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COHOMOLOGIES OF HOMOGENEOUS ENDOMORPHISM BUNDLES OVER LOW DIMENSIONAL KÄHLER C-SPACES

By

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1. Introduction

In this paper, we determine the infinitesimal deformations of an Einstein-Hermitian structure of a homogeneous vector bundle in several cases. In particular, we get the tangent space at the homogeneous structure of the moduli space of Einstein-Hermitian structures as the representation space of a compact Lie group.

A compact simply connected homogeneous Kähler manifold is called a *Kähler C-space*. Such a manifold can be written as G/K where G is a compact semisimple Lie group and K is the centralizer of a toral subgroup of G ([10]). Let G^C be the complexification of G and K^C the complexification of K in G^C. We denote by L the parabolic subgroup of G^C which contains K^C. G/K is diffeomorphic to G^C/L. Thus G/K admits a holomorphic structure from the holomorphic structure of G^C/L. Moreover it admits a G-invariant Kähler metric.

Let (ρ, V) be a complex representation of K. Then (ρ, V) can be extended to a holomorphic representation (ρ_L, V) of L. The homogeneous vector bundle $E_\rho = G\times_\rho V$ is isomorphic to the homogeneous holomorphic vector bundle $E_{\rho_L} = G^C\times_{\rho_L} V$ as C^∞-vector bundles. Thus the homogeneous vector bundle E_ρ has a natural holomorphic structure from the holomorphic structure of E_{ρ_L} ([3]). Moreover if (ρ, V) is irreducible, then E_ρ has a unique G-invariant Einstein-Hermitian structure up to a homothety ([8]).

An irreducible complex representation (ρ, V) is determined by the highest weight. Then a homogeneous vector bundle E_ρ is determined by the highest weight of (ρ, V), if (ρ, V) is irreducible. It is natural to ask how we describe the deformations of the holomorphic structure and the Einstein-Hermitian structure by the highest weight. Also we ask how we describe moduli spaces of holomorphic structures and Einstein-Hermitian structures by the highest weight.

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In the deformation theory of complex structures of complex manifolds, the complex structure of a Kähler C-space is locally stable ([3]). But the holomorphic structure of a homogeneous vector bundle generally is not locally stable. Here the local stability means that any deformation space is trivial. So there is a problem to find sufficient conditions for the local stability of the holomorphic structure of a homogeneous vector bundle.

Let $\mathrm{End}(E_\rho)$ be the endomorphism bundle of E_ρ and $\mathfrak{sl}(E_\rho)$ the subbundle of $\mathrm{End}(E_\rho)$ which consists of trace free endomorphisms. It is well known that the Dolbeault cohomology group $H^{0,1}(G/K, \mathrm{End}(E_\rho))$ is the tangent space of the moduli space of holomorphic structures if $H^{0,0}(G/K, \mathfrak{sl}(E_\rho))=H^{0,2}(G/K, \mathfrak{sl}(E_\rho))=\{0\}$. The moduli space of Einstein-Hermitian structures is an open subset of the moduli space of holomorphic structures. Under the same conditions, $H^{0,1}(G/K, \mathrm{End}(E_\rho))$ is also the tangent space of the moduli space of Einstein-Hermitian structures ([7], [6] and [9]). So we think that it is important to compute these cohomologies for our problems.

In this paper, for a first step of problems above we investigate Kähler C-spaces G/K with $\mathrm{rank}\, G=2$. In § 2 we shall review a construction of Kähler C-spaces and some properties of vector bundles over them. We shall state our results in § 3 and prove them in § 4. Our main results are Theorems 4, 5 and Corollary 6. In the case of $\mathrm{rank}\, G=2$, we compute $H^{0,p}(G/K, \mathrm{End}(E_\rho))$ and $H^{0,p}(G/K, \mathfrak{sl}(E_\rho))$ from the highest weight of (ρ, V) (Theorems 4 and 5). Then we get dimension of the moduli space of Einstein-Hermitian structures of a homogeneous vector bundle in several cases (Corollary 6). The following theorem is an immediate consequence of these results.

THEOREM 1. *Let G/K be a Kähler C-space where G is of type A_2 or B_2. Let (E, h) be an irreducible Einstein-Hermitian homogeneous vector bundle over G/K with* $\mathrm{rank}\, E=r$. *Then the dimension of the moduli space of irreducible Einstein-Hermitian structures of E is as follows:*

(1) *If G is of type A_2 or B_2 and if K is a maximal torus, then the dimension of the moduli space is* 0.

