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CONVERGENCE OF MOMENTS IN THE CENTRAL LIMIT THEOREM FOR STATIONARY ϕ-MIXING SEQUENCES

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1. Introduction and result.

Let $\{X_j, -\infty<j<\infty\}$ be a strictly stationary sequence of random variables centered at expectations with finite variance, which satisfies ϕ-mixing condition

$$(1.1) \qquad \sup |P(A\cap B)-P(A)P(B)|/P(A)\leqq\phi(n)\downarrow 0 \quad (n\to\infty).$$

Here the supremum is taken over all $A\in\mathcal{M}_{-\infty}^{k}$ and $B\in\mathcal{M}_{k+n}^{\infty}$, and $\mathcal{M}_a^b$ denotes the σ-field generated by X_j $(a\leqq j\leqq b)$. Let $S_n=X_1+\cdots+X_n$ and $\sigma_n^2=ES_n^2$, $n=1, 2, \cdots$.

For independent random variables, Brown [1 and 2] has shown that the Lindeberg condition of order $\nu\geqq 2$ is necessary and sufficient for the central limit theorem and the convergence of $E|S_n/\sigma_n|^\nu$ towards the corresponding moment of the normal distribution. For dependent random variables, such a result seems less well-known. We study here the convergence of moments for stationary ϕ-mixing sequences.

THEOREM. *Let* $\{X_j\}$ *satisfy* (1.1). *If* $EX_1^{2m}<\infty$ *for some integer* $m\geqq 2$, *and if*

$$(1.2) \qquad \sigma_n^2=\sigma^2 n(1+o(1))$$

as $n\to\infty(\sigma>0)$, *then*

$$(1.3) \qquad E(S_n/\sigma_n)^{2m}\to\beta_{2m} \quad (n\to\infty),$$

where β_ν *is the* ν*th absolute moment of* $N(0, 1)$.

We remark that under the assumptions of the theorem X_j satisfies the central limit theorem (cf. [4, Theorem 18.5.1]). Also remark that any other conditions beyond (1.1) on the decays of mixing coefficients $\phi(n)$ are not required.

2. Preparatory lemmas.

LEMMA 1 [4, Theorem 17.2.3]. *Suppose that* (1.1) *is satisfied and that* ξ *and* η *are measurable with respect to* $\mathcal{M}_{-\infty}^{k}$ *and* $\mathcal{M}_{k+n}^{\infty}$ $(n\geqq 0)$ *respectively. If* $E|\xi|^p<\infty$ *and* $E|\eta|^q<\infty$ *for* $p, q>1$ *with* $(1/p)+(1/q)=1$, *then*

$$(2.1) \qquad |E(\xi\eta)-E(\xi)E(\eta)| \leqq 2\{\phi(n)E|\xi|^p\}^{1/p}\{E|\eta|^q\}^{1/q}.$$

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LEMMA 2. *Let* $\{X_j\}$ *satisfy* (1.1) *and* $E|X_1|^\nu<\infty$ *for some* $\nu\geqq 2$. *If*

$$\sigma_n\to\infty$$

as $n\to\infty$, *then there is a constant* K, *for which*

$$E|S_n|^\nu\leqq K\sigma_n^\nu,\quad n\geqq 1, \tag{2.2}$$

In this lemma, the assumption (1.2) is not necessarily required. If (1.2) holds, the right-hand side of (2.2) can be replaced by $Kn^{\nu/2}$.

PROOF. We apply the method used in the proof of Lemma 7.4 of Doob [3] to that of Lemma 18.5.1 of Ibragimov-Linnik [4]. Lemma 2 is true for $\nu=2$. We assume therefore that (2.2) holds when ν is an integer $m\geqq 2$ and prove that it then holds for $\nu=m+\delta$, $0<\delta\leqq 1$. Let us write

$$\hat{S}_n=\sum_{j=n+k+1}^{2n+k}X_j,\quad a_n=E|S_n|^{m+\delta}$$

We only prove that, for $\varepsilon>0$ there exist K_1 and k such that

$$E|S_n+\hat{S}_n|^{m+\delta}\leqq(2+\varepsilon)a_n+K_1\sigma_n^{m+\delta}. \tag{2.3}$$

