lambda$ by setting $f(\iota_1)=y^3\otimes y^3$ and $f(\iota_2)=0$, then f is clearly well-defined and has no any extension to $\Lambda\otimes\Lambda$ as we proved in the Case 0.

If $ac\neq 0$ and $b=0$, we may put $a=1$. Then $\operatorname{soc}\Lambda=\langle xy^2, y^3\rangle$ and $x^3=-cy^3\in\operatorname{soc}\Lambda$. So we can take $\{1, x, y, x^2, xy, y^2, xy^2, y^3\}$ as a K-basis of Λ. Define $f: J\to\Lambda\otimes\Lambda$ by setting

$$\iota_1 \longrightarrow xy^2\otimes y^2$$

$$\iota_2 \longrightarrow -c^{-1}xy^2\otimes x^2\,.$$

Then $\iota_2 f(\iota_1)=(y\otimes 1-1\otimes y)(xy^2\otimes y^2)=-xy^2\otimes y^3=(x\otimes 1-1\otimes x)(-c^{-1}xy^2\otimes x^2)=\iota_1 f(\iota_2)$, hence by Lemma 2.5 f is well-defined. Now if $\varphi: \Lambda\otimes\Lambda\to\Lambda\otimes\Lambda$ is an extension of f with $\varphi(1\otimes 1)=1\otimes h_1+x\otimes h_x+y\otimes h_y+x^2\otimes h_{x^2}+xy\otimes h_{xy}+y^2\otimes h_{y^2}+xy^2\otimes h_{xy^2}+y^3\otimes h_{y^3}$. Then

$$\varphi(\iota_1)=xy^2\otimes y^2 \qquad \text{(i)}$$

$$\varphi(\iota_2)=-c^{-1}xy^2\otimes x^2\,. \qquad \text{(ii)}$$

From (i) we get that $-xh_{y^2}=0$ and $h_{y^2}-xh_{xy^2}=y^2$. By the last equation h_{y^2} has nonzero term of y^2, but it contradicts to $-xh_{y^2}=0$.

If $abc\neq 0$, we may put $b=1$. Then $\operatorname{soc}\Lambda=\langle x^3, y^3\rangle$ and $xy^2=-ax^3-cy^3\in\operatorname{soc}\Lambda$. So we can take $\{1, x, y, x^2, xy, y^2, x^3, y^3\}$ as a K-basis of Λ. Define $f: J\to\Lambda\otimes\Lambda$ by setting

$$\iota_1 \longrightarrow c(y^3\otimes y^2)-xy\otimes y^3$$

$$\iota_2 \longrightarrow a(x^2\otimes y^3)\,.$$

Then $\iota_1 f(\iota_2)=(x\otimes 1-1\otimes x)a(x^2\otimes y^3)=a(x^3\otimes y^3)=(ax^3)\otimes y^3=(-cy^3-xy^2)\otimes y^3=(y\otimes 1-1\otimes y)[c(y^3\otimes y^2)-xy\otimes y^3]=\iota_2 f(\iota_1)$, hence by Lemma 2.5 f is well-defined. Now if $\varphi: \Lambda\otimes\Lambda\to\Lambda\otimes\Lambda$ is an extension of f with $\varphi(1\otimes 1)=1\otimes h_1+x\otimes h_x+y\otimes h_y+x^2\otimes h_{x^2}+xy\otimes h_{xy}+y^2\otimes h_{y^2}+x^3\otimes h_{x^3}+y^3\otimes h_{y^3}$, Then

$$\varphi(\iota_1)=c(y^3\otimes y^2)-xy\otimes y^3 \qquad \text{(i)}$$

$$\varphi(\iota_2)=a(x^2\otimes y^3) \qquad \text{(ii)}$$

From (i) we get that $-xh_{y^2}=0$ and $-ch_{y^2}-xh_{y^3}=cy^2$. By the last equation h_{y^2} has nonzero term of y^2, but it contradicts to $-xh_{y^2}=0$.

