

## LOGARITHMIC UNIFORM DISTRIBUTION OF $(\alpha n + \beta \log n)$

By

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### 1. Introduction

A sequence  $\omega = (x_n)_{n=1}^\infty$  of real numbers is said to be uniformly distributed modulo 1 if the proportion of indices  $n \leq N$  such that the fractional parts  $\{x_n\}$  are contained in an interval  $I \subseteq [0, 1)$  is asymptotically equal to the length of  $I$ . Put  $\chi(x; y) = 1$  for  $\{y\} < x$  and  $\chi(x; y) = 0$  otherwise; then  $\omega$  is uniformly distributed if and only if

$$(1) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \chi(x; x_n) = x \quad \text{for } 0 < x < 1.$$

It is well known (cf. the monographs [1] and [3]) that (1) is equivalent to

$$\lim_{N \rightarrow \infty} D_N^*(\omega) = 0,$$

where

$$D_N^*(\omega) = \sup_{0 < x < 1} \left| \frac{1}{N} \sum_{n=1}^N \chi(x; x_n) - x \right|$$

denotes the discrepancy of the sequence  $\omega$ . The systematic study of uniformly distributed sequences was initiated by H. Weyl [9]. Well known examples of uniformly distributed sequences are  $(\alpha n)$  with irrational  $\alpha$  and  $(\sqrt{n})$ ;  $(\log n)$  is known not to be uniformly distributed, but Tsuji [8] proved that

$$(2) \quad \lim_{N \rightarrow \infty} \frac{1}{\sum_{n=1}^N \frac{1}{n}} \sum_{n=1}^N \frac{1}{n} \chi(x; x_n) = x \quad (0 < x < 1)$$

for  $x_n = \log n$ . A sequence  $\omega = (x_n)$  with this property is said to be uniformly distributed with respect to the logarithmic mean. This is equivalent to

$$\lim_{N \rightarrow \infty} D_N(\omega) = 0,$$

where

$$D_N(\omega) = \sup_{0 \leq x < 1} \left| \frac{1}{\sum_{n=1}^N \frac{1}{n}} \sum_{n=1}^N \frac{1}{n} \chi(x; x_n) - x \right|$$

denotes the logarithmic discrepancy of  $\omega$  (cf. [5]). In a recent article E. Hlawka [2] investigated the sequence  $(\alpha n + \beta \log n)$  ( $\beta \neq 0$ ) with respect to the logarithmic mean. He proved upper bounds for exponential sums from which (by the inequality of Erdős-Turan) we may conclude  $D_N(\alpha n + \beta \log n) \leq c(\beta) / \sqrt{\log N}$ . In [7] the first author proved a theorem that gives upper bounds for the discrepancy with respect to general weights and remarked that these estimates in fact give  $D_N(\alpha n + \beta \log n) \leq c(\beta) / \log N$ . Unfortunately the hypotheses of the theorem are not satisfied and it remains an open problem to prove this estimate (which essentially would be best possible; cf. [5]). In the next section we prove  $D_N(\alpha n + \beta \log n) \leq c(\alpha, \beta) / \log N$  provided  $\alpha$  is rational or of finite approximation type. In §3 we obtain  $D_N(\alpha n + \beta \log n) \leq c_1(\beta) (\log \log N)^2 / \log N$  by means of estimates for the exponential sums

$$\sum_{n=1}^N \frac{1}{n} e^{2\pi i h(\alpha n + \beta \log n)} \quad (h = 1, 2, \dots),$$

which are accomplished by a generalization of van der Corput's method.

*Notations.* As usual  $[t]$  denotes the largest integer  $\leq t$  and the fractional part is given by  $\{t\} = t - [t]$ ; furthermore we put  $\psi(t) = \{t\} - \frac{1}{2}$  and  $\|t\| = \min(\{t\}, 1 - \{t\})$ . In §2 we make use of the notations  $\lfloor t \rfloor =$  maximal integer  $< t$  and  $\lceil t \rceil =$  minimal integer  $\geq t$ . It is easy to see that  $\lceil t \rceil = \lfloor t \rfloor + 1$  and  $\lceil t \rceil = -\lfloor -t \rfloor$ .

### 2. Elementary Estimates

We try to establish an upper bound  $c / \log N$  for  $D_N(\omega)$ , where  $\omega = (\alpha n + \beta \log n)_{n=1}^\infty$ . Given  $\beta > 0$  we will prove that we may choose an absolute constant  $c = c(\beta)$  valid for a large class of  $\alpha$ 's.

**THEOREM 2.1.** *Assume that  $0 \leq \{\alpha\} \leq c_0 / \log N$  ( $c_0$  a given positive constant). Then*

$$D_N(\omega) \leq 2 \cdot \frac{c_0 + 2 + \beta + 1/\beta + e^{1/\beta}}{\log N}$$

**PROOF.** Let  $f(u) = \{\alpha\}u + \beta \log u - \{\alpha\}$ . Then  $f'(u) > 0$  and  $uf'(u) = \{\alpha\}u + \beta$  is increasing.

Let  $k$  be the largest integer such that  $\lceil f^{-1}(k) \rceil \leq N$ . Then we have for  $0 < x < 1$ :

$$\sum_{n=1}^N \frac{1}{n} \chi(x; f(n)) = \sum_{j=0}^{k-1} \sum_{\lceil f^{-1}(j) \rceil \leq n < \lceil f^{-1}(j+1) \rceil} \frac{1}{n} \chi(x; f(n)) + \sum_{\lceil f^{-1}(k) \rceil \leq n \leq N} \frac{1}{n} \chi(x; f(n)).$$

The second sum can be estimated in the following way:

$$\begin{aligned} \sum_{\lceil f^{-1}(k) \rceil \leq n \leq N} \frac{1}{n} \chi(x; f(n)) &\leq \frac{N - f^{-1}(k) + 1}{f^{-1}(k)} \leq \frac{N}{f^{-1}(k)} \leq \frac{f^{-1}(k+1)}{f^{-1}(k)} \\ &= e^{(\log f^{-1}(k+1) - \log f^{-1}(k))} = e^{1/(f^{-1}(\xi_k) f'(f^{-1}(\xi_k)))} \\ &= e^{1/(\alpha f^{-1}(\xi_k) + \beta)} \leq e^{1/\beta}, \end{aligned}$$

where  $k \leq \xi_k \leq k+1$ .