The proof of (2.2) then follows on the same line as in Lemma 18.5.1 of [4]. We have

$$\begin{aligned}E|S_n+\hat{S}_n|^{m+\delta}&\leqq E\{|S_n+\hat{S}_n|^m(|S_n|^\delta+|\hat{S}_n|^\delta)\}\leqq E|S_n|^{m+\delta}+E|\hat{S}_n|^{m+\delta}\\&+E\{\sum_{j=0}^{m-1}\binom{m}{j}|S_n|^{j+\delta}|\hat{S}_n|^{m-j}+\sum_{j=1}^{m}\binom{m}{j}|S_n|^j|\hat{S}_n|^{m-j+\delta}\}.\end{aligned} \tag{2.4}$$

Since S_n and $\hat{S}_n$ have the same distribution,

$$E|S_n|^{m+\delta}=E|\hat{S}_n|^{m+\delta}=a_n. \tag{2.5}$$

Using (2.1) with $p=(m+\delta)/(j+\delta)$,

$$E|S_n|^{j+\delta}|\hat{S}_n|^{m-j}\leqq 2a_n[\phi(k)]^{(j+\delta)/(m+\delta)}+E|S_n|^{j+\delta}\,E|S_n|^{m-j}, \tag{2.6}$$

and with $p=(m+\delta/j$,

$$E|S_n|^j|\hat{S}_n|^{m-j+\delta}\leqq 2a_n[\phi(k)]^{j/(m+\delta)}+E|S_n|^jE|S_n|^{m-j+\delta}. \tag{2.7}$$

By Hölder's inequality,

$$E|S_n|^u\leqq(E|S_n|^m)^{u/m},\quad 0<u\leqq m.$$

Thus, since (2.2) is assumed to hold for $\nu=m$ (with some K), for $0\leqq j\leqq m-1$,

$$E|S_n|^{j+\delta}\,E|S_n|^{m-j}\leqq(E|S_n|^m)^{(m+\delta)/m}\leqq K\sigma_n^{m+\delta}. \tag{2.8}$$

and for $1\leqq j\leqq m$,

$$E|S_n|^j\, E|S_n|^{m-j+\delta} \leqq (E|S_n|^m)^{(m+\delta)/m} \leqq K\sigma_n^{m+\delta}. \tag{2.9}$$

From (2.4) through (2.9), we obtain

$$E|S_n+\hat{S}_n|^{m+\delta} \leqq (2+K_2[\phi(k)]^{\delta/(m+\delta)})a_n + K_1\sigma_n^{m+\delta},$$

for some constants K_1 and K_2. To prove (2.3) it suffices to take k so large that $K_2[\phi(k)]^{\delta/(m+\delta)} < \varepsilon$.

We represent the sum S_n in the form

$$S_n = \sum_{i=1}^{k} \xi_i + \sum_{i=1}^{k+1} \eta_i = Z_k + Z'_{k+1},$$

where

$$\xi_i = \sum_{(i-1)(p+q)+1}^{ip+(i-1)q} X_j \quad (1 \leqq i \leqq k)$$

$$\eta_i = \sum_{ip+(i-1)q+1}^{i(p+q)} X_j \quad (1 \leqq i \leqq k)$$

$$\eta_{k+1} = \sum_{k(p+q)+1}^{n} X_j,$$

where $k=[n/(p+q)]$, and $p=p(n)$ and $q=q(n)$ are integer-valued functions such that as $n\to\infty$

$$p\to\infty,\ q\to\infty,\ q=o(p),\ p=o(n),\ nq=o(p^2) \text{ and } n\phi(q)=o(p). \tag{2.10}$$

For such a pair of p and q, see for example [4, Theorem 18.4.1]. Under the requirements imposed on p and q, we shall show that Z'_k is negligible, and that consequently $E(S_n/\sigma_n)^{2m} \sim E(Z_k/\sigma_n)^{2m}$. We note that, because of the stationarity, Lemma 2 is applicable to ξ_i and η_i. In the following, for convenience' sake the conditions of the theorem are assumed to hold and K denotes generic constant.

