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H-SEPARABILITY OF GROUP RINGS
(In memory of Professor Akira Hattori)

By

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Let $k[G]$ be the group ring of a finite group G with a coefficient field k. Assume that the characteristic of k does not divide the order of G. Let H be a subgroup of G, Δ the centralizer of $k[H]$ in $k[G]$ and D the double centralizer of $k[H]$ in $k[G]$. The purpose of this paper is to prove that $k[G]$ is an H-separable extension of D. For this, a unit in the center C of $k[G]$ plays a fundamental role (Lemma 1). Besides, we can prove the well known facts that $k[G]$ is (finitely generated) projective over C and $k[G]$ is a central separable algebra over C, explicitely, by use of this unit.

Denote by g_x and c_x the number and the sum of elements in the conjugate class of G containing the element x of G, respectively.

LEMMA 1. *$u=\Sigma_{c_x}(1/g_x)c_x c_{x^{-1}}$ is a unit in C.*

PROOF. We first prove that $\{(1/g_x)c_x\}$ and $\{c_{x^{-1}}\}$ form a dual base of C over k. Let $c_y c_x=\Sigma_{c_z} c_z a_{zx}$ where a_{zx} are integers. This means that each z_k $(1\leqq k\leqq g_z)$ conjugated to z, appears in $c_y c_x$ a_{zx} times, that is, for fixed k, the number of pairs (i, j) such that $y_i x_j=z_k(1\leqq i\leqq g_y,\ 1\leqq j\leqq g_x)$ is equal to a_{zx}. So, the number of terms $x_j^{-1}=z_k^{-1}y_i(1\leqq j\leqq g_x)$ is $a_{zx}g_z$ in $c_{z^{-1}}c_y$ and $c_{z^{-1}}c_y=\cdots+(a_{zx}g_z/g_x)c_{x^{-1}}+\cdots$. This proves that $((1/g_z)c_{z^{-1}})c_y=\Sigma c_{x^{-1}}a_{zx}$ $((1/g_x)c_{x^{-1}})$ or equivalently $\{(1/g_x)c_x\}$ and $\{c_{x^{-1}}\}$ form a dual base of C over k. Now C is a separable k-algebra in the sense of that, for any field extension L of k, C_L is a semisimple L-algebra. Then $u=\Sigma_{c_x}(1/g_x)c_x c_{x^{-1}}$ is a unit in C by Theorem 71. 6 in [2] p.482.

Let v be the inverse of u in C, $uv=1$.

COROLLARY 2. *$\Sigma_{c_x}(1/g_x)c_x\otimes c_{x^{-1}}v$ is a separability idempotent in $C\otimes_k C$.*

PROOF. It is clear that $c(\Sigma(1/g_x)c_x\otimes c_{x^{-1}}v)=(\Sigma(1/g_x)c_x\otimes c_{x^{-1}}v)c$ for any $c\in C$ and $\Sigma(1/g_x)c_x c_{x^{-1}}v=1$.

Let p be the map of $k[G]$ to C defined by $p(a)=(1/n)\Sigma_{x\in G}xax^{-1}$ for $a\in k[G]$, where n is the order of G. The map p is the projection of $k[G]$ to C. Then p is an element of $\mathrm{Hom}_C(k[G], C)$ which has a left $k[G]$-module structure in the usual way.

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COROLLARY 3. *$\{x\cdot p\}$ and $\{x^{-1}v\}$ $(x\in G)$ form a projective base of $k[G]$ over C.*

PROOF. For the identity 1 of G, we have

$$\Sigma_{x\in G}(x\cdot p)(1)x^{-1}v=\Sigma_{x\in G}\,p(x)x^{-1}v=\Sigma_{x\in G}(1/g_x)c_x x^{-1}v=\Sigma_{c_x}(1/g_x)c_x c_{x^{-1}}v=1.$$

Now, for any $y\in G$, we have

$$\Sigma_{x\in G}(x\cdot p)(y)x^{-1}v=\Sigma_{x\in G}\,p(yx)x^{-1}v=\Sigma_{x\in G}\,p(yx)(yx)^{-1}vy=y.$$

Now consider the two-sided $k[G]$-module $k[G]\otimes_C k[G]$. Then, for each $x\in G$, the element $(1/n)\,\Sigma_{y\in G}\,y\otimes xy^{-1}$ is in

$$(k[G]\otimes_C k[G])^{k[G]}=\{\xi\in k[G]\otimes_C k[G]\,|\,a\xi=\xi a,\ \text{for all } a\in k[G]\}.$$

Therefore the map f_x for $x\in G$, which assigns to each $a\in k[G]$ the element $((1/n)\,\Sigma_{y\in G}\,y\otimes xy^{-1})\,a$ defines a two-sided $k[G]$-homomorphism of $k[G]$ to $k[G]\otimes_C k[G]$. The map l_x for $x\in G$, which assigns to $\Sigma_i\,a_i\otimes b_i$ in $k[G]\otimes_C k[G]$ $\Sigma_i\,a_i x^{-1}vb_i$ in $k[G]$, is a two-sided $k[G]$-homomorphism of $k[G]\otimes_C k[G]$ to $k[G]$. Then it is easily verified that $\Sigma_{x\in G}f_x\circ l_x$ is the identity map of $k[G]\otimes_C k[G]$. Thus we have proved the following corollary.