Since (for positive reals  $A < B$ )

$$\sum_{A \leq n < B} \frac{1}{n} = \log \frac{B}{A} + \frac{\theta}{A} \quad (\text{with } |\theta| \leq 1),$$

we obtain

$$\begin{aligned} S_j &= \sum_{\lceil f^{-1}(j) \rceil \leq n < \lceil f^{-1}(j+x) \rceil} \frac{1}{n} \chi(x; f(n)) = \sum_{f^{-1}(j) \leq n < f^{-1}(j+x)} \frac{1}{n} \\ &= \log f^{-1}(j+x) - \log f^{-1}(j) + \frac{\theta_j}{f^{-1}(j)} \quad (|\theta_j| \leq 1). \end{aligned}$$

Hence, by the mean value theorem,

$$\sum_{j=0}^{k-1} S_j = x \cdot \sum_{j=0}^{k-1} \frac{1}{f^{-1}(\xi_{x,j}) f'(f^{-1}(\xi_{x,j}))} + \theta \left( 1 + \int_0^{f(N)} \frac{du}{f^{-1}(u)} \right)$$

for some  $\xi_{x,j}$  with  $j \leq \xi_{x,j} \leq j+1$ . Since

$$\begin{aligned} \sum_{j=0}^{k-1} \frac{1}{f^{-1}(\xi_{x,j}) f'(f^{-1}(\xi_{x,j}))} &\leq \sum_{j=0}^{k-1} \frac{1}{f^{-1}(j) f'(f^{-1}(j))} \\ &\leq \frac{1}{f'(1)} + \int_0^{f(N)} \frac{du}{f^{-1}(u) f'(f^{-1}(u))} \leq \frac{1}{\beta} + \int_1^N \frac{dt}{t} = \frac{1}{\beta} + \log N, \\ \sum_{j=0}^{k-1} \frac{1}{f^{-1}(\xi_{x,j}) f'(f^{-1}(\xi_{x,j}))} &\geq \sum_{j=0}^{k-1} \frac{1}{f^{-1}(j+1) f'(f^{-1}(j+1))} \\ &= \frac{-1}{f'(1)} + \sum_{j=0}^k \frac{1}{f^{-1}(j) f'(f^{-1}(j))} \geq -\frac{1}{f'(1)} + \int_0^{k+1} \frac{du}{f^{-1}(u) f'(f^{-1}(u))} \\ &\geq -\frac{1}{f'(1)} + \int_0^{f(N)} \frac{du}{f^{-1}(u) f'(f^{-1}(u))} \geq -\frac{1}{\beta} + \log N \end{aligned}$$

and

$$0 < \int_0^{f(N)} \frac{du}{f^{-1}(u)} = \int_1^N \frac{f'(t) dt}{t} = \{\alpha\} \int_1^N \frac{dt}{t} + \beta \int_1^N \frac{dt}{t^2} \leq \{\alpha\} \log N + \beta \leq c_0 + \beta,$$

we obtain

$$\left| \sum_{j=0}^{k-1} S_j - x \sum_{n=1}^N \frac{1}{n} \right| \leq 1 + c_0 + \beta + \frac{1}{\beta} + x \left| \sum_{n=1}^N \frac{1}{n} - \log N \right| \leq 2 + c_0 + \beta + \frac{1}{\beta}.$$

Hence

$$D_N(\omega) \leq 2D_N(f(n)) \leq 2 \cdot \frac{c_0 + 2 + \beta + 1/\beta + e^{1/\beta}}{\sum_{n=1}^N \frac{1}{n}},$$

thus proving the theorem.

REMARK 1. The above arguments essentially reproduce the proof of Satz 3 in [7]. (We want to note that in the estimate for  $D_N^*(P, x_n)$  the integral should be replaced by  $\int_1^N p'(u) f'(u) du$ .)

REMARK 2. By Theorem 2.1. we have

$$D_N \left( \frac{n}{N} + \beta \log n; n=1, \dots, N \right) \leq 2 \cdot \frac{3 + \beta + 1/\beta + e^{1/\beta}}{\log N},$$

since  $\log N/N \leq 1/e < 1 = c_0$ .

In order to prove that  $D_N(\omega) \leq c(\beta)/\log N$  uniformly for all  $\alpha$  it would suffice to prove this for all rational  $\alpha$ 's. This follows immediately from

$$|D_N(\omega) - D_N(\omega')| \leq \varepsilon$$

where  $\omega = (x_n)$ ,  $\omega' = (x'_n)$  such that  $|x_n - x'_n| \leq \varepsilon$  (special case of [5], Satz 6). Unfortunately we did not succeed in establishing the existence of such a bound  $c(\beta)$  and we must content ourselves with

THEOREM 2.2. Let  $\alpha = p/q$  ( $0 < p < q$ ;  $p, q$  integers) and  $\beta > 0$ . Then for  $\omega = (x_n) = ((p/q)n + \beta \log n)$

$$\left| \sum_{n=q}^N \frac{1}{n} \chi(x; x_n) - x \sum_{n=q}^N \frac{1}{n} \right| \leq K(\beta) = \frac{2}{\beta} + 2 + 3 \frac{e^{2/\beta}}{e^{1/\beta} - 1} \quad (0 \leq x < 1)$$

and

$$D_N(\omega) \leq \frac{1 + K(\beta) + \log q}{\log N}.$$

PROOF. Put  $b = e^{1/\beta} > 1$  and let  $l, k$  be the largest integers such that  $b^l \leq q$ ,  $b^k \leq N$ , respectively. Then

$$\begin{aligned} \sum_{n=q}^N \frac{1}{n} \chi(x; x_n) &= - \sum_{b^l \leq n < q} \frac{1}{n} \chi(x; x_n) + \sum_{j=l}^{k-1} \sum_{b^j \leq n < b^{j+1}} \frac{1}{n} \chi(x; x_n) + \sum_{b^k \leq n \leq N} \frac{1}{n} \chi(x; x_n) \\ &= -I + II + III. \end{aligned}$$

We have for the first and the third term

$$I \leq \sum_{b^l \leq n < b^{l+1}} \frac{1}{n} \leq \log b + 1, \quad III \leq \sum_{b^k \leq n < b^{k+1}} \frac{1}{n} \leq \log b + 1.$$

For the remaining term we proceed as follows:

$$\begin{aligned} II &= \sum_{j=l}^{k-1} \sum_{r=0}^{q-1} \sum_{b^j \leq mq+r < b^{j+1}} \frac{1}{mq+r} \chi \left( x; \frac{rp}{q} + \log_b (mq+r) \right) \\ &= \sum_{j=l}^{k-1} \sum_{\substack{\lfloor \frac{rp}{q} \rfloor < x \\ 0 \leq r < q}} \left( \sum_{b^j \leq mq+r < b^{j+1} - \lfloor \frac{rp}{q} \rfloor + x} \frac{1}{mq+r} + \sum_{b^{j+1} - \lfloor \frac{rp}{q} \rfloor \leq mq+r < b^{j+1}} \frac{1}{mq+r} \right) \\ &\quad + \sum_{j=l}^{k-1} \sum_{\substack{\lfloor \frac{rp}{q} \rfloor \geq x \\ 0 \leq r < q}} \sum_{b^{j+1} - \lfloor \frac{rp}{q} \rfloor \leq mq+r < b^{j+1} + x - \lfloor \frac{rp}{q} \rfloor} \frac{1}{mq+r} \\ &= \sum_{j=l}^{k-1} \sum_{\substack{\lfloor \frac{rp}{q} \rfloor < x \\ 0 \leq r < q}} \left( \frac{1}{q} \log b^x + \frac{\theta_{jr} + \theta'_{jr}}{b^j} \right) + \sum_{j=l}^{k-1} \sum_{\substack{\lfloor \frac{rp}{q} \rfloor \geq x \\ 0 \leq r < q}} \left( \frac{1}{q} \log b^x + \frac{\theta''_{jr}}{b^j} \right) \\ &= (k-l)x \log b + \frac{3\theta'q}{1-\frac{1}{b}} \frac{1}{b^l} = x(\log N - \log q) + \theta'' \log b + \frac{3\theta''b^2}{b-1}, \end{aligned}$$