For case (3), if $a=0$, we define $f: J\to\Lambda\otimes\Lambda$ by setting $f(\iota_1)=0$ and $f(\iota_2)=x^3\otimes x^3$, then f is clearly well-defined and has no extension to $\Lambda\otimes\Lambda$ as we proved in the Case 0.

If $c=0$, similarly define $f: J\to\Lambda\otimes\Lambda$ by setting $f(\iota_1)=y^3\otimes y^3$ and $f(\iota_2)=0$, then f is clearly well-defined and has no any extension to $\Lambda\otimes\Lambda$ as we proved in the Case 0.

If $ac\neq 0$ and $b=0$, we may put $a=1$ and then $\operatorname{soc}\Lambda=\langle xy^2, y^3\rangle$ with $x^3=-cy^3\in\operatorname{soc}\Lambda$. So we can take $\{1, x, y, x^2, xy, y^2, xy^2, y^3\}$ as a K-basis of Λ. Define $f: J\to\Lambda\otimes\Lambda$ by setting

$$\iota_1\longrightarrow 0$$

$$\iota_2\longrightarrow y^3\otimes x^2+x^2\otimes y^3\,.$$

Then $\iota_2 f(\iota_1)=0=(x\otimes 1-1\otimes x)(y^3\otimes x^2+x^2\otimes y^3)=-y^3\otimes x^3+x^3\otimes y^3=\iota_1 f(\iota_2)$, hence by Lemma 2.5 f is well-defined. If $\varphi: \Lambda\otimes\Lambda\to\Lambda\otimes\Lambda$ is an extension of f with $\varphi(1\otimes 1)=1\otimes h_1+x\otimes h_x+y\otimes h_y+x^2\otimes h_{x^2}+xy\otimes h_{xy}+y^2\otimes h_{y^2}+xy^2\otimes h_{xy^2}+y^3\otimes h_{y^3}$. Then

$$\varphi(\iota_1)=0 \qquad \text{(i)}$$

$$\varphi(\iota_2)=y^3\otimes x^2+x^2\otimes y^3\,. \qquad \text{(ii)}$$

From (i) we get $-xh_{y^2}=0$ and from (ii) $h_{y^2}-yh_{y^3}=x^2$. By the last equation h_{y^2} has nonzero term of x^2, but it contradicts to $-xh_{y^2}=0$.

If $abc\neq 0$, we may put $a=1$. Then $\operatorname{soc}\Lambda=\langle xy^2, y^3\rangle$ and $x^3=-bxy^2-cy^3\in\operatorname{soc}\Lambda$. So we can take $\{1, x, y, x^2, xy, y^2, xy^2, y^3\}$ as a K-basis of Λ. Define $f: J\to\Lambda\otimes\Lambda$ by setting

$$\iota_1\longrightarrow 0$$

$$\iota_2\longrightarrow y^3\otimes x^2-(c^{-1}x^2+bc^{-1}xy)\otimes x^3\,.$$

Then $\iota_2 f(\iota_1)=0=(x\otimes 1-1\otimes x)\{y^3\otimes x^2-(c^{-1}x^2+bc^{-1}xy)\otimes x^3\}=\iota_1 f(\iota_2)$ hence by Lemma 2.5 f is well-defined. If $\varphi: \Lambda\otimes\Lambda\to\Lambda\otimes\Lambda$ is an extension of f with $\varphi(1\otimes 1)=1\otimes h_1+x\otimes h_x+y\otimes h_y+x^2\otimes h_{x^2}+xy\otimes h_{xy}+y^2\otimes h_{y^2}+xy^2\otimes h_{xy^2}+y^3\otimes h_{y^3}$. Then

$$\varphi(\iota_1)=0 \qquad \text{(i)}$$

$$\varphi(\iota_2)=y^3\otimes x^2-(c^{-1}x^2+bc^{-1}xy)\otimes x^3\,. \qquad \text{(ii)}$$

From (i) we get $-xh_{y^2}=0$ and from (ii) $h_{y^2}-yh_{y^3}=x^2$. By the last equation h_{y^2} has nonzero term of x^2, but it contradicts to $-xh_{y^2}=0$.

