

## HYPERSURFACES WITH HARMONIC CURVATURE

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### Introduction.

A Riemannian curvature tensor is said to be *harmonic* if it satisfies

$$R_{ijk} = R_{ikj},$$

where  $R_{ijk}$  denotes the covariant derivative of Ricci tensor  $R_{ij}$ . This condition is essentially weaker than that for the parallel Ricci tensor. In fact Derdziński [2] gave an example of a 4-dimensional Riemannian manifold with harmonic curvature whose Ricci tensor is not parallel.

Recently E. Ômachi [5] investigated compact hypersurfaces with harmonic curvature in a Euclidean space or a sphere and gave a classification of such hypersurfaces provided that the mean curvature is constant.

This paper is concerned with hypersurfaces with harmonic curvature isometrically immersed into a Riemannian manifold of constant curvature. In the first section, a concept of Codazzi type for a symmetric  $(0, 2)$ -tensor is introduced and a sufficient condition for a symmetric tensor of Codazzi type to be parallel is given. A similar condition for a symmetric tensor of Codazzi type is also treated by S. Y. Cheng and S. T. Yau [1]. In the second section, the result proved in the first section is applied to hypersurfaces with harmonic curvature immersed in a Riemannian manifold of constant curvature, in which Ômachi's result [5] is generalized without the assumption of compactness. Finally we study also the case where the assumption that the mean curvature is constant is omitted.

### § 1. Symmetric tensor of Codazzi type.

Let  $M$  be an  $n$ -dimensional Riemannian manifold and let  $\{e_1, \dots, e_n\}$  be a local orthonormal frame field defined on  $M$ , and  $\{\omega_1, \dots, \omega_n\}$  denotes its dual field. Here and in the sequel, indices  $i, j, \dots$  run over the range  $\{1, 2, \dots, n\}$  unless otherwise stated. Then the structure equation of  $M$  are given by

$$\begin{aligned}d\omega_i + \sum_j \omega_{ij} \wedge \omega_j &= 0, \quad \omega_{ij} + \omega_{ji} = 0, \\d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} &= \Omega_{ij},\end{aligned}$$

where

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$$\Omega_{ij} = (1/2) \Sigma_{k,l} R_{ijkl} \omega_k \wedge \omega_l.$$

Let  $\Sigma_{i,j} \phi_{ij} \omega_j \otimes \omega_j$  be a symmetric  $(0, 2)$ -tensor field on  $M$ . Then the covariant derivative  $\phi_{ijk}$  of  $\phi_{ij}$  is denifined by

$$\Sigma_k \phi_{ijk} \omega_k = d\phi_{ij} - \Sigma_k \phi_{kj} \omega_{ki} - \Sigma_k \phi_{ik} \omega_{kj}.$$

$\phi_{ij}$  is said to be of *Codazzi type* if it satisfies the so-called Codazzi equation

$$\phi_{ijk} = \phi_{ikj}.$$

For a symmetric tensor  $\phi_{ij}$ , symmetric tensor  $\phi_{ij}^m$  ( $m=1, 2, 3, \dots$ ) are definid inductively as follows:

$$\begin{aligned} \phi_{ij}^1 &= \phi_{ij}, \\ \phi_{ij}^{m+1} &= \Sigma_k \phi_{ik} \phi_{kj}^m. \end{aligned}$$

Let  $\text{tr } \phi = \Sigma_i \phi_{ii}$  and  $\text{tr } \phi^m = \Sigma_i \phi_{ii}^m$ .

Now we shall give a sufficient condition for a symmetric tensor of Codazzi type to be parallel. First of all, the following fact is easily proved.