(2) *If $G/K\cong SU(3)/S(U(1)\times U(2))\cong P_2\boldsymbol{C}$, then the dimension of the moduli spoce is*

$$\frac{1}{2}\sum_{k=1}^{r-1}(2k+1)(k+2)(k-1).$$

(3) *If $G/K\cong Sp(2)/(U(1)\times Sp(1))\cong P_3\boldsymbol{C}$, then the dimension of the moduli space is*

$$\frac{1}{3}\sum_{k=1}^{r-1}(2k+3)(2k+1)(2k-1).$$

(4) *If* $G/K \cong SO(5)/(U(1)\times SO(3)) \cong Q_3(C)$, *then the dimension of the moduli space is*

$$\frac{1}{2}\sum_{k=1}^{r-1}(2k-1)(k+2)(k-1).$$

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2. Preliminaries

In this section, we review a construction of Kähler C-spaces and some properties of homogeneous bundles over them. We refer the reader to the books [1] and [5] for the representation theory of compact Lie group.

Following Wang ([10]), we call a compact simply connected homogeneous Kähler manifold a *Kähler C-space.* A vector bundle E over a homogeneous space G/K is said to be *homogeneous* if it is associated to the principal K-bundle $G\to G/K$.

Let G be a compact simply connected semisimple Lie group. Let T_0 be a toral subgroup of G and K the centralizer of T_0 in G. Then G/K is a compact simply connected homogeneous manifold. Let T be a maximal torus of G which contains T_0. Then T is contained in K and we put $l=\dim T$. We denote by Δ the set of nonzero roots of G relative to T. Let $\Pi=\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ be a fundamental system of Δ. We may assume that Π is the system of simple roots of Δ for a suitable order of the Lie algebra of T. In this order, we denote the set of positive roots by Δ^+. Let Δ_{Π_1} be the set of nonzero roots of K relative to T and $\Pi_1=\{\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_r}\}$ the subset of Π which generates Δ_{Π_1}. If we denote the set of positive roots of Δ_{Π_1} by $\Delta^+_{\Pi_1}$, then we have $\Delta^+_{\Pi_1}=\Delta^+\cap\Delta_{\Pi_1}$.

Let G^C and K^C be complexifications of G and K, respectively. Let L be the parabolic subgroup of G^C such that its Lie algebra is generated by the Lie algebra of K^C and $\{E_\alpha ; \alpha\in\Delta^+\backslash\Delta^+_{\Pi_1}\}$. Here E_α denotes the root vector of $\alpha\in\Delta$. Then we see

$$G/K \cong G^C/L$$

as C^∞-manifolds. So G/K is a homogeneous complex manifold by this holomorphic structure. Moreover it has a G-invariant Kähler metric ([2]). Thus we get a Kähler C-space G/K. Conversely every Kähler C-space can be described as above. Also we can construct the Kähler C-space from a pair (Π, Π_1), where Π is a fundamental system of roots and Π_1 is a subset of Π ([10] and [2]).

Let (ρ, V) be an irreducible finite dimensional complex representation of K with highest weight $\hat{\rho}$. We denote by $\{\varpi_1, \varpi_2, \cdots, \varpi_l\}$ the system of fundamental weights of Π. Then the highest weight $\hat{\rho}$ of (ρ, V) can be written as follows:

$$\hat{\rho}=n_1\varpi_1+n_2\varpi_2+\cdots+n_l\varpi_l,$$

where $n_1, n_2, \cdots, n_l$ are integers and if $\alpha_i\in\Pi_1$ then $n_i\geqq 0$.

In this case we can uniquely extend (ρ, V) to a holomorphic representation (ρ_L, V) of L ([3]). We put

$$E_\rho=G\times_\rho V,$$

$$E_{\rho_L}=G^C\times_{\rho_L}V.$$

Then

$$E_\rho\cong E_{\rho_L}$$

as C^∞-vector bundles. We regard E_ρ as a holomorphic vector bundle by the isomorphism above unless otherwise stated. Also if (ρ, V) is irreducible, there is a unique G-invariant Hermitian structure h up to a homothety and (E_ρ, h) is an irreducible Einstein-Hermitian vector bundle ([8]). Therefore we consider E_ρ as an irreducible Einstein-Hermitian vector bundle if (ρ, V) is irreducible. For more details about an Einstein-Hermitian vector bundle, we refer the reader to [7].

By $\mathrm{End}(E_\rho)$ we denote the endomorphism bundle of E_ρ. Let $\mathfrak{sl}(E_\rho)$ be the subbundle of E_ρ which consists of trace free endomorphisms. By definition of $\mathrm{End}(E_\rho)$ and $\mathfrak{sl}(E_\rho)$,

$$\mathrm{End}(E_\rho)\cong G\times_{\rho\otimes\rho*}\mathrm{End}(V), \tag{1}$$

$$\mathfrak{sl}(E_\rho)\cong G\times_{\rho\otimes\rho*}\mathfrak{sl}(V), \tag{2}$$

where $\mathrm{End}(V)$ is the linear space of endomorphisms and $\mathfrak{sl}(V)$ is the subspace of $\mathrm{End}(V)$ consisting of trace free endomorphisms. Thus

$$\mathrm{End}(V)=V\otimes V^*, \tag{3}$$

where V^* is the dual space of V. And K acts $\mathrm{End}(V)$ by the tensor product representation $(\rho\otimes\rho^*, V\otimes V^*)$ where (ρ^*, V^*) is the dual representation of (ρ, V). By the way $\mathfrak{sl}(E)$ is invariant by K. Thus K acts $\mathfrak{sl}(V)$ via the action for $\mathrm{End}(V)$.

