TSUKUBA J. MATH
Vol. 6 No. 2 (1982). 293—298

V-RINGS RELATIVE TO HEREDITARY TORSION THEORIES

By

Yasuhiko TAKEHANA

A ring R is called a right V-ring in case every simple right R-module is injective. Villamayor has characterized a right V-ring as one each right ideal of which is an intersection of maximal right ideals. The main purpose of this paper is to give torsion theoretical generalizations of right V-rings. Theorem 2 generalizes Theorem 2.1 in [6], stating that any simple module in $\mathcal{T}$ is $\mathcal{T}$-injective if and only if $J(M)=0$ holds for any M in $\mathcal{T}$, where $\mathcal{T}$ denotes a class of modules closed under cyclic submodules, homomorphic images and extensions.

Applying Theorem 2 for the Goldie and the Lambek torsion theories, we obtain Corollaries 5 and 6. We consider in Corollary 5 a ring R (called a right V(G)-ring) for which every singular simple right R-module is injective, and in Corollary 6 a right V(L)-ring for which every dense right ideal is an intersection of maximal right ideals. We characterize V-rings in terms of V(G)-rings or V(L)-rings in Proposition 8 which is closely related to Theorem 8 in [7]. In Theorem 9 it is proved that commutative V(G)-rings turn out to be V-rings. In this connection two examples are given to show that neither commutative V(L)-rings nor V(G)-rings are V-rings.

Throughout this paper R is a ring with a unit, every right R-module is unital and Mod-R is the category of right R-modules. For a right R-module M, $Z(M)$, $E(M)$ and $J(M)$ denote the singular submodule of M, the injective hull of M and the intersection of all maximal submodules of M. A right R-module M is called $\mathcal{T}$-injective for a subclass $\mathcal{T}$ of Mod-R if $\mathrm{Hom}_R(-, M)$ preserves the exactness for every exact sequence of right R-modules $0\to A\to B\to C\to 0$ with $C\in\mathcal{T}$.

LEMMA 1. *A right R-module M is $\mathcal{T}$-injective if and only if* $\mathrm{Hom}_R(-, M)$ *preserves the exactness for every exact sequence $0\to I\to R\to R/I\to 0$ with $R/I\in\mathcal{T}$, where $\mathcal{T}$ denotes a subclass of* Mod-R *closed under cyclic submodules and cyclic homomorphic images.*

Received December 22, 1981. Revised June 10, 1982.

PROOF. This is proved similarly as in the well known proof of Baer's criterion for injectivity.

The following theorem including its proof is a slight modification of Theorem 2.1 in [6].

THEOREM 2. *Let* $\mathcal{T}$ *denote a subclass of* Mod-R *closed under cyclic submodules, homomorphic images and extensions. Then the following conditions are equivalent.*

(1) *Any simple module in* $\mathcal{T}$ *is* $\mathcal{T}$*-injective.*

(2) $J(M)=0$ *holds for any* M *in* $\mathcal{T}$.

(3) *If* I *is a right ideal of* R *with* R/I *in* $\mathcal{T}$, *then* I *is an intersection of maximal right ideals of* R.

PROOF. (1)→(2): Let M in $\mathcal{T}$ and $0\neq x\in M$. By Zorn's lemma there is a submodule Y of M which is maximal among the submodules X of M with $x\notin X$. Let $D=Y+xR$. Then D/Y is a simple submodule of M/Y with D/Y and M/Y in $\mathcal{T}$. Then by (1) $M/Y=(D/Y)\oplus(K/Y)$ for some submodule K of M containing Y. Since $0\neq x+Y\in D/Y$, we have $x\notin K$, and so $K=Y$ by the maximality of Y. We conclude that Y is a maximal submodule of M and $x\notin Y$.

(2) → (3): Obvious.

(3) → (1): Let I be a right ideal of R with R/I in $\mathcal{T}$, S a simple module in $\mathcal{T}$ and $f\in \mathrm{Hom}_R(I, S)$. In view of Lemma 1, it suffices to show that f has an extension $R\rightarrow S$. We may assume f is an epimorphism. Putting $K=\mathrm{Ker}\,(f)$, we have $R/K\in\mathcal{T}$, for I/K and R/I are in $\mathcal{T}$ and $\mathcal{T}$ is closed under extensions. Thus by the assumption there exists a maximal right ideal L of R with $L\supset K$ and $L\not\supset I$. Then $L+I=R$ and $L\cap I=K$, and so $R/K=(L/K)\oplus(I/K)$. It now easily follows that f has an extension $R\rightarrow S$.

We call a ring satisfying the equivalent conditions of the preceding theorem a right V($\mathcal{T}$)-ring.

COROLLARY 3. *Let* R *be a right* V($\mathcal{T}$)*-ring and* $\mathcal{T}$ *a hereditary torsion class of* Mod-R. *Then* $L^2=L$ *holds for any right ideal* L *of* R *with* R/L *in* $\mathcal{T}$.