Finally we shall prove for the Case III: $\Lambda=K[x, y]/((x, y)^4, f_1, f_2, f_3)$ where f_1, f_2, f_3 are K-linearly independent and $f_i=a_ix^3+b_ix^2y+c_ixy^2+d_iy^3$ with a_i, b_i, c_i and $d_i\in K$ for $i=1, 2, 3$. By Corollary 2.2 we know that there exist two nontrivial elements α and β belonging to $\operatorname{soc}\Lambda$ such that $\alpha=ax^2+bxy+cy^2$ with $a, b, c\in K$ and $\beta\in\operatorname{rad}^3\Lambda$.

Now define $f: J\to\Lambda\otimes\Lambda$ by putting $f(\iota_1)=\alpha\otimes\alpha$ for $i=1, 2$, then f is clearly well-defined.

If $a\neq 0$, we may put $a=1$ and then $x^2=\alpha-bxy-cy^2$. So we can take $\{1, x, y, xy, y^2, \alpha, \beta\}$ as a K-basis of Λ. If $\varphi: \Lambda\otimes\Lambda\to\Lambda\otimes\Lambda$ is an extension of f with $\varphi(1\otimes 1)=1\otimes h_1+x\otimes h_x+y\otimes h_y+xy\otimes h_{xy}+y^2\otimes h_{y^2}+\alpha\otimes h_\alpha+\beta\otimes h_\beta$. Then

$$\varphi(\iota_1)=\alpha\otimes\alpha \tag{1}$$

$$\begin{aligned}\varphi(\iota_2)=&-1\otimes yh_1-x\otimes yh_x+y\otimes(h_1-yh_y)+xy\otimes(h_x-yh_{xy})\\&+y^2\otimes(h_y-yh_{y^2})-\alpha\otimes yh_\alpha-\beta\otimes yh_\beta+xy^2\otimes h_{xy}+y^3\otimes h_{y^2}\\=&\,\alpha\otimes\alpha\,.\end{aligned} \tag{2}$$

Since $xy^2=k_1\beta$ and $y^3=k_2\beta$ for some $k_1, k_2\in K$ we have by (2) that $-yh_\alpha=\alpha$, a contradiction.

If $c\neq 0$, we may put $c=1$ and then $y^2=\alpha-ax^2-bxy$. So we can take $\{1, x, y, x^2, xy, \alpha, \beta\}$ as a K-basis of Λ. If $\varphi: \Lambda\otimes\Lambda\to\Lambda\otimes\Lambda$ is an extension of f with $\varphi(1\otimes 1)=1\otimes h_1+x\otimes h_x+y\otimes h_y+x^2h_{x^2}+xy\otimes h_{xy}+\alpha\otimes h_\alpha+\beta\otimes h_\beta$. Then

$$\varphi(\iota_1)=\alpha\otimes\alpha \tag{1}$$

$$\varphi(\iota_2)=\alpha\otimes\alpha\,. \tag{2}$$

By (1) we have similarly that $-xh_\alpha=\alpha$, a contradiction.

If $a=0=c$, then $\alpha=bxy$ and we may put $b=1$. So we can take $\{1, x, y, x^2, y^2, \alpha, \beta\}$ as a K-basis of Λ. If $\varphi: \Lambda\otimes\Lambda\to\Lambda\otimes\Lambda$ is an extension of f with $\varphi(1\otimes 1)=1\otimes h_1+x\otimes h_x+y\otimes h_y+x^2\otimes h_{x^2}+y^2\otimes h_{y^2}+\alpha\otimes h_\alpha+\beta\otimes h_\beta$. Then