where all  $\theta$ 's are non specified numbers with  $|\theta| \leq 1$ . Combining the estimates for I, II and III we obtain

$$\begin{aligned} \sum_{n=q}^N \frac{1}{n} \chi(x; x_n) &= x(\log N - \log q) + \theta' \left( 2 \log b + 1 + \frac{3b^2}{b-1} \right) \\ &= x \sum_{n=q}^N \frac{1}{n} + \theta \left( 2 \log b + 2 + \frac{3b^2}{b-1} \right) \quad (|\theta| \leq 1). \end{aligned}$$

To prove the second part of the theorem we note that

$$\begin{aligned} \left| \sum_{n=1}^N \frac{1}{n} \chi(x; x_n) - x \sum_{n=1}^N \frac{1}{n} \right| &\leq \left| \sum_{n=1}^{q-1} \frac{1}{n} \chi(x; x_n) - x \sum_{n=1}^{q-1} \frac{1}{n} \right| + \left| \sum_{n=q}^N \frac{1}{n} \chi(x; x_n) - x \sum_{n=q}^N \frac{1}{n} \right| \\ &\leq 1 + \log q + 2 \log b + 2 + \frac{3b^2}{b-1} \end{aligned}$$

(for  $N \geq q$ , whereas the result is trivial for  $N < q$ ).

REMARK 1. Taking  $c_0 = \{\alpha\} \log q$  in the last line of the proof of Theorem 2.1. we have

$$\sum_{n=1}^q \frac{1}{n} D_q(\omega) \leq 2 \left( \{\alpha\} \log q + 2 + \beta + \frac{1}{\beta} + e^{1/\beta} \right).$$

Hence, instead of taking the trivial bound  $1 + \log q$ , we may write

$$\left| \sum_{n=1}^{q-1} \frac{1}{n} \chi(x; x_n) - x \sum_{n=1}^{q-1} \frac{1}{n} \right| \leq \frac{1}{q} + \sum_{n=1}^q \frac{1}{n} D_q(\omega) \leq 1 + 2 \left( \{\alpha\} \log q + 2 + \beta + \frac{1}{\beta} + e^{1/\beta} \right).$$

This gives  $D_N(\omega) \leq 2(\{\alpha\} \log q + (3 + \beta + 1/\beta + K(\beta) + e^{1/\beta}))/\log N$ . As a special case we obtain a uniform bound for all  $\alpha = p/q$  with  $0 < p/q \leq 1/\log q$ .

REMARK 2. Since

$$\begin{aligned} \left| \sum_{n=1}^N \frac{1}{n} \chi(x; x_n) - x \sum_{n=1}^N \frac{1}{n} \right| &= \left| \sum_{n=1}^N \frac{1}{n} (\chi(x; x_n) - x) \right| \leq \left| \frac{1}{N} \sum_{n=1}^N (\chi(x; x_n) - x) \right| \\ &\quad + \sum_{n=1}^{N-1} \frac{1}{n+1} \left| \frac{1}{n} \sum_{j=1}^n \chi(x; x_j) - x \right| \leq \sum_{n=1}^{N-1} \frac{1}{n+1} D_n^*(\omega) + 1, \end{aligned}$$

we have

$$(1) \quad D_N(\omega) \leq \frac{1}{\log N} \left( \sum_{n=1}^{N-1} \frac{1}{n+1} D_n^*(\omega) + 1 \right),$$

where  $D_n^*$  denotes the usual discrepancy with respect to the arithmetic mean. In the following let  $\alpha$  be an irrational number of finite approximation type  $\eta \geq 1$ , i.e. for all  $\varepsilon > 0$  there is a constant  $c(\alpha, \varepsilon)$  such that

$$\|q\alpha\| \geq \frac{c(\alpha, \varepsilon)}{q^{\eta+\varepsilon}}$$

for all integers  $q \geq 1$ . For such  $\alpha$ 's rather good estimates of  $D_N^*(\alpha n)$  are known. In order to utilize these we relate  $D_N^*(\omega)$  to  $D_N^*(\alpha n)$ . For every positive integer  $h$  we have

$$\begin{aligned} \left| \sum_{n=1}^N e^{2\pi i h(\alpha n + \beta \log n)} \right| &= \left| \sum_{n=1}^N e^{2\pi i h \alpha n} \cdot e^{2\pi i h \beta \log n} \right| \leq \left| \sum_{n=1}^N e^{2\pi i h \alpha n} \right| \\ &\quad + \sum_{n=1}^{N-1} 2\pi h \beta (\log(n+1) - \log n) \left| \sum_{j=1}^n e^{2\pi i h \alpha j} \right| \\ &\leq \frac{1}{2\|h\alpha\|} (1 + 2\pi\beta h \log N), \end{aligned}$$

since  $\left| \sum_{j=1}^n e^{2\pi i h \alpha j} \right| \leq \frac{1}{|\sin \pi h \alpha|} \leq \frac{1}{2\|h\alpha\|}$ .

Now we use the inequality of Erdős and Turán (cf. [3], p. 112) together with the estimates

$$\sum_{h=1}^m \frac{1}{\|h\alpha\|} \leq c'(\alpha, \varepsilon) m^{\eta+\varepsilon}, \quad \sum_{h=1}^m \frac{1}{h\|h\alpha\|} \leq c''(\alpha, \varepsilon) \cdot m^{\eta-1+\varepsilon}$$

(cf. [3], p. 123) to obtain

$$\begin{aligned} D_N^*(\omega) &\leq c_0 \left( \frac{1}{m} + \sum_{h=1}^m \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^N e^{2\pi i h(\alpha n + \beta \log n)} \right| \right) \\ &\leq c_0 \left( \frac{1}{m} + \frac{1}{N} \sum_{h=1}^m \frac{1}{2h \|h\alpha\|} + \frac{\log N}{N} \sum_{h=1}^m \frac{\pi\beta}{\|h\alpha\|} \right) \\ &\leq c_1(\alpha, \varepsilon, \beta) \left( \frac{1}{m} + \frac{m^{\eta-1+\varepsilon}}{N} + \frac{\log N}{N} m^{\eta+\varepsilon} \right). \end{aligned}$$

Choosing  $m = \lfloor N^{\frac{1}{\eta+1}} \rfloor$  we obtain for every  $\varepsilon > 0$ :  $D_N^*(\omega) \leq c_2(\alpha, \varepsilon, \beta) N^{-\frac{1}{\eta+1} + \varepsilon}$ . Inserting this in (1) yields

$$(2) \quad D_N(\omega) \leq \frac{c_3(\alpha, \beta)}{\log N}$$

provided that  $\alpha$  is of finite approximation type.