LEMMA 1. 1. *Let  $\phi, \phi^2, \dots, \phi^r$  be a symmetric  $(0, 2)$ -tensor of Codazzi type. Then  $\phi^{r+1}$  is also of Codazzi type if and only if*

$$(1. 1) \quad \Sigma_l \phi_{ilk} \phi_{lj}^r - \Sigma_l \phi_{ulj} \phi_{lk}^r = 0.$$

For a symmetric  $(0, 2)$ -tensor  $\phi$  of Codazzi type, we define a subset  $M_\phi$  of  $M$  consisting of points  $p$  so that there exists a neighborhood  $U_p$  of  $p$  such that the multiplicity of each principal carvature is constant on  $U_p$ . The  $M_\phi$  is an open and dense subset of  $M$ . In each connected component of  $M_\phi$ , the distinct eigenvalues of  $\phi$  are considered as smooth functions. Let  $\lambda$  be one of such eigenfunctions, and the eigendistribution which is denoted by  $\Delta_\lambda$  is the set of all eigenvectors corresponding to  $\lambda$ . Derdziński [3] showed that the eigendistributions of  $\phi$  are all involutive. We shall give a necessary and sufficient condition of the eigen distributions of  $\phi$  to be parallel.

LEMMA 1. 2. *Let  $\phi$  be a symmetric  $(0, 2)$ -tensor of Codazzi type. Then  $\phi^2$  is also of Codazzi type if and only if the eigendistributions of  $\phi$  are all parallel.*

PROOF. Let  $\{e_1, \dots, e_n\}$  be a local orthonormal frame field consisting of the eigenvector field of  $\phi$ , and  $\lambda_i$  denotes the eigenfunction corresponding to  $e_i$ . Since  $\phi$  is of Codazzi type, Lemma 1. 1 implies that  $\phi^2$  is also of Codazzi type if and only if

$$\Sigma_l \phi_{ilk} \phi_{lj} - \Sigma_l \phi_{ulj} \phi_{lk} = 0,$$

that is

$$(1. 2) \quad (\lambda_j - \lambda_k) \phi_{jki} = 0.$$

If  $\lambda_j \neq \lambda_k$  then  $d\phi_{jk} = 0$ . By the definition of covariant derivative, (1. 2) is equivalent to the equation

$$(\lambda_j - \lambda_k)(\Sigma_i \phi_{ik} \omega_{ij} + \Sigma_i \phi_{ji} \omega_{ik}) = 0,$$

that is

$$(1. 3) \quad (\lambda_j - \lambda_k)^2 \omega_{jk} = 0,$$

which implies that each eigendistribution is parallel. This proves Lemma 1. 2.

Under this preparation we shall now prove the following.

**THEOREM 1. 3.** *Let  $M$  be an  $n$ -dimensional Riemannian manifold and  $\phi$  a symmetric  $(0, 2)$ -tensor defined on  $M$ . If  $\phi$  and  $\phi^2$  are both of Codazzi type then the following assertions are true:*

- (1)  $\phi^r$  ( $r=1, 2, 3, \dots$ ) are all of Codazzi type.
- (2) Let  $\{e_1, \dots, e_n\}$  be a frame which diagonalizes the tensor  $\phi$  so that  $\phi_{ij} = \lambda_i \delta_{ij}$ . If  $\lambda_i \neq \lambda_j$ , then  $R_{ijij} = 0$ .
- (3) In addition, if  $\text{tr } \phi$  is constant, then  $\text{tr } \phi^r$  ( $r=1, 2, 3, \dots$ ) are all constant and  $\phi$  is parallel.

**PROOF.** By taking the frame  $\{e_1, \dots, e_n\}$ , (1. 1) is simplified to

$$(\lambda_j^r - \lambda_k^r) \phi_{ijk} = 0.$$

This can be written as

$$(1. 4) \quad (\Sigma_{s=0}^{r-1} \lambda_j^s \lambda_k^{r-1-s}) (\lambda_j - \lambda_k) \phi_{ijk} = 0.$$