$$\varphi(\iota_1)=\alpha\otimes\alpha \tag{1}$$

$$\varphi(\iota_2)=\alpha\otimes\alpha\,. \tag{2}$$

Then by (1) we have that $h_x=xh_{x^2}$, $h_y=xy+xh_{xy}$ and $-xh_{y^2}=0$. By (2) we have that $h_x=xy+yh_{xy}$, $h_y=yh_{y^2}$ and $-yh_{x^2}=0$. If h_{xy} has nonzero term of y, $h_x=xy+yh_{xy}\neq xh_{x^2}$ since yh_{xy} has nonzero term of y^2, it is a contradiction. If h_{xy} has nonzero term of x, $h_y=xy+xh_{xy}\neq yh_{y^2}$ since xh_{xy} has nonzero term of x^2, also a contradiction. Finally if both the terms of x and y in h_{xy} are zero, then $h_y=xy+xh_{xy}\neq yh_{y^2}$ because h_{y^2} has no term of x by $xh_{y^2}=0$. This is also a contradiction.

§ 3. Miscellaneous results.

As was easily seen in [6] and [7] for commutative algebra Λ over an algebraically closed field K it holds that the both composition lengths of

$(\mathrm{rad}\,\Lambda)/(\mathrm{rad}\,\Lambda)^2$ and $(\mathrm{rad}\,\Lambda)^2/(\mathrm{rad}\,\Lambda)^3 \leqq 2$ implies that Λ is generalized biserial. We shall generalize the above fact for non-commutative algebras.

PROPOSITION 3.1. *A local K-algebra Λ is generalized biserial if and only if so is $\Lambda/(\mathrm{rad}\,\Lambda)^3$.*

PROOF. At first we would like to remark Λ is not assumed to be commutative. It is enough to proved only the "if" part. If $\dim_K(\mathrm{rad}\,\Lambda)/(\mathrm{rad}\,\Lambda)^2 =1$, then Λ is uniserial. So we can take two elements x and y of Λ such that $\{x+(\mathrm{rad}\,\Lambda)^2,\ y+(\mathrm{rad}\,\Lambda)^2\}$ generates $(\mathrm{rad}\,\Lambda/(\mathrm{rad}\,\Lambda)^2$.

If $(ax+by)^2 \equiv 0 \bmod (\mathrm{rad}\,\Lambda)^3$ for any $a, b \in K$, then $\Lambda/(\mathrm{rad}\,\Lambda)^3$ is isomorphic to the exterior algebra over 2-dimensional K-vector space and $xy \equiv -yx \bmod (\mathrm{rad}\,\Lambda)^3$. Consequently $\Lambda/(\mathrm{rad}\,\Lambda)^3$ is selfinjective. Now $(\mathrm{rad}\,\Lambda)^3/(\mathrm{rad}\,\Lambda)^4$ is generated by $\{x+(\mathrm{rad}\,\Lambda)^4,\ x^2y+(\mathrm{rad}\,\Lambda)^4,\ xyx+(\mathrm{rad}\,\Lambda)^4,\ yxy+(\mathrm{rad}\,\Lambda)^4,\ yx^2 +(\mathrm{rad}\,\Lambda)^4,\ y^2x+(\mathrm{rad}\,\Lambda)^4,\ xy^2+(\mathrm{rad}\,\Lambda)^4,\ y^3+(\mathrm{rad}\,\Lambda)^4\}$. But each of them belongs to $(\mathrm{rad}\,\Lambda)^4$ because $x^2, y^2 \equiv 0 \bmod (\mathrm{rad}\,\Lambda)^3$ and $xy \equiv -yx \bmod (\mathrm{rad}\,\Lambda)^3$ and hence $(\mathrm{rad}\,\Lambda)^3 \subset (\mathrm{rad}\,\Lambda)^4$. It follows from Nakayama Lemma that $(\mathrm{rad}\,\Lambda)^3=0$.