The argument used in the proof of Theorem 2.1. may be refined to give the following result:

**THEOREM 2.3.** *Assume  $f$  to be a twice continuously differentiable (real-valued) function defined on  $[1, \infty)$  such that*

$$f(1) = 0, \quad 0 < c_0 \leq f'(u) \leq c_1, \quad |f''(u)| \leq \frac{c_2}{u} \quad (u \geq 1).$$

Then for  $0 \leq x < 1$

$$\sum_{n=1}^N \frac{1}{n} \chi(x; f(n)) = x \sum_{n=1}^N \frac{1}{n} + \sum_{j=0}^{\lfloor f^{-1}(N) \rfloor} \frac{\{-f^{-1}(j+x)\} - \{-f^{-1}(j)\}}{f^{-1}(j)} + O(1),$$

where the  $O$ -constant only depends on  $c_0, c_1, c_2$ .

**PROOF.** Let  $k$  be the largest integer such that  $\lfloor f^{-1}(k) \rfloor \leq N$ . Then

$$\sum_{n=1}^N \frac{1}{n} \chi(x; f(n)) = \sum_{j=0}^{k-1} \sum_{\lfloor f^{-1}(j) \rfloor \leq n < \lfloor f^{-1}(j+1) \rfloor} \frac{1}{n} \chi(x; f(n)) + \sum_{\lfloor f^{-1}(k) \rfloor \leq n \leq N} \frac{1}{n} \chi(x; f(n)).$$

The second sum can be estimated in the following way:

$$\begin{aligned} \sum_{\lfloor f^{-1}(k) \rfloor \leq n \leq N} \frac{1}{n} \chi(x; f(n)) &\leq \frac{N - f^{-1}(k) + 1}{f^{-1}(k)} < \frac{f^{-1}(k+1) - f^{-1}(k) + 1}{f^{-1}(k)} \\ &\leq \frac{1}{f^{-1}(k) f'(f^{-1}(\xi_k))} + \frac{1}{f^{-1}(k)} \leq \frac{1}{c_0} + 1. \end{aligned}$$

Applying the formula

$$\sum_{l=1}^m \frac{1}{l} = \gamma + \log m + \frac{1}{2m} + O\left(\frac{1}{m^2}\right)$$

we obtain

$$\begin{aligned} \sum_{\lceil f^{-1}(j) \rceil \leq n < \lceil f^{-1}(j+1) \rceil} \frac{1}{n} \chi(x; f(n)) &= \sum_{\lceil f^{-1}(j) \rceil \leq n \leq \lfloor f^{-1}(j+x) \rfloor} \frac{1}{n} \\ &= \frac{1}{\lceil f^{-1}(j) \rceil} + \log \lfloor f^{-1}(j+x) \rfloor - \log \lceil f^{-1}(j) \rceil \\ &\quad + \frac{1}{2 \lfloor f^{-1}(j+x) \rfloor} - \frac{1}{2 \lceil f^{-1}(j) \rceil} + O\left(\frac{1}{\lceil f^{-1}(j) \rceil^2}\right) \end{aligned}$$

where the  $O$ -constants are absolute noes. We note, that the above formula is valid even if  $\lfloor f^{-1}(j+x) \rfloor < \lceil f^{-1}(j) \rceil$ . Next we observe that

$$(3) \quad \sum_{j=0}^{\infty} \frac{1}{\lfloor f^{-1}(j) \rfloor^2} = O(1).$$

Hence

$$\begin{aligned} \sum_{n=1}^N \frac{1}{n} \chi(x; f(n)) &= \sum_{j=0}^{k-1} (\log f^{-1}(j+x) - \log f^{-1}(j)) \\ &\quad + \frac{\lceil f^{-1}(j+x) \rceil - f^{-1}(j+x)}{f^{-1}(j)} - \frac{\lceil f^{-1}(j) \rceil - f^{-1}(j)}{f^{-1}(j)} + O(1), \end{aligned}$$

using Taylor's theorem, formula (3) and  $\lceil t \rceil = \lfloor t \rfloor + 1$ . The  $O$ -constants depend on  $c_0, c_1, c_2$ . Since  $\lceil t \rceil - t = -\lfloor -t \rfloor - t = \{-t\}$ , the result may be written as

$$\sum_{j=0}^{k-1} (\log f^{-1}(j+x) - \log f^{-1}(j)) + \sum_{j=0}^{\lfloor f^{-1}(N) \rfloor} \frac{\{-f^{-1}(j+x)\} - \{-f^{-1}(j)\}}{f^{-1}(j)} + O(1).$$

By Taylor's theorem,

$$\begin{aligned} \log f^{-1}(j+x) - \log f^{-1}(j) &= x \frac{1}{f^{-1}(j)f'(f^{-1}(j))} + \frac{x^2}{2} \left( -\frac{f^{-1}(\xi)f''(f^{-1}(\xi)) + f'(f^{-1}(\xi))}{f^{-1}(\xi)^2 f'(f^{-1}(\xi))^3} \right) \end{aligned}$$

(for some  $\xi$  with  $j \leq \xi \leq j+x$ ). The absolute value of the last expression is bounded above by

$$\frac{f^{-1}(\xi)|f''(f^{-1}(\xi))| + c_1}{f^{-1}(\xi)^2 c_0^3} \leq \frac{c_2 + c_1}{f^{-1}(j)^2 c_0^3} = O\left(\frac{1}{f^{-1}(j)^2}\right).$$

Hence (applying (3)) we obtain by Euler's summation formula

$$\sum_{j=0}^{k-1} (\log f^{-1}(j+x) - \log f^{-1}(j)) = x \int_1^{k-1} \frac{dt}{f^{-1}(t)f'(f^{-1}(t))} + O(1).$$

The proof of the theorem is now completed by observing that



$$\int_1^{k-1} \frac{dt}{f^{-1}(t)f'(f^{-1}(t))} = \log f^{-1}(k-1) - \log f^{-1}(1) = \sum_{n=1}^N \frac{1}{n} + O(1).$$

REMARK. The above theorem may be applied to the sequence  $\omega = (\alpha n + \beta \log n)$  ( $\alpha, \beta \geq 0$ ); we just have to consider  $f(u) = (1 + \{\alpha\})u + \beta \log u - (1 + \{\alpha\})$ . The existence of a bound  $D_N(\omega) \leq c'(\beta)/\log N$  (uniformly in  $\alpha$ ) is equivalent to

$$\left| \sum_{j=0}^M \frac{\{-f^{-1}(j+x)\} - \{-f^{-1}(j)\}}{f^{-1}(j)} \right| \leq c(\beta) \text{ for all } M, \alpha \text{ and } x.$$

### 3. Exponential Sums

In the following we give a refinement of [2], Satz 1 in the case of the logarithmic mean.

