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# QF-3′ RINGS AND MORITA DUALITY

By

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In [2] we proved that a one-sided artinian ring is QF-3 if and only if its double dual functors preserve monomorphisms. Here with the aid of [3] we prove that the double dual functor over an arbitrary ring preserves monomorphisms of left modules if and only if it is a left QF-3′ ring. In view of this theorem results in [3] and [4] provide an analogue for QF-3′ rings of the Morita-Tachikawa representation theorem for QF-3 rings ([9], Chapter 5). Also we apply it to obtain a characterization of Morita duality between Grothendieck categories that serves to generalize Onodera's theorem [7] that cogenerator rings are self injective, by showing that injectivity is redundant in the classical bimodule characterization of Morita duality for categories of modules.

We denote both the dual functors $\mathrm{Hom}_R(\_, R_R)$ and $\mathrm{Hom}_R(\_, {}_RR)$ by $(\ )^*$. Recall that there is a natural transformation $\sigma: 1_{R\text{-}Mod} \longrightarrow (\ )^{**}$, defined via the usual evaluation maps $\sigma_M: M \longrightarrow M^{**}$. An $R$-module $M$ is called *R-reflexive* (*R-torsionless*) in case $\sigma_M$ is an isomorphism (a monomorphism). Also recall that $R$ is left QF-3′ if the injective envelope $E({}_RR)$ of ${}_RR$ is $R$-torsionless.

1. Theorem. *For any ring R, the following are equivalent:*
   (a) *R is left* QF-3′;
   (b) *The double dual functor* $(\ )^{**}$ *preserves monomorphisms in* $R$-Mod;
   (c) *If* i: $R \longrightarrow E$ *is the inclusion of R into its injective envelope E in* $R$-Mod, *then* i** *is a monomorphism.*

Proof. That (b) implies (c) is immediate, and (c) implies (a) is easy (see [3], Proposition 1.2). Assume that $R$ is a left QF-3′ ring. Since $E=E({}_RR)$ is torsionless there is a sequence

$$R \xrightarrow{i} E \xrightarrow{j} R^X$$

for some set $X$ where $i$ is the inclusion and $j$ is a monomorphism.
Let $p_x: R^X \longrightarrow R$ be the canonical projections and let $b_x = p_x \circ j \circ i(1) \in R$ for each

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$x \in X$. Then if $K=\sum\{b_x R: x \in X\}$ it follows that the left annihilator of $K$ in $R$ is zero. Now suppose $\alpha: M \longrightarrow N$ is a monomorphism in $R$-Mod and consider the induced sequence

$$N^* \xrightarrow{\alpha^*} M^* \xrightarrow{\beta} \text{Coker } \alpha^* \longrightarrow 0.$$

If $f \in M^*$, then since $E$ is injective there exists $\bar{f} \in \text{Hom}_R(N, E)$ such that $\bar{f} \circ \alpha = i \circ f$. Then considering the diagram

$$\begin{array}{ccccccc} O \longrightarrow & M & \xrightarrow{\alpha} & N & & & \\ & f \downarrow & & \bar{f} \downarrow & & & \\ & R & \xrightarrow{i} & E & \xrightarrow{j} & R^X & \xrightarrow{p_x} R \end{array}$$

it follows easily that

$$\alpha^*(p_x \circ j \circ \bar{f}) = p_x \circ j \circ \bar{f} \circ \alpha = p_x \circ j \circ i \circ f = f b_x.$$

Hence $M^* K \subseteq \text{Im } \alpha^*$ so $(\text{Coker } \alpha^*)K = \beta(M^*)K = \beta(M^* K) = 0$. Thus if $\phi \in (\text{Coker } \alpha^*)^*$ we have $\phi(\text{Coker } \alpha^*)K = \phi((\text{Coker } \alpha^*)K) = 0$ so, since the left annihilator of $K$ is zero, $\phi = 0$. But then since $(\text{Coker } \alpha^*)^* = 0$ we see that $M^{**} \xrightarrow{\alpha^{**}} N^{**}$ is monic.

The following theorem follows immediately from ([3], Theorem 1.4) and Theorem 1.