Now we may assume that there exists an element y of Λ such that $x^2 \not\equiv 0 \bmod (\mathrm{rad}\,\Lambda)^3$. By the assumption $\dim_K(\mathrm{rad}\,\Lambda)^2/(\mathrm{rad}\,\Lambda)^3 \leqq 2$, we proceed at first the proof for the following

Case 1: $(\mathrm{rad}\,\Lambda)^2/(\mathrm{rd}\,\Lambda)^3$ is spanned by $\{x^2+(\mathrm{rad}\,\Lambda)^3,\ xy+(\mathrm{rad}\,\Lambda)^3\}$. We may put $y^2 \equiv ax^2+bxy$ and $yx \equiv cx^2+dxy \bmod (\mathrm{rad}\,\Lambda)^3$ with a, b, c and $d \in K$. Since $(\mathrm{rad}\,\Lambda)^3 = (\mathrm{rad}\,\Lambda)(\mathrm{rad}\,\Lambda)^2$ it holds that $(\mathrm{rad}\,\Lambda)^3/(\mathrm{rad}\,\Lambda)^4 = \langle x^3+(\mathrm{rad}\,\Lambda)^4,\ x^2y+ (\mathrm{rad}\,\Lambda)^4,\ yx^2+(\mathrm{rad}\,\Lambda)^4,\ yxy+(\mathrm{rad}\,\Lambda)^4\rangle$. But $(\mathrm{rad}\,\Lambda)^3/(\mathrm{rad}\,\Lambda)^4$ is spanned by $\{x^3+(\mathrm{rad}\,\Lambda)^4,\ x^2y+(\mathrm{rad}\,\Lambda)^4\}$. Because we have that $xy^2 \equiv x(ax^2+bxy) \equiv ax^3+ bx^2y$, $yx^2 \equiv (yx)x \equiv (cx^2+dxy)x \equiv cx^3+dxyx$, but $xyx \equiv x(yx) \equiv x(cx^2+dxy) \equiv cx^3+dx^2y$ and $yxy \equiv (yx)x \equiv (cx^2+dxy)x \equiv cx^2y+dxy^2 \bmod (\mathrm{rad}\,\Lambda)^4$.

Now we shall proceed the proof by induction on exponents of rad Λ. Suppose that $(\mathrm{rad}\,\Lambda)^{n-2}/(\mathrm{rad}\,\Lambda)^{n-1}$ is spanned by $\{x^{n-2}+(\mathrm{rad}\,\Lambda)^{n-1},\ x^{n-3}y+(\mathrm{rad}\,\Lambda)^{n-1}\}$. Then $(\mathrm{rad}\,\Lambda)^{n-1}/(\mathrm{rad}\,\Lambda)^n = \langle x^{n-1}+(\mathrm{rad}\,\Lambda)^n,\ x^{n-2}y+(\mathrm{rad}\,\Lambda)^n,\ yx^{n-3}+(\mathrm{rad}\,\Lambda)^n,\ yx^{n-3}y+(\mathrm{red}\,\Lambda)^n\rangle$. But $yx^{n-2} \equiv (yx)x^{n-3} \equiv (cx^2+dxy)x^{n-3} \equiv cx^{n-1}+dx(yx)x^{n-4} \equiv cx^{n-1}+dx(cx^2+dxy)x^{n-4} \equiv (c+cd)x^{n-1}+d^2x^2(yx)x^{n-5} \equiv (c+cd+cd^2)x^{n-1}+ d^3x^3(yx)x^{n-6} \equiv \cdots \equiv c(1+d+d^2+\cdots+d^{n-3})x^{n-1}+d^{n-2}x^{n-2}y$, $yx^{n-3}y \equiv [c(1+d+d^2 +\cdots+d^{n-4})x^{n-2}+d^{n-3}x^{n-3}y]y = c(1+d+d^2+\cdots+d^{n-4})x^{n-2}y+d^{n-3}x^{n-3}y^2$ and $x^{n-3}y^2 \equiv x^{n-3}(ax^2+bxy) \equiv ax^{n-1}+bx^{n-2}y \bmod (\mathrm{rad}\,\Lambda)^n$, hence $(\mathrm{rad}\,\Lambda)^{n-1}/(\mathrm{rad}\,\Lambda)^n$ is spanned by $\{x^{n-1}+(\mathrm{rad}\,\Lambda)^n,\ x^{n-2}y+(\mathrm{rad}\,\Lambda)^n\}$.