THEOREM 3.1. For reals  $\alpha, \beta$  ( $\beta \neq 0$ ) and positive integers  $h$  we have

$$\left| \sum_{n=A}^B \frac{1}{n} e^{2\pi i h(\alpha n + \beta \log n)} \right| \leq C(\beta) \left( \frac{\sqrt{h}}{A} + \frac{1}{\sqrt{h}} \right),$$

and

$$\left| \sum_{n=1}^N \frac{1}{n} e^{2\pi i h(\alpha n + \beta \log n)} \right| \leq \log h + 1 + 2C(\beta),$$

where  $A, B$  and  $N$  denote positive integers and  $C(\beta)$  is a constant depending continuously on  $\beta$ .

PROOF. We begin by showing how to deduce the second formula from the first.

$$\begin{aligned} \left| \sum_{n=1}^N \frac{1}{n} e^{2\pi i h(\alpha n + \beta \log n)} \right| &\leq \sum_{n=1}^h \frac{1}{n} + \left| \sum_{n=h+1}^N \frac{1}{n} e^{2\pi i h(\alpha n + \beta \log n)} \right| \\ &\leq 1 + \log h + C(\beta) \left( \frac{\sqrt{h}}{h+1} + \frac{1}{\sqrt{h}} \right) \leq 1 + \log h + 2C(\beta). \end{aligned}$$

For the main part of the proof we require the following lemma:

LEMMA 1. Let  $A < B$  be positive reals. Then for arbitrary reals  $\alpha, \beta$  ( $\beta \neq 0$ ) we have

$$\begin{aligned} \left| \int_A^B \frac{1}{u} e^{2\pi i h(\alpha u + \beta \log u)} du \right| &\leq \left( 1 + \frac{1}{\sqrt{\pi |\beta|}} \right) \frac{16}{\sqrt{\pi h |\beta|}}, \\ \left| \int_A^B \frac{1}{u^2} e^{2\pi i h(\alpha u + \beta \log u)} du \right| &\leq \left( 1 + \frac{\sqrt{2}}{\sqrt{\pi |\beta|}} \right) \frac{32}{A \sqrt{\pi h |\beta|}} \end{aligned}$$

for all positive integers  $h$ .

Proof of the Lemma. We may restrict ourselves to the case  $\beta > 0$ ; otherwise we may

take the complex conjugate of the integral. If  $\alpha u + \beta \neq 0$  for all  $u$  with  $A \leq u \leq B$  then

$$\left| \int_A^B \frac{1}{u} e^{2\pi i h(\alpha u + \beta \log u)} du \right| = \left| \int_A^B \frac{(e^{2\pi i h(\alpha u + \beta \log u)})'}{2\pi i h(\alpha u + \beta)} du \right|.$$

By the second mean value theorem we obtain

$$\begin{aligned} \left| \int_A^B \frac{(e^{2\pi i h(\alpha u + \beta \log u)})'}{\alpha u + \beta} du \right| &\leq \left| \int_A^B \frac{(\cos 2\pi h(\alpha u + \beta \log u))'}{\alpha u + \beta} du \right| \\ &+ \left| \int_A^B \frac{(\sin 2\pi h(\alpha u + \beta \log u))'}{\alpha u + \beta} du \right| \leq 2^3 \max_{A \leq u \leq B} \frac{1}{|\alpha u + \beta|}. \end{aligned}$$

Assume that  $u_0 = -\beta/\alpha \leq A$ . Then for  $0 < \varepsilon < 1$  we have

$$\begin{aligned} \left| \int_A^B \frac{1}{u} e^{2\pi i h(\alpha u + \beta \log u)} du \right| &\leq \int_A^{(1+\varepsilon)A} \frac{du}{u} + \left| \int_{(1+\varepsilon)A}^{(1+\varepsilon)B} \frac{1}{u} e^{2\pi i h(\alpha u + \beta \log u)} du \right| + \int_B^{(1+\varepsilon)B} \frac{du}{u} \\ &\leq 2 \log(1+\varepsilon) + \frac{4}{\pi h \varepsilon \beta}, \end{aligned}$$

since

$$\max_{(1+\varepsilon)A \leq u \leq (1+\varepsilon)B} \frac{1}{|\alpha u + \beta|} \leq \frac{1}{|\alpha(1+\varepsilon)u_0 + \beta|} = \frac{1}{\varepsilon \beta}.$$

Let us now assume that  $B \leq u_0$ . Then for  $0 < \varepsilon < 1$  we have

$$\begin{aligned} \left| \int_A^B \frac{1}{u} e^{2\pi i h(\alpha u + \beta \log u)} du \right| &\leq \int_{(1-\varepsilon)A}^A \frac{du}{u} + \left| \int_{(1-\varepsilon)A}^{(1-\varepsilon)B} \frac{1}{u} e^{2\pi i h(\alpha u + \beta \log u)} du \right| + \int_{(1+\varepsilon)B}^B \frac{du}{u} \\ &\leq 2 \log \frac{1}{1-\varepsilon} + \frac{4}{\pi h \varepsilon \beta}, \end{aligned}$$

since

$$\max_{(1-\varepsilon)A \leq u \leq (1-\varepsilon)B} \frac{1}{|\alpha u + \beta|} \leq \frac{1}{|\alpha(1+\varepsilon)u_0 + \beta|} = \frac{1}{\varepsilon \beta}.$$

Hence for arbitrary  $A, B, \varepsilon$  with  $0 < A < B$ ,  $0 < \varepsilon < 1$  we obtain

$$\left| \int_A^B \frac{1}{u} e^{2\pi i h(\alpha u + \beta \log u)} du \right| \leq 2 \log \frac{1+\varepsilon}{1-\varepsilon} + \frac{8}{\pi h \varepsilon \beta}.$$

Choosing  $\varepsilon = 1/\sqrt{\pi h \beta}$  for  $h \geq 4/\pi \beta$  we obtain the upper bound  $8(\varepsilon + 1/\pi h \varepsilon \beta) = 16/\sqrt{\pi h \beta}$ , since

$$\log \frac{1+\varepsilon}{1-\varepsilon} = \log \left( 1 + \frac{2\varepsilon}{1-\varepsilon} \right) \leq \frac{2\varepsilon}{1-\varepsilon} \leq 4\varepsilon \quad \text{for } 0 < \varepsilon \leq \frac{1}{2}.$$

For  $1 \leq h < 4/\pi \beta$  we choose  $\varepsilon = 1/2$  and obtain the bound  $8(1/2 + 2/\pi h \beta)$

$< 8(2/\sqrt{\pi h \beta} + 2/\pi \sqrt{h \beta})$ . Thus the first part of the Lemma is proved.

For the second integral let us start again with the case  $\alpha u + \beta \neq 0$  for all  $u$  with  $A \leq u \leq B$ . Then, as above, we obtain

$$\left| \int_A^B \frac{1}{u^2} e^{2\pi i h(\alpha u + \beta \log u)} du \right| \leq \frac{1}{2\pi h} \cdot 2^4 \max_{A \leq u \leq B} \frac{1}{|u(\alpha u + \beta)|},$$

since  $1/u(\alpha u + \beta)$  consists of at most two monotone pieces.