PROOF. Since $L/(L^2)$ is a homomorphic image of a direct sum of copies of R/L, $L/(L^2)\in\mathcal{T}$. As $\mathcal{T}$ is a closed under extensions, $R/(L^2)$ is in $\mathcal{T}$, and so L^2 is an intersection of maximal right ideals of R by the preceding theorem. It now follows from the same argument as in the proof of Corollary 2.2 in [6] that $L^2=L$ holds.

As is easily seen from Theorem 2.4 in [4], a ring R is a right noetherian V-ring if and only if every semisimple right R-module is injective. This result can be generalized as follows (The proof of (2)→(1) of the following proposition is a modification of the proof of (4)→(2) in [4, Theorem 2.4]).

PROPOSITION 4. *Let* $\mathcal{T}$ *denote a hereditary torsion class of* Mod-R *and* $\mathcal{L}=\{I\subset R\,;\,R/I\in\mathcal{T}\}$. *Then the following conditions are equivalent.*

(1) *R is a right* V($\mathcal{T}$)*-ring and $\mathcal{L}$ satisfies the ascending chain conditions.*

(2) *Every semisimple module in $\mathcal{T}$ is $\mathcal{T}$-injective.*

PROOF. (1)→(2): This follows from Theorem 2 together with Proposition 14.2 of [3].

(2)→(1): In view of Theorem 2 it suffices to prove that $\mathcal{L}$ satisfies the ascending chain conditions. Suppose that

$$I_1\subsetneqq I_2\subsetneqq I_3\cdots\cdots\subsetneqq I_j\subsetneqq\cdots\cdots$$

is a strictly ascending chain in $\mathcal{L}$ and $I=\cup I_j$. Since R is a V($\mathcal{T}$)-ring, for each j, there exists a maximal right ideal L_j of R with $L_j\supseteqq I_{j-1}$ and $L_j\not\supseteqq I_j$. Putting $H_j=L_j\cap I_j$, we have $I_j\supsetneqq H_j\supseteqq I_{j-1}$ and I_j/H_j is a simple module in $\mathcal{T}$, for I_j/H_j is a homomorphic image of R/I_{j-1}. Thus the sequence $0\to I_j/H_j\to I/H_j\to I/I_j\to 0$ splits by the assumption, and so there exists a canonical projection $h_j\colon I\to I/H_j\to I_j/H_j$ for each j. Let f denote a mapping from I to $\oplus(I_j/H_j)$ defined by $f(x)=(h_j(x))$ for $x\in I$. Since $h_i(x)=0$ for $x\in H_j$ and $i>j$, f is of course well defined. By the assumption, $\oplus(I_j/H_j)$ is $\mathcal{T}$-injective, and so f is extended to a mapping from R to $\oplus(I_j/H_j)$ since R/I is in $\mathcal{T}$. But $g(1)\in(I_1/H_1)\oplus(I_2/H_2)\oplus\cdots\oplus(I_j/H_j)$ for some j. This contradicts to the fact that $h_i(x)\neq 0$ for each i and $x\in I_i-H_i$.

Recall a fundamental property of the Goldie or the Lambek torsion theory. For their definitions see [3]. Letting $G(M)$ ($L(M)$) denote the Goldie (the Lambek) torsion submodule of a right R-module M, respectively, there hold for a module M (1) G and L are left exact radicals, (2) $G(M)/Z(M)=Z(M/Z(M))$, (3) $G(M)=M$ ($L(M)=M$) if and only if $Z(M)$ is large in M ($\mathrm{Hom}_R(M, E(R))=0$), (4) $Z(M)\supset L(M)$, (5) if $Z(R)=0$ then $G(M)=Z(M)=L(M)$ and (6) if M is $\mathcal{T}$-injective then M is injective, where $\mathcal{T}=\{M\in\text{Mod-}R\,;\,G(M)=M\}$.

Now we apply Theorem 2 for the Goldie or the Lambek torsion class.

COROLLARY 5. *The following conditions are equivalent.*

(1) *Any singular simple right R-module is injective.*

(2) *$J(M)=0$ holds for each right R-module M with $Z(M)$ large in M.*

(3) *If I is a right ideal of R with $Z(R/I)$ large in R/I, then I is an intersection of maximal right ideals of R.*

A right ideal I of R is called dense if $\mathrm{Hom}_R(R/I, E(R))=0$.

COROLLARY 6. *The following conditions are equivalent.*

(1) *If S is a simple right R-module with $\mathrm{Hom}_R(S, R)=0$ and I a dense right ideal of R, then for any $f\in\mathrm{Hom}_R(I, S)$, f is extended to a mapping from R into S.*

(2) *$J(M)=0$ holds for any right R-module M with $\mathrm{Hom}_R(M, E(R))=0$.*

(3) *Any dense right ideal of R is an intersection of maximal right ideals of R.*

COROLLARY 7. *Suppose that $Z(R_R)=0$, then the following conditions are equivalent.*

(1) *If S is a simple right R-module with $\mathrm{Hom}_R(S, R)=0$, then S is injective.*

(2) *$J(M)=0$ holds for any singular right R-module M.*

(3) *Any large right ideal of R is an intersection of maximal right ideals of R.*

PROOF. This follows from Corollaries 5 and 6 together with the fact that if $Z(R_R)=0$, then the Lambek torsion theory coincides with the Goldie torsion theory.