Case 2: $xy+(\mathrm{rad}\,\Lambda)^3$ is linearly dependent to $x^2+(\mathrm{rad}\,\Lambda)^3$ and $(\mathrm{rad}\,\Lambda)^2/(\mathrm{rad}\,\Lambda)^3$ is spanned by $\{x^2+(\mathrm{rad}\,\Lambda)^3,\ yx+(\mathrm{rad}\,\Lambda)^3\}$. Then we may put that $y^2 \equiv ax^2+$

$byx \bmod (\mathrm{rad}\,\Lambda)^3$ with $a, b\in K$. Considering $(\mathrm{rad}\,\Lambda)^3/(\mathrm{rad}\,\Lambda)^4=\langle x^3+(\mathrm{rad}\,\Lambda)^4, yx^2+(\mathrm{rad}\,\Lambda)^4, y^2x+(\mathrm{rad}\,\Lambda)^4\rangle$, but $y^2x\equiv(ax^2+byx)x\equiv ax^3+byx^2 \bmod (\mathrm{rad}\,\Lambda)^4$, so $(\mathrm{rad}\,\Lambda)^3/\mathrm{rad}\,\Lambda)^4$ is spanned by $\{x^3+(\mathrm{rad}\,\Lambda)^4, yx^2+(\mathrm{rad}\,\Lambda)^4\}$.

Now we shall proceed the proof by induction on exponents of rad Λ. Suppose that $(\mathrm{rad}\,\Lambda)^{n-2}/(\mathrm{rad}\,\Lambda)^{n-1}$ is spanned by $\{x^{n-2}+(\mathrm{rad}\,\Lambda)^{n-1}, yx^{n-3}+(\mathrm{rad}\,\Lambda)^{n-1}\}$. Then $(\mathrm{rad}\,\Lambda)^{n-1}/(\mathrm{rad}\,\Lambda)^n=\langle x^{n-1}+(\mathrm{rad}\,\Lambda)^n, yx^{n-2}+(\mathrm{rad}\,\Lambda)^n, y^2x^{n-3}+(\mathrm{rad}\,\Lambda)^n\rangle$. But $y^2x^{n-3}\equiv(ax^2+byx)x^{n-3}\equiv ax^{n-1}+byx^{n-2} \bmod (\mathrm{rad}\,\Lambda)^n$, hence $(\mathrm{rad}\,\Lambda)^{n-1}/(\mathrm{rad}\,\Lambda)^n$ is spanned by $\{x^{n-1}+(\mathrm{rad}\,\Lambda)^n, yx^{n-2}+(\mathrm{rad}\,\Lambda)^n\}$.

Case 3: $xy+(\mathrm{rad}\,\Lambda)^3$ and $yx+(\mathrm{rad}\,\Lambda)^3$ are both linearly dependent to $x^2+(\mathrm{rad}\,\Lambda)^3$. Then $(\mathrm{rad}\,\Lambda)^2/(\mathrm{rad}\,\Lambda)^3$ is spanned by $\{x^2+(\mathrm{rad}\,\Lambda)^3, y^2+(\mathrm{rad}\,\Lambda)^3\}$, and so it is clear that $(\mathrm{rad}\,\Lambda)^n/(\mathrm{rad}\,\Lambda)^{n+1}$ is spanned by $\{x^n+(\mathrm{rad}\,\Lambda)^{n+1}, y^n+(\mathrm{rad}\,\Lambda)^{n+1}\}$ for $n\geqq 3$. It completes the proof.