2. THEOREM. *For any ring R, the following are equivalent:*
(1) *R is left* QF-3′ *and its own maximal left quotient ring;*
(2) *The double dual functor ( )** is left exact on R*-Mod;
(3) *If* $0 \longrightarrow R \xrightarrow{i} E_1 \xrightarrow{j} E_2$ *is exact with* $E_1$ *and* $E_2$ *injective in R*-Mod *then* $0 \longrightarrow R \xrightarrow{i^{**}} E_1^{**} \xrightarrow{j^{**}} E_2^{**}$ *is also exact.*

Let $D: \mathfrak{a} \rightleftarrows \mathfrak{a}' : D'$ be a pair of contravariant functors between abelian categories $\mathfrak{a}$ and $\mathfrak{a}'$ that are *adjoint on the right*, i. e., there are isomorphisms

$$\eta_{A,A'}: \text{Hom}_{\mathfrak{a}}(A, D'(A')) \longrightarrow \text{Hom}_{\mathfrak{a}'}(A', D(A)),$$

natural in $A \in |\mathfrak{a}|$ and $A' \in |\mathfrak{a}'|$. Associated with $\eta_{A,A'}$ are the arrows of right adjunction $\tau: 1_{\mathfrak{a}} \longrightarrow D'D$ and $\tau': 1_{\mathfrak{a}'} \longrightarrow DD'$ defined by $\tau_A = \eta^{-1}{}_{A, D(A)}(1_{D(A)})$ and $\tau'_{A'} = \eta_{D'(A'), A'}(1_{D'(A')})$, respectively. These satisfy, for each $A \in |\mathfrak{a}|$, $A' \in |\mathfrak{a}'|$,

$$D(\tau_A) \circ \tau'_{D(A)} = 1_{D(A)} \text{ and } D'(\tau'_{A'}) \circ \tau_{D'(A')} = 1_{D'(A')}.$$

We recall that any pair of such functors $D: \mathfrak{a} \leftrightarrows \mathfrak{a}' : D'$ which are adjoint on the right are left exact ([8], Corollary 3. 2. 3).

We call an object $A$ of $\mathfrak{a}$ ($A'$ of $\mathfrak{a}'$) *reflexive* (respectively, *torsionless*) in case $\tau_A$ ($\tau'_{A'}$) is an isomorphism (respectively, a monomorphism); and we note that (as in [1], Section 23) $D$ and $D'$ define a duality between the *full subcategories of* *reflexive objects* $\mathfrak{a}_0\subseteq\mathfrak{a}$ and $\mathfrak{a}'_0\subseteq\mathfrak{a}'$. Then as in [3] we say that the pair $D:\mathfrak{a}\rightleftarrows\mathfrak{a}':D'$ defines a *Morita duality* in case $D$ and $D'$ are exact and the subcategories $\mathfrak{a}_0\subseteq\mathfrak{a}$ and $\mathfrak{a}_0'\subseteq\mathfrak{a}$, are closed under subobjects and quotient objects and contain sets of generators for $\mathfrak{a}$ and $\mathfrak{a}'$, respectively.

According to ([3], Proposition 2. 3) the functors $D$ and $D'$ of a Morita duality are faithful as well as exact. We shall now show that these conditions imply the closure condition for reflexive objects (as is well known if $\mathfrak{a}$ and $\mathfrak{a}'$ are module categories).

3. Lemma. *Let* $D:\mathfrak{a}\leftrightarrows\mathfrak{a}':D$ *be a right adjoint pair of contravariant functors between abelian categories. Then* $D$ *and* $D'$ *are faithful if and only if all objects in* $\mathfrak{a}$ *and* $\mathfrak{a}'$ *are torsionless.*

Proof. If $D$ is faithful and $0\longrightarrow K\xrightarrow{f}A\xrightarrow{\tau_A}D'D(A)$ is exact, then since $D(\tau_A)\circ\tau'_{D(A)}=1_{D(A)}$, $D(f)=0$ so $f=0$ also. On the other hand, if all objects of $A$ are torsionless and $f\in\mathrm{Hom}_{\mathfrak{a}}(A, B)$, $f\neq0$, then $D'D(f)\circ\tau_A=\tau_B\circ f\neq0$ so $D'D(f)\neq0$, hence $D(f)\neq0$, so $D$ is faithful.