Since  $\phi$  and  $\phi^2$  are both of Codazzi type, using (1. 1) we have

$$(1. 5) \quad (\lambda_j - \lambda_k) \phi_{ijk} = 0.$$

From (1. 4) and (1. 5), the first assertion follows immediately. In the next place, the assertion (2) is considered. By Lemma 1. 2, eigendistributions of  $\phi$  are mutually orthogonal and parallel. Hence  $p \in M_\phi$  has a Riemannian product neighborhood  $U_1 \times \dots \times U_l$  where the tangent space of each  $U_i$  is spanned by eigenvectors of  $\phi$  with the same eigenvalue. If  $\lambda_j \neq \lambda_k$  then  $e_i$  and  $e_j$  belong to the distinct eigendistributions, hence  $R_{ijij} = 0$ . Since  $M_\phi$  is dense, the assertion (2) holds at every point in  $M$ . We now prove the assertion (3). Since  $\phi^k$  ( $k=1, 2, 3, \dots$ ) are all of Codazzi type, we can use (1. 1). Contracting (1. 1) with respect to  $j$  and  $i$  we have

$$\{1/(r+1)\}(\operatorname{tr} \phi^{r+1})_k - \Sigma_l (\operatorname{tr} \phi)_l \phi_{lk}^r = 0.$$

Suppose that  $\operatorname{tr} \phi$  is constant, then

$$(\operatorname{tr} \phi^{r+1})_k = 0 \quad (k=1, 2, \dots, n).$$

Hence  $\operatorname{tr} \phi^r$  is constant on  $M$ . Next we prove that  $\phi$  is parallel. Since  $\phi$  is of Codazzi type, the well-known Bochner formula is reduced to the following relation

$$(1.6) \quad (1/2) \Delta (\operatorname{tr} \phi^2) = \Sigma_{i,j,k} (\phi_{ijk})^2 + \Sigma_i \lambda_i (\operatorname{tr} \phi)_{ii} + (1/2) \Sigma_{i,j} R_{ijij} (\lambda_i - \lambda_j)^2,$$

where  $\Delta$  denotes the Laplace operator (cf. [1]). Since  $\operatorname{tr} \phi$  and  $\operatorname{tr} \phi^2$  are constant and  $\Sigma_{i,j} R_{ijij} (\lambda_i - \lambda_j)^2$  is equal to zero, (1.6) implies that  $\phi$  is parallel.

To show that  $\phi$  is parallel, we assume that  $\operatorname{tr} \phi$  is constant. If  $M$  has positive or negative sectional curvature, then this condition can be omitted.

**COROLLARY 1.4.** *Let  $M$  be a connected Riemannian manifold with positive or negative sectional curvature. If  $\phi$  and  $\phi^2$  are both of Codazzi type, then  $\phi$  coincides with the Riemannian metric on  $M$  up to scalar multiple.*

**PROOF.** From (2) of Theorem 1.3, all the eigenvalues of  $\phi$  are the same, that is  $\lambda_1 = \dots = \lambda_n$  at every point. So there exists a function  $f$  defined on  $M$  such that  $\phi_{ij} = f \delta_{ij}$ . Since  $\phi$  is of Codazzi type, it is easy to verify  $f$  is a constant function.

## §2. Hypersurfaces with harmonic curvature.

This section is devoted to the study of hypersurfaces with harmonic curvature immersed into a Riemannian manifold of constant curvature.

Let  $M$  be an  $n$ -dimensional Riemannian manifold with harmonic curvature isometrically immersed into a Riemannian manifold of constant curvature  $c$ . Then the second fundamental form  $h$  is a symmetric  $(0, 2)$ -tensor of Codazzi type. Let  $\{e_1, \dots, e_n\}$  be a frame which diagonalizes the second fundamental form  $h$  so that  $h_{ij} = \lambda_i \delta_{ij}$ . Then the Gauss equation says

$$R_{ijkl} = c(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + h_{ik} h_{jl} - h_{il} h_{kj}.$$