LEMMA 3. *As* $n\to\infty$

$$EZ_k^{2l} = ES_n^{2l} + o(\sigma_n^{2l}),\quad l=1, 2, \cdots, m, \tag{2.11}$$

PROOF. We first show that

$$EZ'^{2l}_{k+1} = o(\sigma_n^{2l}). \tag{2.12}$$

We have

$$EZ'^{2l}_{k+1} = EZ'^{2l}_k + \sum_{j=1}^{2l-1} \binom{2l}{j} EZ'^{j}_k \eta_{k+1}^{2l-j} + E\eta_{k+1}^{2l}. \tag{2.13}$$

By Minkowski's inequiality, Lemma 2 and (2.10),

$$EZ'^{2l}_k \leqq k^{2l} E\eta_1^{2l} \leqq K(k^2q)^l = o(\sigma_n^{2l}), \tag{2.14}$$

by Lemma 2.

(2.15) $$E\eta_{k+1}^{2l} \leqq K(n-k(p+q))^l = o(\sigma_n^{2l}),$$

and by Hölder's inequality, (2.14) and (2.15),

(2.16) $$|EZ'^j_k \eta_{k+1}^{2l-j}| \leqq (EZ'^{2l}_k)^{j/2l}(E\eta_{k+1}^{2l})^{(2l-j)/2l} = o(\sigma_n^{2l}).$$

Then (2.12) follows from (2.13)-(2.16).

$$EZ_k^{2l} = E(S_n - Z'_{k+1})^{2l} = ES_n^{2l} + \sum_{j=0}^{2l-1} (-1)^{2l-j} ES_n^j Z'^{2l-j}_{k+1},$$

and by Hölder's inequality, Lemma 2 and (2.12),

$$|ES_n^j Z'^{2l-j}_{k+1}| \leqq (ES_n^{2l})^{j/2l}(EZ'^{2l}_{k+1})^{(2l-j)/2l} = o(\sigma_n^{2l}),$$

for $j=0, 1, \cdots, 2l-1$. Thus the lemma is proved.

Let $\tau_i^2 = EZ_i^2$ for $i=1, 2, \cdots, k$. Then (2.11) implies that

(2.17) $$\tau_k^2 = \sigma_n^2 (1+o(1)).$$

Since

$$EZ_i^{2l} = E(S_{i(p+q)} - Z'_i)^{2l},$$

it follows from the proof of Lemma 3 that

(2.18) $$EZ_i^{2l} = ES_{i(p+q)}^{2l} + o(\sigma_{i(p+q)}^{2l}),$$

which together with (2.10) implies that

(2.19) $$EZ_i^{2l} \leqq K(ip)^l,$$

for $i=1, 2, \cdots, k$, $l=1, 2, \cdots, m$. Also (2.10) and (2.18) imply that

(2.20) $$\tau_i^2 = \sigma_{ip}^2(1+o(1)), \quad i=1, 2, \cdots, k.$$

3 Proof of Theorem.

$E(S_n/\sigma_n)^2=1$, $n=1, 2, \cdots$. Assume inductively that as $n\to\infty$

(3.1) $$E(S_n/\sigma_n)^{2l} \to \beta_{2l}, \quad l=1, 2, \cdots, m-1.$$

In view of (2.11) and (2.17), the assumption (3.1) is equivalent to the one that as $n\to\infty$

(3.2) $$E(Z_k/\tau_k)^{2l} \to \beta_{2l}, \quad l=1, 2, \cdots, m-1.$$

Using (2.11) again, we have only to prove under the assumption (3.2) that as $n\to\infty$

(3.3) $$E(Z_k/\tau_k)^{2m} \to \beta_{2m}.$$

We have

(3.4) $$EZ_k^{2m}=\sum_{i=1}^{k}\sum_{j=0}^{2m-1}\binom{2m}{j}EZ_{i-1}^{j}\xi_i^{2m-j},\quad \text{where } Z_0=0,$$
$$=\sum_{i=1}^{k}E\xi_i^{2m}+2m\sum_{i=1}^{k}EZ_{i-1}^{2m-1}\xi_i+\sum_{i=1}^{k}\binom{2m}{2}EZ_{i-1}^{2m-2}\xi_i^2+\sum_{i=1}^{k}\sum_{j=1}^{2m-3}\binom{2m}{j}EZ_{i-1}^{j}\xi_i^{2m-j}.$$