If  $u_0 = -\beta/\alpha \leq A$ , then for  $0 < \varepsilon < 1$  we have

$$\begin{aligned} \left| \int_A^B \frac{1}{u^2} e^{2\pi i h(\alpha u + \beta \log u)} du \right| &\leq \int_A^{(1+\varepsilon)A} \frac{du}{u^2} + \left| \int_{(1+\varepsilon)A}^{(1+\varepsilon)B} \frac{1}{u^2} e^{2\pi i h(\alpha u + \beta \log u)} du \right| + \int_B^{(1+\varepsilon)B} \frac{du}{u^2} \\ &\leq \frac{\varepsilon}{1+\varepsilon} \cdot \frac{1}{A} + \frac{8}{(1+\varepsilon)A\pi h \varepsilon \beta} + \frac{\varepsilon}{1+\varepsilon} \cdot \frac{1}{B} \leq \frac{2}{(1+\varepsilon)A} \left( \varepsilon + \frac{4}{\pi h \varepsilon \beta} \right), \end{aligned}$$

since

$$\max_{(1+\varepsilon)A \leq u \leq (1+\varepsilon)B} \frac{1}{|u(\alpha u + \beta)|} \leq \frac{1}{(1+\varepsilon)A} \cdot \frac{1}{|\alpha(1+\varepsilon)u_0 + \beta|} = \frac{1}{(1+\varepsilon)A\varepsilon\beta}.$$

Similarly, for  $B \leq u_0$  we obtain the bound  $2/(1-\varepsilon)A \cdot (\varepsilon + 4/\pi h \varepsilon \beta)$ . Hence for arbitrary  $A, B, \varepsilon$  with  $0 < A < B, 0 < \varepsilon < 1$  we may take the upper bound  $4/(1-\varepsilon^2)A \cdot (\varepsilon + 4/\pi h \varepsilon \beta)$ . For  $h \geq 8/\pi\beta$  we choose  $\varepsilon = 2/\sqrt{\pi h \beta} \leq 1/\sqrt{2}$  and obtain the upper bound  $8/A \cdot 4/\sqrt{\pi h \beta}$ ; choosing  $\varepsilon = 1/\sqrt{2}$  for  $h < 8/\pi\beta$  gives the bound  $8/A \cdot (1/\sqrt{2} + 4\sqrt{2}/\pi h \beta)$ . Since (for  $1 \leq h < 8/\pi\beta$ )  $1 + \sqrt{2/\pi\beta} > \sqrt{\pi h \beta}/4\sqrt{2} + \sqrt{2/\pi h \beta} = \sqrt{\pi h \beta} (1/\sqrt{2} + 4\sqrt{2}/\pi h \beta)/4$ , this yields the second inequality of the Lemma.

In the proof of the previous lemma we have shown the following estimates

LEMMA 2. *If  $\alpha u + \beta \neq 0$  for  $0 < A \leq u \leq B$  then*

$$\begin{aligned} \left| \int_A^B \frac{1}{u} e^{2\pi i h(\alpha u + \beta \log u)} du \right| &\leq \frac{4}{\pi h} \max_{A \leq u \leq B} \frac{1}{|\alpha u + \beta|} \\ \left| \int_A^B \frac{1}{u^2} e^{2\pi i h(\alpha u + \beta \log u)} du \right| &\leq \frac{8}{A\pi h} \max_{A \leq u \leq B} \frac{1}{|\alpha u + \beta|}. \end{aligned}$$

We continue now with the proof of the theorem. By Euler's summation formula

$$\begin{aligned} \sum_{n=A}^B \frac{1}{n} e^{2\pi i h(\alpha n + \beta \log n)} &= \frac{\theta}{A} + \int_A^B \frac{1}{u} e^{2\pi i h(\alpha u + \beta \log u)} du \\ &\quad + \int_A^B \psi(u) \left( -\frac{1}{u^2} + \frac{1}{u} 2\pi i h \left( \alpha + \frac{\beta}{u} \right) \right) e^{2\pi i h(\alpha u + \beta \log u)} du \end{aligned}$$

for some complex number  $\theta$  with  $|\theta| \leq 1$ . We may assume that  $1 \leq \alpha < 2$  and  $\beta > 0$ , since for  $\beta < 0$  we may take the complex conjugate of the exponential sum and the sum remains un-

changed if we replace  $\alpha$  by  $\alpha + k$  for integral  $k$ . By Lemma 2

$$\left| \int_A^B \frac{1}{u} e^{2\pi i h(\alpha u + \beta \log u)} du \right| \leq \frac{4}{\pi h(\alpha + \beta)}.$$

The trivial estimate  $|\psi(u)| \leq 1/2$  yields

$$\left| \int_A^B \frac{\psi(u)}{u^2} e^{2\pi i h(\alpha u + \beta \log u)} du \right| \leq \frac{1}{2A}.$$

Using the Fourier expansion of  $\psi(u)$  we obtain

$$\begin{aligned} & \int_A^B \psi(u) \frac{2\pi i h}{u} \left( \alpha + \frac{\beta}{u} \right) e^{2\pi i h(\alpha u + \beta \log u)} du \\ &= \int_A^B \sum_{m \neq 0} \frac{2\pi i h}{m} \left( \frac{\alpha}{u} + \frac{\beta}{u^2} \right) e^{2\pi i m u} e^{2\pi i h(\alpha u + \beta \log u)} du \\ &= \sum_{m \neq 0} \frac{2\pi i h}{m} I_m \left( I_m = \int_A^B \left( \frac{\alpha}{u} + \frac{\beta}{u^2} \right) e^{2\pi i h \left( \left( \alpha + \frac{m}{h} \right) u + \beta \log u \right)} du \right). \end{aligned}$$

Put

$$\begin{aligned} M_1 &= \left\{ m \in \mathbf{Z}: m < - \left( 1 + h \left( \alpha + \frac{\beta}{A} \right) \right) \text{ or } m > -h\alpha + 1, m \neq 0 \right\}, \\ M_2 &= \left\{ m \in \mathbf{Z}: - \left( 1 + h \left( \alpha + \frac{\beta}{A} \right) \right) \leq m \leq -h\alpha + 1 \right\}. \end{aligned}$$

For

$$m < - \left( 1 + h \left( \alpha + \frac{\beta}{A} \right) \right) \text{ we have } \alpha + \frac{m}{h} < -\frac{1}{h} - \frac{\beta}{A} < 0$$

and  $(\alpha + m/h)A + \beta < -A/h < 0$ . Hence for these  $m$

$$\max_{A \leq u \leq B} \frac{1}{\left| \left( \alpha + \frac{m}{h} \right) u + \beta \right|} = \frac{1}{\left| \left( \alpha + \frac{m}{h} \right) A + \beta \right|}.$$