We have

$$(2.1) \quad R_{ij} = c(n-1) \delta_{ij} + h_{ij} \operatorname{tr} h - \Sigma_l h_{il} h_{lj},$$

where  $R_{ij}$  denotes the Ricci tensor of  $M$ . Hence the covariant derivative  $R_{ijk}$  of  $R_{ij}$  satisfies

$$(2.2) \quad R_{ijk} = h_{ijk} \operatorname{tr} h + h_{ij} (\operatorname{tr} h)_k \\ - \Sigma_l h_{ilk} h_{lj} - \Sigma_l h_{il} h_{ljk}.$$

Subtracting the equation which exchanges the index  $k$  with  $j$  in (2.2), since  $M$  has harmonic curvature we have

$$(2.3) \quad h_{ik} (\operatorname{tr} h)_k - h_{ik} (\operatorname{tr} h)_j = \Sigma_l h_{ilk} h_{lj} - \Sigma_l h_{ilj} h_{lk}.$$

The following theorem is an extension of Ômach's result [5].

**THEOREM 2.1.** *Let  $M$  be a hypersurface with harmonic curvature isometrically immersed into a Riemannian manifold of constant curvature  $c$ . If the mean curvature is constant, then the principal curvatures are all constant and the number of distinct principal curvatures is less than or equal to 2. Moreover if the ambient space is simply connected and  $M$  ( $\dim M \geq 3$ ) is connected and complete, then  $M$  is totally umbilical or a Riemannian product of two totally umbilical constantly curved submanifolds.*

**PROOF.** Since the mean curvature is constant, from (2.3) we have

$$(2.4) \quad \Sigma_l h_{ilk} h_{lj} - \Sigma_l h_{ilj} h_{lk} = 0.$$

This implies that  $h$  and  $h^2$  are both of Codazzi type. Hence (3) of Theorem 1.3 implies that the principal curvatures are all constant on  $M$ . On the other hand, (2) of Theorem 1.3 implies that  $\lambda_i = \lambda_j$  or  $R_{ijij} = 0$ . By the Gauss equation  $R_{ijij} = c + \lambda_i \lambda_j$ , we have

$$(2.5) \quad (c + \lambda_i \lambda_j) (\lambda_i - \lambda_j)^2 = 0.$$

It is a simple algebraic fact that (2.5) implies  $M$  has at most two distinct principal curvatures. Now we suppose that the ambient space is simply connected and  $M$  is connected and complete. If the principal curvatures are all the same, then  $M$  is totally umbilical. If  $M$  has two distinct principal curvatures, using the argument of K. Nomizu and B. Smith [4] and the rigidity of such an immersion, we concluded that  $M$  is a Riemannian product of two totally umbilical constantly curved submanifolds. This proves Theorem 2.1.

Using Theorem 2.1, we obtain the following result.

**THEOREM 2.2.** *Let  $M$  be a connected hypersurface with harmonic curvature isometrically immersed into a Riemannian manifold of constant curvature. If the multiplicity of the each principal curvature is everywhere greater than or equal to 2, then  $M$  satisfies one of the following conditions.*

- (1) *The second fundamental form of  $M$  is degenerate everywhere.*

- (2) *The principal curvatures are all constant and the number of distinct principal curvatures is less than or equal to 2.*

In order to prove this theorem, two lemmas are first of all prepared.

LEMMA 2.3. *In the assumption of Theorem 2.2, if the second fundamental form  $h$  is nondegenerate at the point  $p$ , then  $d(\operatorname{tr} h)=0$  at  $p$ .*

PROOF. Since the second fundamental form  $h$  is nondegenerate at  $p$ , all the principal curvatures are not equal to zero. From (2.3), we have

$$(2.6) \quad \lambda_i \delta_{ij}(\operatorname{tr} h)_k - \lambda_k \delta_{ik}(\operatorname{tr} h)_j = (\lambda_j - \lambda_k) h_{jki}.$$

Since the principal curvatures are nonsimple, for a fixed index  $k$ , there exists an index  $j$  such that  $j \neq k$  and  $\lambda_j = \lambda_k$ . In (2.6) putting  $i=j$  then

$$\lambda_j(\operatorname{tr} h)_k = (\lambda_j - \lambda_k) h_{jkj}.$$

Since  $\lambda_j = \lambda_k$  and  $\lambda_j \neq 0$ , we have  $(\operatorname{tr} h)_k = 0$ . This implies  $d(\operatorname{tr} h)=0$  at  $p$ .