COROLLARY 3.2. *Let Λ be a generalized biserial local algebra. If* $\dim_K(\mathrm{rad}\,\Lambda)^n/(\mathrm{rad}\,\Lambda)^{n+1}\leqq 1$ *for some integer $n>0$, then* $\dim_K(\mathrm{rad}\,\Lambda)^s/(\mathrm{rad}\,\Lambda)^{s+1}\leqq 1$ *for all $s\geqq n$.*

PROOF. Use the spanning systems of $(\mathrm{rad}\,\Lambda)^s/(\mathrm{rad}\,\Lambda)^{s+1}$ which we have obtained in the proof of the previous proposition for $s\geqq n$.

Let $\Lambda=\Lambda_0\oplus\Lambda_1\oplus\cdots\oplus\Lambda_n$ be a positively $\mathbf{Z}$-grading of K-algebra Λ such that $\mathrm{rad}\,\Lambda=\Lambda_1\oplus\cdots\oplus\Lambda_n$, $\Lambda_0=K$ and $\Lambda_i\Lambda_j\subset\Lambda_{i+j}$ for $i, j\geqq 0$. If $\dim_K(\mathrm{rad}\,\Lambda)/(\mathrm{rad}\,\Lambda)^2=t$, we have a minimal system $\{x_1, x_2, \cdots, x_t\}$ of generators of rad Λ. Put

(1) $\qquad x_j=\sum_{i=1}^n\alpha_{j,i}\quad$ for $j=1, 2, \cdots, t$ and $\alpha_{j,i}\in\Lambda_i$.

In the proof of [7, Proposition 4.1] we proved the fact stated in the following proposition for $t=2$ and Λ being commutative. Now we shall prove

PROPOSITION 3.3. *There is a set $B=\{\alpha_{j_k,s_k}\in\Lambda_{s_k}\,|\,1\leqq j_k, s_k\leqq n$ and $k=1, 2, \cdots, t\}$ such that B is a minimal system of generators of* rad Λ.

PROOF. Let us denote by $\bar{a}$ the residue class of $a\in\Lambda$ modulo $(\mathrm{rad}\,\Lambda)^2$. From (1) we have that $\bar{x}_j=\sum_{i=1}^n\bar{\alpha}_{j,i}$ for $j=1, 2, \cdots, t$. If follows that there exists a set $\{\bar{\alpha}_{j_k,s_k}\,|\,k=1, 2, \cdots, t\}$ such that it becomes a K-basis of $(\mathrm{rad}\,\Lambda)/(\mathrm{rad}\,\Lambda)^2$. Then it is clear that $\{\alpha_{j_k,s_k}\,|\,k=1, 2, \cdots, t\}$ generates rad Λ, since rad Λ is nilpotent.

Acknowledgements

The author would like to express his sincere thanks to Professor H. Tachikawa and Doctor T. Wakamatsu for their invaluable comments and encouragements during the preparation of this paper.

References

[1] Asashiba, H., The selfinjectivity of a local algebra A and the condition $\mathrm{Ext}_A^1(DA, A)=0$, In: Proc. Tsukuba Conf. 1990, CMS Conference Proceedings, No. 11 (to appear).
[2] Cartan, H. and Eilenberg, S., Homological Algebra, Princeton, 1956.
[3] Hoshino, M., On algebras with radical cube zero, Arch. Math. **52** (1989), 226-232.
[4] Nakayama, T., On algebras with complete homology, Abh. Math. Sem. Univ. Hamburg **22** (1958), 300-307.
[5] Tachikawa, H., Quasi-Frobenius rings and generalizations, Lecture Notes in Math., No. 351, Springer-Verlag, Berlin/Heidelberg/New York, 1973.
[6] Zeng, Q., Vanishing of Hochschild's cohomologies $H^i(A \otimes A)$ and gradability of a local commutative algebra A, Tsukuba J. Math., Vol. 14, No. 2 (1990), 263-273.
[7] Zeng, Q., Vanishing of Hochschild cohomologies and directed graphs with polynomial weights, to appear in J. Algebra.

Institute of Mathematics
University of Tsukuba
Ibaraki 305, Japan