4. Proposition. *A right adjoint pair of contravariant functors* $D:\ \mathfrak{a}\rightleftarrows\mathfrak{a}':\ D'$ *between abelian categories defines a Morita duality if and only if* $\mathfrak{a}$ *and* $\mathfrak{a}'$ *contain generating sets of reflexive objects and* $D$ *and* $D'$ *are faithful and exact.*

Proof. From Lemma 3 and exactness we obtain a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B_0 & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D'D(A) & \longrightarrow & D'D(B_0) & \longrightarrow & D'D(C) & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

with exact rows and columns when $B_0$ is reflexive. Thus the Five Lemma apples.

We don't know whether, in the presence of reflexive generating sets, the closure properties for $\mathfrak{a}_0$ and $\mathfrak{a}_0'$ imply that $D$ and $D'$ are exact and faithful. Of course they do if $\mathfrak{a}$ and $\mathfrak{a}'$ are the categories of modules over a pair of rings (or even if they are functor categories [10]).

We now turn to the general setting of contravariant functors $D:\ \mathfrak{a}\rightleftarrows\mathfrak{a}':\ D'$, adjoint on the right, where $\mathfrak{a}$ and $\mathfrak{a}'$ are Grothendieck categories. With $\mathfrak{a}_0$ and $\mathfrak{a}_0'$ as above we assume that these contain generators $V\in\mathfrak{a}_0$, $V'\in\mathfrak{a}_0'$. Then letting $U=V\oplus D'V'$ and $U'=V'\oplus DV$ (so that $DU\cong U'$ and $D'U'\cong U$), $R=\mathrm{End}_{\mathfrak{a}}(U)$ ($\cong\mathrm{End}_{\mathfrak{a}'}(U')^{\mathrm{op}}$), $S=\mathrm{Hom}_{\mathfrak{a}}(U,\_)$ and $S'=\mathrm{Hom}_{\mathfrak{a}'}(U',\_)$, we have, as in ([6], Theorem 8.1) and ([3], Theorem 3.1), functors

$$\begin{array}{ccc} R\text{-Mod} & \underset{S}{\overset{T}{\rightleftarrows}} & \mathfrak{a} \\ (\ \ )^*\big\downarrow\big\uparrow(\ \ )^* & & D\big\downarrow\big\uparrow D' \\ \text{Mod-}R & \underset{S'}{\overset{T'}{\rightleftarrows}} & \mathfrak{a}' \end{array}$$

where $T(T')$ is a left adjoint of $S(S')$, $T$ and $T'$ are exact, and $TS$ and $T'S'$ are equivalent to the identity functors on $\mathfrak{a}$ and $\mathfrak{a}'$, respectively. Also, as in ([3], Theorem 3.1), $S'\circ D\circ T\cong(\ \ )^*$ and $S\circ D'\circ T'\cong(\ \ )^*$ so $D\circ T\cong T'\circ(\ \ )^*$ and $D'\circ T'\cong T\circ(\ \ )^*$. Thus $\mathrm{Ker}\,T'\subseteq\mathrm{Ker}(\ \ )^*$ and $\mathrm{Ker}\,T\subseteq\mathrm{Ker}(\ \ )^*$.

5. LEMMA. *Let $D$, $D'$, $\mathfrak{a}$, $\mathfrak{a}'$, $U$, $U'$ and $R$ be as above. Then the following are equivalent:*

(a) *$D$ and $D'$ are faithful;*

(b) *$U$ and $U'$ are cogenerators in $\mathfrak{a}$ and $\mathfrak{a}'$, respectively;*

(c) *$\mathrm{Ker}\ T=\mathrm{Ker}(\ \ )^*$ and $\mathrm{Ker}\ T'=\mathrm{Ker}(\ \ )^*$.*

PROOF. If $\alpha\in\mathrm{Hom}_{\mathfrak{a}}(A,B)$, we have a commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{a}}(A,U) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathfrak{a}'}(U',DA) \\ \mathrm{Hom}(\alpha,U)\big\uparrow & & \big\uparrow\mathrm{Hom}(U',D(\alpha)) \\ \mathrm{Hom}_{\mathfrak{a}}(B,U) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathfrak{a}'}(U',DB) \end{array}$$