LEMMA 2.4. *In the assumption of Theorem 2.2, if the second fundamental form is degenerate at one point, then it is degenerate everywhere.*

PROOF. Suppose that there exist two points  $p$  and  $q$  on  $M$  so that the second fundamental form is nondegenerate at  $p$  and degenerate at  $q$ , and consider a curve  $\tau = x_t$  ( $0 \leq t \leq 1$ ) such that  $x_0 = p$  and  $x_1 = q$ . Putting

$$\delta = \inf_{t \in [0,1]} \{\det A_{x_t} = 0\},$$

where  $A_{x_t}$  is the shape operator at  $x_t$ , then by the continuity of the shape operator, we see that

$$(2.7) \quad \det A_{x_\delta} = 0.$$

On the other hand, for all  $s$  ( $0 \leq s \leq \delta$ ), there exist an open subset  $U_s$  such that  $U_s \supset \{x_t : 0 \leq t \leq p\}$  and  $\det A_y \neq 0$  for all  $y \in U_s$ . Since the second fundamental form is nondegenerate on  $U_s$ , from the Lemma 2.3, the mean curvature is constant on  $U_s$ . Applying Theorem 2.1, we see that the principal curvatures are all constant on  $U_s$ . Hence

$$\det A_{x_s} = \det A_p \quad (0 \leq s \leq \delta),$$

so we have

$$(2.8) \quad \det A_{x_\delta} = \lim_{s \rightarrow \delta \rightarrow 0} \det A_{x_s} = \det A_p \neq 0.$$

From (2.7) and (2.8), we can make a contradiction. This proves Lemma 2.4.

PROOF OF THEOREM 2. 2. If the second fundamental form is nondegenerate everywhere, then by Lemma 2. 3, the mean curvature is constant on  $M$ . Applying Theorem 2. 1, we see that  $M$  satisfies the second condition of Theorem 2. 2. If the second fundamental form is degenerate at some point, then by Lemma 2. 4, the second fundamental form is degenerate everywhere. This proves Theorem 2. 2.

Finally we study hypersurfaces with harmonic curvature assuming no other conditions. We obtain the following result.

THEOREM 2. 5. *Let  $M$  be a hypersurface with harmonic curvature isometrically immersed into a Riemannian manifold of constant curvature  $c$  and  $p \in M$  be a critical point of the mean curvature  $H$ .*

- (1) *If  $c=0$ , then the number of distinct principal curvatures does not exceed 4 at  $p$ .*
- (2) *If  $c \neq 0$ , then the number of distinct principal curvatures does not exceed 3 at  $p$ .*

PROOF. Let  $p$  be a critical point of  $H$ . The covariant derivative  $h_{ijkl}$  of  $h_{ijk}$  is defined by

$$\Sigma_l h_{ijkl} \omega_l = dh_{ijk} - \Sigma_l h_{ijk} \omega_{li} - \Sigma_l h_{ilk} \omega_{lj} - \Sigma_l h_{ijl} \omega_{lk}.$$

From (2. 3), we have

$$(2. 7) \quad \begin{aligned} h_{ijm}(\text{tr } h)_k + h_{ij}(\text{tr } h)_{km} - h_{ikm}(\text{tr } h)_j - h_{ik}(\text{tr } h)_{jm} \\ = \Sigma_l h_{likm} h_{lj} + \Sigma_l h_{ilk} h_{ljm} - \Sigma_l h_{iljm} h_{lk} - \Sigma_l h_{ilj} h_{lkm}. \end{aligned}$$