For  $m > -h\alpha + 1$  ( $m \neq 0$ ) we have  $\alpha + m/h > 0$  and

$$\max_{A \leq u \leq B} \frac{1}{\left| \left( \alpha + \frac{m}{h} \right) u + \beta \right|} = \frac{1}{\left( \alpha + \frac{m}{h} \right) A + \beta}.$$

Thus Lemma 2 gives

$$|I_m| \leq \left( \frac{4\alpha}{\pi h} + \frac{8\beta}{A\pi h} \right) \frac{1}{\left| \left( \alpha + \frac{m}{h} \right) A + \beta \right|} \leq \frac{4(\alpha + 2\beta)}{A\pi \left| m + h \left( \alpha + \frac{\beta}{A} \right) \right|} \text{ for } m \in M_1.$$

For  $m \in M_2$  we obtain

$$|I_m| \leq \frac{16}{\sqrt{\pi h \beta}} \left(1 + \frac{\sqrt{2}}{\sqrt{\pi \beta}}\right) (\alpha + 2\beta)$$

by Lemma 1. As  $\text{card}(M_2) \leq 3 + h\beta/A$ , we derive

$$\begin{aligned} \left| \sum_{m \neq 0} \frac{2\pi i h}{m} I_m \right| &\leq 2\pi h \left( \sum_{m \in M_1} \frac{4(\alpha + 2\beta)}{\pi A |m + h(\alpha + \frac{\beta}{A})| |m|} + \frac{3 + h\beta/A}{h\alpha/2} \frac{16}{\sqrt{\pi h \beta}} \right) \\ &\quad \times \left(1 + \frac{\sqrt{2}}{\sqrt{\pi \beta}}\right) (\alpha + 2\beta) \\ &= \frac{8(\alpha + 2\beta)}{A} \left( h \sum_{m \in M_1} \frac{1}{|m| |m + h(\alpha + \frac{\beta}{A})|} + \frac{8(3 + h\beta/A)A\pi}{\alpha \sqrt{h\beta\pi}} \left(1 + \frac{2}{\pi\beta}\right) \right). \end{aligned}$$

Put  $\mu = h(\alpha + \beta/A)$ , then

$$\begin{aligned} \sum_{m \in M_1} \frac{1}{|m| |m + \mu|} &= \sum_{m > \mu + 1} \frac{1}{m(m - \mu)} + \sum_{0 < m < h\alpha - 1} \frac{1}{m(\mu - m)} + \sum_{m > 0} \frac{1}{m(m + \mu)} \\ &\leq 2 \sum_{m=1}^{\infty} \frac{1}{m(m + \mu)} + \sum_{0 < m < h\alpha - 1} \frac{1}{m(\mu - m)}. \end{aligned}$$

We have  $\mu > 1$  (since  $h, \alpha \geq 1$ ), and so

$$\sum_{m=1}^{\infty} \frac{1}{m(m + \mu)} \leq \sum_{m=1}^{\infty} \frac{1}{m(m + [\mu])} = \frac{1}{[\mu]} \sum_{m=1}^{[\mu]} \frac{1}{m} \leq 2 \frac{1 + \log \mu}{\mu}.$$

Since

$$\sum_{0 < m < h\alpha - 1} \frac{1}{m(\mu - m)} = \frac{1}{\mu} \sum_{0 < m < h\alpha - 1} \left( \frac{1}{\mu - m} + \frac{1}{m} \right) \leq \frac{2}{\mu} \sum_{0 < m < [\mu]} \frac{1}{m} \leq 2 \frac{1 + \log \mu}{\mu},$$

we obtain

$$\sum_{m \in M_1} \frac{1}{|m| |m + \mu|} \leq 6 \frac{1 + \log \mu}{\mu} \leq 6 \frac{1 + \log(h(\alpha + \beta))}{h\alpha}.$$

Hence

$$\left| \sum_{m \neq 0} \frac{2\pi i h}{m} I_m \right| \leq \frac{8(\alpha + 2\beta)}{A} \left( 6h \frac{1 + \log(h(\alpha + \beta))}{h\alpha} + \frac{8(3 + h\beta/A)A\pi}{\alpha \sqrt{h\beta\pi}} \left(1 + \frac{\sqrt{2}}{\sqrt{\pi\beta}}\right) \right),$$

and so

$$\begin{aligned} \left| \sum_{n=A}^B \frac{1}{n} e^{2\pi i h(\alpha n + \beta \log n)} \right| &\leq \frac{1}{A} + \frac{4}{\pi h(\alpha + \beta)} + \frac{1}{2A} \\ &\quad + \frac{8(\alpha + 2\beta)}{A} \left( 6h \frac{1 + \log(h(\alpha + \beta))}{h\alpha} + \frac{8(3 + h\beta/A)A\pi}{\alpha \sqrt{h\beta\pi}} \left(1 + \frac{\sqrt{2}}{\sqrt{\pi\beta}}\right) \right) \leq C(\beta) \cdot \left( \frac{\sqrt{h}}{A} + \frac{1}{\sqrt{h}} \right), \end{aligned}$$

where  $C(\beta)$  is a constant only depending on  $\beta$ . Thus the proof of the theorem is complete.

COROLLARY. For reals  $\alpha, \beta (\beta \neq 0)$  we have

$$D_N(\alpha n + \beta \log n) \leq C_1(\beta) \frac{(\log \log N)^2}{\log N}.$$

PROOF. We choose  $m = [\log N] + 1$  in the inequality of Erdős-Turan for the logarithmic mean (cf. [4], Th. 1):

$$D_N(x_n) \leq 4 \left( \frac{1}{m} + \sum_{h=1}^m \frac{1}{h} \left| \frac{1}{\sum_{n=1}^N \frac{1}{n}} \sum_{n=1}^N \frac{1}{n} e^{2\pi i h x_n} \right| \right).$$

A simple calculation yields the desired result.

Exponential sums for functions of a similar type as  $\alpha n + \beta \log n$  can be related to the exponential sums considered above by the following theorem.

THEOREM 3.2. Let  $p(n)$  be positive weights and let  $(x_n), (a_n)$  be sequences of real numbers and assume that  $|a_{n+1} - a_n| \leq c/n^{1+\delta}P(n)$  (with positive constants  $c$  and  $\delta$ ), where  $P(n) = \sum_{k=1}^n p(k)$ . Then we have for positive values  $h$

$$\left| \sum_{n=1}^N p(n) e^{2\pi i h(x_n + a_n)} \right| \leq 2P(m) + \left| \sum_{n=1}^N p(n) e^{2\pi i h x_n} \right| + 2\pi c \sum_{n=m+1}^N \frac{1}{n^{1+\delta/2}} \left| \frac{1}{P(n)} \sum_{k=1}^n p(k) e^{2\pi i h x_k} \right|,$$

where  $m = [h^{2/\delta}] + 1$ .