We call a ring satisfying the equivalent conditions of Corollary 5 (Corollary 6) a right V(G)-ring (a right V(L)-ring) respectively. It is clear that right V(G)-rings are right V(L)-rings.

PROPOSITION 8. *The following assertions hold.*

(1) *R is a right V-ring if and only if R is a right V(G)-ring and every minimal right ideal of R is injective.*

(2) *R is a right and left V-ring if and only if R is a right and left V(L)-ring with $Z(R_R)=Z(_RR)=0$ and every minimal one-sided ideal of R is injective.*

PROOF. (1) The "only if" part is clear. For the "if" part it is sufficient to observe that $J(M)=0$ holds for each cyclic singular right R-module M in view of Theorem 8 in [1]. But this is a direct consequence of Corollary 5.

(2) It is well known that the left Goldie torsion theory coincides with the left Lambek torsion theory in case R is left nonsingular. On the other hand, each right V-ring is left nonsingular by Lemma 2.3 in [6]. Thus (2) follows from (1).

Next we consider commutative V(G)-rings.

THEOREM 9. *Each commutative* V(G)*-ring is a* V*-ring.*

PROOF. It is well known that if R is commutative, then R is a V-ring if and only if R is a Von-Neumann regular ring. It is sufficient to prove that $I^2=I$ holds for every right ideal I of R. If I is a large right ideal of R, then $I^2=I$ holds by an application of Corollary 3 for the Goldie torsion theory. Now let L be a right ideal of R and J a complement of L in R (i.e. J is maximal in $\{J\subset R;\ J\cap L=0\}$). Then it is well known that $L+J$ is large in R. Thus $L+J=(L+J)^2=L^2+J\cdot L+L\cdot J+J^2=L^2+J^2$, and so $L^2=L$ as desired.

The following example is given to show that V(G)-rings are not necessarily V-rings.

EXAMPLE 1. Let k be a field, $R=\begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$, $M=\begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix}$ and $K=\begin{pmatrix} 0 & 0 \\ k & k \end{pmatrix}$. Then it is easily verified that M is a unique proper large right ideal of the ring R and $Z(R)=0$. Since M is a maximal right ideal of R, R is a V(G)-ring by Corollary 7. But $J(R)=M\cap K\neq 0$, and so R is not a V-ring.

In Theorem 8 in [7], R. Yue Chi Ming showed that R is a V-ring if and only if R satisfies the following conditions (1) $J(M)=0$ holds for any cyclic singular right R-module M and (2) every minimal right ideal is injective. It is easily verified that the above condition (1) is equivalent to the condition (3) of Corollary 7.

The following example shows that a ring satisfying the condition (1) above is not always a V(G)-ring and a commutative V(L)-ring is not always a V-ring.

EXAMPLE 2. Let k be a field, $R=\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix};\ a,\ b\in k\right\}$ and $M=\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$. Then R is a commutative ring and has only one non-trivial right ideal M. Since $J(R)=Z(R)=M\neq 0$, R is not a V-ring. It is clear that R is a V(L)-ring and satisfies the condition (3) of Corollary 7.

Finally we consider another generalization of V-rings which is suggested by Theorem 6 in [7].

THEOREM 10. *Let $\mathfrak{T}$ denote a subclass of* Mod-R *closed under cyclic submodules and homomorphic images. Then the following conditions are equivalent.*

(1) *Any simple right R-module is $\mathfrak{T}$-injective.*

(2) *$J(N)=0$ holds for any right R-module N such that there exists a simple submodule S of N with N/S in $\mathfrak{T}$.*

(3) *If K is a maximal right subideal of a right ideal P of R with R/P in $\mathcal{T}$, then K is an intersection of maximal right ideals of R.*

PROOF. (1)→(2): Let N be a module and S a simple submodule of N with N/S in $\mathcal{T}$. Then it is similarly proved as in the proof of Theorem 2 that $J(N/S)=0$. By the assumption, $N=S\oplus H$ holds for some submodule H ($\cong N/S$) of N. Thus $J(N)\subseteqq J(S)\oplus J(H)=0$, as desired.

(2)→(3): Obvious.

(3)→(1): It is similarly proved as in the proof of (3)→(1) of Theorem 2.

COROLLARY 11. *The following conditions are equivalent.*

(1) *Any simple right R-module is injective.*

(2) *$J(N)=0$ holds for any right R-module N such that there exists a simple submodule S with $Z(N/S)=N/S$.*

(3) *If K is a maximal right subideal of a large right ideal P of R, then K is an intersection of maximal right ideals of R.*

PROOF. Put $\mathcal{T}=\{M\in\text{Mod-}R\,;\,Z(M)=M\}$ in Theorem 10.

Acknowledgement

The author wishes to thank the referee for his remark on Proposition 8.

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Institute of Mathematics,
University of Tsukuba
Sakura-mura, Niihari-gun,
Ibaraki, 305 Japan