Subtracting the equation which exchanges the index  $m$  with  $i$  in (2. 7), since  $h$  is of Codazzi type, we have

$$\begin{aligned} h_{ij}(\text{tr } h)_{km} - h_{mj}(\text{tr } h)_{ki} - h_{ik}(\text{tr } h)_{jm} + h_{mk}(\text{tr } h)_{ji} \\ = \Sigma_l (h_{likm} - h_{lkm}) h_{lj} - \Sigma_l (h_{ljim} - h_{ljmi}) h_{lk} + 2 \Sigma_l h_{ilk} h_{ljm} - 2 \Sigma_l h_{ilj} h_{lkm}. \end{aligned}$$

Applying the Ricci formula we obtain

$$\begin{aligned} h_{ij}(\text{tr } h)_{km} - h_{mj}(\text{tr } h)_{ki} - h_{ik}(\text{tr } h)_{jm} + h_{mk}(\text{tr } h)_{ji} \\ = \Sigma_{l,s} R_{skim} h_{sl} h_{lj} + 2 \Sigma_{l,s} R_{slim} h_{sk} h_{lj} - \Sigma_{l,s} R_{sjim} h_{sl} h_{lk} + 2 \Sigma_l h_{ilk} h_{ljm} - 2 \Sigma_l h_{ilj} h_{lkm}. \end{aligned}$$

It simplifies to

$$(2. 8) \quad \begin{aligned} \lambda_j \delta_{ij}(\text{tr } h)_{km} - \lambda_m \delta_{mj}(\text{tr } h)_{ki} - \lambda_i \delta_{ik}(\text{tr } h)_{jm} + \lambda_m \delta_{mk}(\text{tr } h)_{ji} \\ = (\lambda_j - \lambda_k)^2 R_{jkim} + 2 \Sigma_l h_{ilk} h_{ljm} - 2 \Sigma_l h_{ilj} h_{lkm}. \end{aligned}$$

On putting  $k=i$  and  $m=j$  ( $i \neq j$ ) we have

$$(2.9) \quad -\lambda_j(\operatorname{tr} h)_{ii} - \lambda_i(\operatorname{tr} h)_{jj} = (\lambda_j - \lambda_i)^2 R_{jii} + 2\Sigma_i h_{iil} h_{jil} - 2\Sigma(h_{ilj})^2.$$

Since  $p$  is a critical point of  $\operatorname{tr} h$ , (2.3) implies that

$$(2.10) \quad (\lambda_j - \lambda_k) h_{jki} = 0.$$

We now suppose that  $\lambda_i \neq \lambda_j$ . From (2.10) it follows that  $h_{ijl} = 0$  ( $l=1, 3, \dots, n$ ). Hence  $\Sigma_i h_{iil} h_{jil}$  and  $\Sigma_i (h_{ijl})^2$  are equal to zero, so we obtain

$$(2.11) \quad \lambda_j(\operatorname{tr} h)_{ii} + \lambda_i(\operatorname{tr} h)_{jj} = (\lambda_i - \lambda_j)^2 R_{ijij}.$$

Denoting that

$$\begin{aligned} x_k &= (\operatorname{tr} h)_{kk}, \\ c_{ij} &= (\lambda_i - \lambda_j)^2 R_{ijij} = (\lambda_i - \lambda_j)^2 (c + \lambda_i \lambda_j), \end{aligned}$$

then (2.11) simplifies to

$$(2.12) \quad \lambda_j x_i + \lambda_i x_j = c_{ij}.$$

Now we assume that there exist nonzero distinct principal curvatures  $\lambda_1, \lambda_2, \lambda_3$  and  $\lambda_4$ , then we have

$$\begin{aligned} \lambda_1 x_2 + \lambda_2 x_1 &= c_{12}, \\ \lambda_1 x_4 + \lambda_4 x_1 &= c_{14}, \\ \lambda_2 x_4 + \lambda_4 x_2 &= c_{24}, \end{aligned}$$

from which it follows that

$$(2.13) \quad x_4 = (1/2 \lambda_1 \lambda_2) (\lambda_1 e_{24} + \lambda_2 c_{14} - \lambda_4 c_{12}).$$