By Lemma 2 and (2.20),

(3.5) $$\sum_{i=1}^{k}E\xi_i^{2m}\leqq Kkp^m=o(\tau_k^{2m}).$$

By Lemmas 1, 2, (2.10), (2.19) and (2.20),

(3.6) $$\sum_{i=1}^{k}|EZ_{i-1}^{2m-1}\xi_i|$$
$$\leqq 2[\phi(q)]^{(2m-1)/2m}\sum_{i=1}^{k}(EZ_{i-1}^{2m})^{(2m-1)/2m}(E\xi_i^{2m})^{1/2m}$$
$$\leqq K[\phi(q)]^{1/2}p^m\sum_{i=1}^{k}(i-1)^{(2m-1)/2}$$
$$\leqq K'[k\phi(q)]^{1/2}(kp)^m=o(\tau_k^{2m}).$$

For $j=1,\cdots,2m-3$, by Lemma 2, (2.19) and (2.20),

(3.7) $$\sum_{i=1}^{k}(EZ_{i-1}^{2m})^{j/2m}(E\xi_i^{2m})^{(2m-j)/2m}\leqq Kk^{-(2m-j-2)/2}(kp)^m=o(\tau_k^{2m}),$$

and so

(3.8) $$\sum_{i=1}^{k}|EZ_{i-1}^{j}\xi_i^{2m-j}|=o(\tau_k^{2m}).$$

Further, by Lemmas 1, 2, (2.19) and (2.20),

(3.9) $$\left|\sum_{i=1}^{k}EZ_{i-1}^{2m-2}\xi_i^2-\sum_{i=1}^{k}EZ_{i-1}^{2m-2}E\xi_i^2\right|$$
$$\leqq 2[\phi(q)]^{(m-1)/m}\sum_{i=1}^{k}(EZ_{i-1}^{2m})^{(m-1)/m}(E\xi_i^{2m})^{1/m}$$
$$\leqq K[\phi(q)]^{(m-1)/m}(kp)^m=o(\tau_k^{2m}).$$

Consequently, by (3.4)-(3.9), as $n\to\infty$

(3.10) $$EZ_k^{2m}=\sum_{i=1}^{k}\binom{2m}{2}EZ_{i-1}^{2m-2}E\xi_i^2+o(\tau_k^{2m}).$$

By (2.20) and (3.2),

(3.11) $$\sum_{i=1}^{k}\binom{2m}{2}E(Z_{i-1}/\tau_{i-1})^{2m-2}E(\xi_i^2)\tau_{i-1}^{2m-2}$$
$$=\left\{\binom{2m}{2}\beta_{2m-2}+o(1)\right\}\sigma_p^2\sum_{i=1}^{k}\tau_{i-1}^{2m-2}+O(1)$$

$$\sim\{\tbinom{2m}{2}\beta_{2m-2}+o(1)\}\sigma_p^{2m}\sum_{i=1}^{k}(i-1)^{m-1}+O(1)$$
$$\sim\{\tbinom{2m}{2}\beta_{2m-2}+o(1)\}\sigma_p^{2m}(k^m/m)+O(1)$$
$$\sim\{(2m-1)\beta_{2m-2}+o(1)\}\tau_k^{2m}+O(1)\sim\beta_{2m}\tau_k^{2m}.$$

Hence, by (3.10) and (3.11), (3.3) follows, and the proof of the theorem is completed.

REMARK. If $E|X_1|^{\nu'}<\infty(\nu'>\nu>2)$, then by Lemma 2, $\{|S_n/\sigma_n|^\nu,\ n\geqq 1\}$ is uniformly integrable. By the central limit theorem we have, without the assumption (1.2),

$$E|S_n/\sigma_n|^\nu\to\beta_\nu \quad (n\to\infty).$$

References

[1] Brown, B. M., Moments of a stopping rule related to the central limit theorem. Ann. Math. Statist. **40** (1969) 1236–1249.

[2] Brown, B. M., Characteristic functions, moments, and the central limit theorem. Ann. Math. Statist. **41** (1970) 658–664.

[3] Doob, J. L., Stochastic Processes, Wiley, New York, 1953.

[4] Ibragimov, I. A. and Linnik, Yu. V., Independent and Stationary Sequences of Random Variables, Wolters-Noordhoff, Groningen, 1971.

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