COROLLARY 4. *$k[G]\otimes_C k[G]$ is a two-sided $k[G]$-direct summand of the direct sum of n-copies of $k[G]$.*

If this is the case, then it holds that $k[G]\otimes_C k[G]\cong \mathrm{Hom}_C(k[G],k[G])$ and $k[G]$ is C-finitely generated projective, see [3] p. 112. Therefore $k[G]$ is a central separable C-algebra by Theorem 2.1 [1].

Let H be a subgroup of G and $G=\Sigma_{i=1}^{r}y_iH$ a coset decomposition of G by H. Denote by h_x and d_x the number and the sum of elements in the H-conjugate class of G containing the element x of G, respectively. Let $\varDelta$ be the centralizer of $k[H]$ in $k[G]$. Then $\{d_x\}$ is a k-base of $\varDelta$. By the same way as in Lemma 1, it can be verified that $\{(1/h_x)d_x\}$ and $\{d_{x^{-1}}\}$ form a dual base of $\varDelta$ over k. Let q be the map of $\varDelta$ to C defined by $q(a)=(1/r)\,\Sigma_i\,y_i a y_i^{-1}$, $a\in\varDelta$. It can be shown that q does not depend on the choice of y_i, and q is the projection of $\varDelta$ to C.

PROPOSITION 5. *$\{(1/h_x)d_x\cdot q\}$ and $\{d_{x^{-1}}v\}$ form a projective base of $\varDelta$ over C.*

PROOF. If we notice that $q(d_x)=(h_x/g_x)c_x$, the calculation is similar to the proof in Corollary 3 and we shall omit it.

Let D be the centralizer of $\varDelta$ in $k[G]$. Then $D\supset k[H]$ and the centralizer of D in $k[G]$ is equal to $\varDelta$.

PROPOSITION 6. *$k[G]$ is an H-separable extension of D.*

PROOF. For a representative x of an H-conjugate class of G, define

$$s_x\colon k[G]\longrightarrow k[G]\otimes_D k[G]\quad\text{by}\quad s_x(a)=((1/r)\,\Sigma_i\,y_i\otimes(1/h_x)d_x y_i^{-1})a$$

and

$$t_x\colon k[G]\otimes_D k[G] \longrightarrow k[G] \quad \text{by} \quad t_x(\Sigma_i\, a_i\otimes b_i)=\Sigma_i\, a_i d_{x^{-1}} v b_i,$$

respectively. As $(1/r)\Sigma_i\, y_i\otimes(1/h_x)d_x y_i^{-1}$ is in $(k[G]\otimes_D k[G])^{k[G]}$ and $d_{x^{-1}}v$ is in Δ, s_x and t_x are two-sided $k[G]$-homomorphisms, respectively. If we notice that $\Sigma_{d_x}(1/h_x)d_x y_i^{-1} d_{x^{-1}}v$ is contained in D, it is easily verified that $\Sigma\, s_x\circ t_x$ is the identity map of $k[G]\otimes_D k[G]$, where the sum is taken over all the H-conjugate classes of G. Therefore $k[G]\otimes_D k[G]$ is a two-sided $k[G]$-direct summand of a direct sum of finite copies of $k[G]$ and $k[G]$ is an H-separable extension of D.

Even if the characteristic of k divides the order of G, if the index of H in G is a unit in k, $k[G]$ is always a separable extesion of $k[H]$ by Proposition 3.1 [4]. In this case, it happens that $k[G]$ may or not be an H-separable extension of D. Let k be a field of characteristic two. Take $G=S_3$ the symmetric group of degree three and $H=\langle(12)\rangle$. Then $G=H+(13)H+(23)H$ is a coset decomposition of G by H. Put $x_1=(12)$, $x_2=(13)+(23)$ and $y=(123)+(132)$. Then we have $\Delta=k1+kx_1+kx_2+ky$ and $D=k[G]^{\Delta}=D$. The projection q of Δ to C is given by $q(a)=(1/3)(1\cdot a\cdot 1+(13)a(13)+(23)a(23))$ for $a\in\Delta$. Then $\{q, x_2\cdot q, y\cdot q\}$ and $\{1+y, x_2, 1\}$ form a projective base of Δ over C. Define maps $s_i\colon k[G]\to k[G]\otimes_D k[G]$ $(i=1, 2, 3)$ by $s_1(a)=(1/3)(1\otimes 1+(13)\otimes(13)+(23)\otimes(23))a$, $s_2(a)=(1/3)(1\otimes x_2+(13)\otimes x_2(13)+(23)\otimes x_2(23))\,a$ and $s_3(a)=(1/3)(1\otimes y+(13)\otimes y(13)+(23)\otimes y(23))a$, respectively. Also define maps $t_i\colon k[G]\otimes_D k[G]\to k[G]$ $(i=1, 2, 3)$ by $t_1(\Sigma\, a_i\otimes b_i)=\Sigma\, a_i(1+y)b_i$, $t_2(\Sigma\, a_i\otimes b_i)=\Sigma\, a_i x_2 b_i$ and $t_3(\Sigma\, a_i\otimes b_i)=\Sigma\, a_i b_i$, respectively. Then $\Sigma_{i=1}^3\, s_i\circ t_i$ is the identity map of $k[G]\otimes_D k[G]$ and $k[G]$ is an H-separable extension of D. Next, take $G=S_4$ and $H=\langle(13), (1234)\rangle$. Then the center C of $k[G]$ is a local ring of dimension five over k. On the other hand we can see easily that Δ is eight dimensional over k. Therefore Δ is not C-projective and $k[G]$ is not an H-separable extension of D.

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