On the other hand, we also have

$$(2.14) \quad x_4 = (1/2 \lambda_1 \lambda_3) (\lambda_1 c_{34} + \lambda_3 c_{14} - \lambda_4 c_{13}).$$

Combining (2.13) together with (2.14) we obtain

$$\lambda_1 \lambda_2 c_{34} + \lambda_3 \lambda_4 c_{12} = \lambda_3 \lambda_1 c_{24} + \lambda_2 \lambda_4 c_{31}.$$

Because of  $c_{ij} = (\lambda_i - \lambda_j)^2 (c + \lambda_i \lambda_j)$ , it is reduced to

$$\begin{aligned} \lambda_1 \lambda_2 (\lambda_3 - \lambda_4)^2 c + \lambda_3 \lambda_4 (\lambda_1 - \lambda_2)^2 c - \lambda_1 \lambda_3 (\lambda_2 - \lambda_4)^2 c - \lambda_2 \lambda_4 (\lambda_3 - \lambda_1)^2 c \\ = \lambda_1 \lambda_2 \lambda_4 \lambda_3 \{ (\lambda_2 - \lambda_4)^2 + (\lambda_3 - \lambda_1)^2 - (\lambda_3 - \lambda_4)^2 - (\lambda_1 - \lambda_2)^2 \}, \end{aligned}$$

which is rewritten as

$$c(\lambda_1 - \lambda_4)(\lambda_3 - \lambda_2)(\lambda_2 \lambda_3 + \lambda_1 \lambda_4) = 2 \lambda_1 \lambda_3 \lambda_3 \lambda_4 (\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4).$$

First of all we consider the case  $c=0$ , in which

$$\lambda_1 \lambda_2 \lambda_2 \lambda_4 (\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4) = 0.$$



Since  $\lambda_1, \lambda_2, \lambda_3$  and  $\lambda_4$  are all the nonzero distinct principal curvatures, it is impossible. Hence the number of nonzero distinct principal curvatures is less than 4. So the number of distinct principal curvatures is at most 4.

Next we consider the case  $c \neq 0$ . In this case we have

$$(\lambda_1 - \lambda_4)(\lambda_3 - \lambda_2)(c\lambda_2\lambda_3 + c\lambda_1\lambda_4 - 2\lambda_1\lambda_2\lambda_3\lambda_4) = 0,$$

hence

$$(2.15) \quad \lambda_2\lambda_3 + \lambda_1\lambda_4 = (2/c)\lambda_1\lambda_2\lambda_3\lambda_4.$$

Similarly we have

$$(2.16) \quad \lambda_1\lambda_3 + \lambda_2\lambda_4 = (2/c)\lambda_1\lambda_2\lambda_3\lambda_4.$$

From (2.15) and (2.16) we see that

$$(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_4) = 0,$$

which leads to a contradiction, hence the number of nonzero distinct principal curvatures are less than 4. But if  $M$  has three distinct principal curvatures  $\lambda_1, \lambda_2$  and  $\lambda_3$  at  $p$  and one of them, say  $\lambda_1$  is equal to zero, then (2.12) implies that

$$\begin{aligned} \lambda_2 x_1 &= c_{12} = c\lambda_2^2, \\ \lambda_3 x_1 &= c_{13} = c\lambda_2^2. \end{aligned}$$

Hence  $\lambda_2 = \lambda_3$  or  $\lambda_2\lambda_3 = 0$ , this makes a contradiction. Therefore the number of distinct principal curvatures at  $p$  is at most 3.

**COROLLARY 2.6.** *Let  $M$  be a compact hypersurface with harmonic curvature isometrically immersed into a Riemannian manifold of constant curvature  $c$ .*

- (1) *If  $c = 0$ , then there exists a point  $p \in M$  such that the number of distinct principal curvatures is at most 4.*
- (2) *If  $c \neq 0$ , then there exists a point  $p \in M$  such that the number of distinct principal curvatures is at most 3.*

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