Now $D(\alpha)\neq 0$ if and only if $\mathrm{Hom}(U',D(\alpha))\neq 0$ (since $U'$ is a generator) if and only if $\mathrm{Hom}(\alpha,U)\neq 0$. It follows that (a) and (b) are equivalent. Now since $D\circ T\cong T'\circ(\ \ )^*$, and $D'\circ T'\cong T\circ(\ \ )^*$, it is clear that if $D$ and $D'$ are faithful, then $\mathrm{Ker}\,T=\mathrm{Ker}\,D\circ T=\mathrm{Ker}\,T'\circ(\ \ )^*\subseteq\mathrm{Ker}(\ \ )^*$ and similarly $\mathrm{Ker}\,T'\subseteq\mathrm{Ker}(\ \ )^*$. Thus (a) implies (c). Suppose $\mathrm{Ker}\,T=\mathrm{Ker}(\ \ )^*$. If $\alpha\in\mathrm{Hom}_{\mathfrak{a}}(A,B)$ and $D(\alpha)=0$, then $(S(\alpha))^*\cong S'DTS(\alpha)\cong S'D(\alpha)=0$ so $\alpha\cong TS(\alpha)=0$, also. Thus (c) implies (a).

We denote the full subcategories of $R$-Mod and Mod-$R$ whose objects are the torsion modules, i.e., those modules $M$ with $M^*=0$, by $R$-Tors and Tors-$R$, respectively. Then, if $R$ is QF-3′, an $R$-module $M$ is torsion if and only if $\mathrm{Hom}(M,E(R))=0$, and $R$-Tors and Tors-$R$ are then localizing subcategories of $R$-Mod and Mod-$R$, respectively (see [3], Proposition 1.1).

6. THEOREM. *Every right adjoint pair of contravariant faithful functors*

*$D: \mathfrak{a} \rightleftarrows \mathfrak{a}' : D'$ between Grothendieck categories with reflexive generators defines a Morita duality.*

PROOF. Suppose that $D: \mathfrak{a} \rightleftarrows \mathfrak{a}' : D', U, U', R, T$, and $T'$ are as in Lemma 5 and that $D$ and $D'$ are faithful. Then by Lemma 5, $U$ and $U'$ are generator-cogenerators, so by Morita's ([6], Theorems 8.3 and 5.6) $R$ is QF-3′ and its own maximal quotient ring. (See also [8], Theorem 4.13.4 or [5], Proposition 4.3.1). Thus by Theorem 2 the $R$-double duals are left exact. But by Lemma 5 we also have Ker $T$=Ker( )* so by ([8], Theorem 4.4.9) we may identify $T$: $R$-Mod$\longrightarrow \mathfrak{a}'$ and $T'$: Mod-$R \longrightarrow \mathfrak{a}'$ with the canonical functors $T$: $R$-Mod$\longrightarrow R$-Mod/$R$-Tors and $T'$: Mod-$R\longrightarrow$Mod-$R$/Tors-$R$ and conclude that $(D, D')$ define a Morita duality by ([3], Theorem 2.6).

Specializing Theorem 6 to the case of module categories yields the following generalization of Onodera's theorem that cogenerator rings are injective [7] and provides a new characterization of Morita duality between categories of modules.

7. COROLLARY. *If $R$ and $S$ are rings and ${}_RU_S$ is a bimodule with $S=\mathrm{End}({}_RU)$ and $R=\mathrm{End}(U_S)$ and if ${}_RU$ and $U_S$ are cogenerators, then ${}_RU$ and $U_S$ are injective.*

PROOF. Apply Theorem 6 to the functors $D=\mathrm{Hom}_R(\_, U)$ and $D'=\mathrm{Hom}_S(\_, U)$.

8. REMARKS.

(1) One can apply the technique used to prove Theorem 1 to the sequence $U \xrightarrow{i} E({}_RU) \xrightarrow{j} U^X$ to give a direct proof that if ${}_RU_S$ a balanced bimodule and ${}_RU$ and $U_S$ are cogenerators, then ${}_RU$ is injective.

(2) In ([4], Theorem 1), conditions (i) and (ii) and the last part of (iii) easily imply that $R$ is QF-3′ so by Theorem 1, as we speculated in ([4], Remark (a)), we can delete the first part of condition (iii) from the statement of that theorem; in view of Theorems 1 and 2 and ([3], Theorem 3.1) it now becomes an analogue for QF-3′ rings of the Morita-Tachikawa representation theorems for QF-3 rings ([9], Theorems 5.3 and 5.8).

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