PROOF. Since  $|e^{iu} - e^{iv}| \leq |u - v|$  we have

$$\begin{aligned} \left| \sum_{n=1}^N p(n) e^{2\pi i h(x_n + a_n)} \right| &\leq \sum_{n=1}^m p(n) + \left| \sum_{n=m+1}^N \left( \sum_{k=1}^n p(k) e^{2\pi i h x_k} \right) (e^{2\pi i h a_n} - e^{2\pi i h a_{n+1}}) \right. \\ &\quad \left. - \sum_{k=1}^m p(k) e^{2\pi i h x_k} e^{2\pi i h a_{m+1}} + \sum_{k=1}^N p(k) e^{2\pi i h x_k} e^{2\pi i h a_{N+1}} \right| \\ &\leq 2 \sum_{n=1}^m p(n) + \sum_{n=m+1}^N \left| \sum_{k=1}^n p(k) e^{2\pi i h x_k} \right| 2\pi h |a_n - a_{n+1}| + \left| \sum_{n=1}^N p(n) e^{2\pi i h x_n} \right| \\ &\leq 2 \sum_{n=1}^m p(n) + \sum_{n=m+1}^N \left| \sum_{k=1}^n p(k) e^{2\pi i h x_k} \right| \frac{2\pi c \cdot n^{\delta/2}}{n^{1+\delta}P(n)} + \left| \sum_{n=1}^N p(n) e^{2\pi i h x_n} \right|, \end{aligned}$$

thus proving the theorem.

Assuming  $|a_{n+1} - a_n| \leq c p(n)/P(n)$  instead of  $|a_{n+1} - a_n| \leq c/n^{1+\delta}P(n)$  we obtain (for  $m=0$ )

$$\left| \frac{1}{P(N)} \sum_{n=1}^N p(n) e^{2\pi i h(x_n + a_n)} \right| \leq \frac{2\pi h c}{P(N)} \sum_{n=1}^N p(n) \left| \frac{1}{P(n)} \sum_{k=1}^n p(k) e^{2\pi i h x_k} \right| + \left| \frac{1}{P(N)} \sum_{n=1}^N p(n) e^{2\pi i h x_n} \right|.$$

Hence  $\lim_{N \rightarrow \infty} P(N)^{-1} \sum_{n=1}^N p(n) e^{2\pi i h x_n} = 0$  implies  $\lim_{N \rightarrow \infty} P(N)^{-1} \sum_{n=1}^N p(n) e^{2\pi i h (x_n + a_n)} = 0$  provided that  $\lim_{N \rightarrow \infty} P(N) = \infty$ . Thus Weyl's criterion (cf. [1], p. 55) gives

**THEOREM 3.3.** *Let  $p(n)$  be positive weights with  $\lim_{N \rightarrow \infty} \sum_{n=1}^N p(n) = \infty$ . If  $(x_n)$  is uniformly distributed with respect to the weights  $p(n)$  and if  $|a_{n+1} - a_n| = O(p(n) / \sum_{k=1}^n p(k))$  then  $(x_n + a_n)$  is also uniformly distributed with respect to  $p(n)$ .*

**COROLLARY.** *For  $\omega = (\alpha n + \beta \log n + a_n)$  with reals  $\alpha, \beta$  ( $\beta \neq 0$ ) and  $|a_{n+1} - a_n| \leq c/n^{1+\delta}$  ( $c, \delta$  positive constants) we have*

$$D_N(\omega) \leq C(\beta, c, \delta) \frac{(\log \log N)^2}{\log N}$$

where  $D_N$  denotes the discrepancy with respect to the logarithmic mean. ( $C(\beta, c, \delta)$  may be chosen to depend continuously on  $\beta$ .)

**PROOF.** From Theorem 3.1. and Theorem 3.2. we easily deduce (for  $x_n = \alpha n + \beta \log n$ )

$$\left| \sum_{n=1}^N \frac{1}{n} e^{2\pi i h (x_n + a_n)} \right| \leq c_1(\beta, c, \delta)(\log h + 1).$$

As in the proof of the Corollary after Theorem 3.1., the assertion follows from the inequality of Erdős and Turan.

**REMARK 1.** In the special case of the arithmetic mean  $p(n) = 1$  the result of Theorem 3.3. can be found in [6].

**REMARK 2.** From Theorem 2.3. and the corollary after Theorem 3.1. it follows that (for  $N \geq 2$ )

$$\sum_{j=0}^{[f^{-1}(N)]} \frac{\{-f^{-1}(j+x)\} - \{-f^{-1}(j)\}}{f^{-1}(j)} \leq C(\beta)(\log \log N)^2,$$

where  $f^{-1}$  denotes the inverse function of  $\alpha x + \beta \log x - \alpha$  ( $1 \leq \alpha < 2, \beta > 0$ ) and  $C(\beta)$  is a constant only depending on  $\beta$ . In the following we give an upper bound for  $D_N(\omega^*)$ , where  $\omega^* = (f^{-1}(n))_{n=1}^\infty$ . We define  $\varepsilon(y)$  by

$$x = \frac{y}{\alpha} - \frac{\beta}{\alpha} \log \frac{y}{\alpha} + \varepsilon(y), \quad \alpha x + \beta \log x = y + \alpha.$$

Since  $dy/dx = \alpha + \beta/x$ , we obtain

$$\varepsilon'(y) = \frac{x}{\alpha x + \beta} - \frac{1}{\alpha} + \frac{\beta}{\alpha} \frac{1}{y} = \frac{\beta}{\alpha} \frac{\alpha x + \beta - y}{y(\alpha x + \beta)} = \frac{\beta}{\alpha} \frac{\beta(1 - \log x) + \alpha}{y(\alpha x + \beta)};$$

thus

$$|\varepsilon'(y)| \leq \frac{\beta}{\alpha} \frac{\beta \log x + \beta + \alpha}{\alpha x \cdot y} \leq \left(\frac{\beta}{\alpha}\right)^2 \frac{\log(y/\alpha + 1) + 1 + \alpha/\beta}{(y + \alpha)y} (\alpha + \beta) \\ \leq \beta^2(1 + \beta) \left(1 + \frac{1}{\beta}\right) \frac{\log(y + 1) + 1}{y^2}$$

(where we have used the trivial estimates  $\alpha x \leq y + \alpha < (\alpha + \beta)x$ ). By the mean value theorem we have

$$|\varepsilon(n + 1) - \varepsilon(n)| \leq \beta(1 + \beta)^2 \frac{\log(n + 2) + 1}{n^2} \leq \frac{3\beta(1 + \beta)^2}{n^{3/2}}.$$

Applying the last corollary to the sequence  $\omega^* = (n/\alpha - \beta/\alpha \cdot \log n + \beta/\alpha \cdot \log \alpha + \varepsilon(n))$  yields

$$D_N(\omega^*) \leq \max_{1 \leq \alpha \leq 2} C \left( -\frac{\beta}{\alpha}, 3\beta(1 + \beta)^2, \frac{1}{2} \right) \frac{(\log \log N)^2}{\log N} = c_0(\beta) \frac{(\log \log N)^2}{\log N}.$$

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