Finally we denote the Dolbeault cohomology groups of $\mathrm{End}(E_\rho)$ and $\mathfrak{sl}(E_\rho)$ by $H^{p,q}(G/K, \mathrm{End}(\rho))$ and $H^{p,q}(G/K, \mathfrak{sl}(E_\rho))$, respectively. We set

$$h^{p,q}(\mathrm{End}(E_\rho))=\dim H^{p,q}(G/K, \mathrm{End}(E_\rho)),$$

$$h^{p,q}(\mathfrak{sl}(E_\rho))=\dim H^{p,q}(G/K, sl(E_\rho)).$$

3. Main Results

We continue with the notation and the situation in §2. Let G/K be a Kähler C-space where G is a compact simply connected semisimple Lie group and K is the centralizer of a toral subgroup of G.

LEMMA 2. *Let (ρ, V) be an irreducible complex representation of K. Then the restriction of $(\rho\otimes\rho^*, V\otimes V^*)$ to the center of K is trivial.*

PROOF. Let Z_K be the center of K and K' the semisimple part of K. Then

$$\varphi: Z_K\times K'\to K, \quad (z, k)\mapsto zk$$

is a Lie group homomorphism with kernel $Z_K\cap K'$. Then $(\rho\circ\varphi, V)$ is a representation of the direct product Lie group $Z_K\times K'$ on V. We note that

$$\rho\circ\varphi|_{Z_K}=\rho|_{Z_K}, \quad \rho\circ\varphi|_{K'}=\rho|_{K'}.$$

Because of irreducibility of (ρ, V), $(\rho\circ\varphi, V)$ is also irreducible. So there are irreducible representations (ρ_{Z_K}, V_{Z_K}) of Z_K and $(\rho_{K'}, V_{K'})$ of K' such that

$$(\rho_{Z_K}\otimes\rho_{K'}, V_{Z_K}\otimes V_{K'})\cong(\rho\circ\varphi, V),$$

where $(\rho_{Z_K}\otimes\rho_{K'}, V_{Z_K}\otimes V_{K'})$ denotes the exterior tensor product representation of (ρ_{Z_K}, V_{Z_K}) and $(\rho_{K'}, V_{K'})$. By irreducibility of (ρ_{Z_K}, V_{Z_K}), V_{Z_K} is a one dimensional space. Then the tensor product $(\rho_{Z_K}\otimes\rho^*_{Z_K}, V_{Z_K}\otimes V^*_{Z_K})$ is isomorphic to the trivial representation. Q.E.D.

COROLLARY 3. *Let G be a compact semisimple Lie group and $K=T$ be a maximal torus of G. Let (ρ, V) be an irreducible complex representation of T. Then*

$$\mathrm{End}(E_\rho)\cong G/T\times \boldsymbol{C},$$

$$\mathfrak{sl}(E_\rho)\cong G/T\times\{0\}.$$

In particular,

$$h^{0,p}(\mathrm{End}(E_\rho))=\begin{cases} 1, & \text{for } p=0 \\ 0, & \text{for } p\geqq 1,\end{cases}$$

$$h^{0,p}(\mathfrak{sl}(E_\rho))=0, \quad \text{for } p\geqq 0.$$

PROOF. We note that any irreducible complex representation space of a torus is one dimensional. From Lemma 1 it is easy to see that $\mathrm{End}(E_\rho)$ is trivial. And also it is easy to see that Hodge numbers of $\mathrm{End}(E_\rho)$ and $\mathfrak{sl}(E_\rho)$ are as stated above (for example, by means of Bott's generalized Borel-Weil theorem ([3, Theorem IV′]). Q.E.D.

Next we consider the case of rank $G=2$. In this case the fundamental system of roots Π is $\{\alpha_1, \alpha_2\}$. And if G/K is a Kähler C-space then the corresponding Π_1 as in section 2 is $\{\alpha_1, \alpha_2\}$, $\{\alpha_1\}$, $\{\alpha_2\}$ or ϕ. Furthermore a compact simple Lie group G is of type A_2, B_2, or G_2 in this case. If G is of classical type then corresponding Kähler C-spaces are

$$G/K\cong SU(3)/S(U(2)\times U(1))\cong P_2C \quad \text{if } G=A_2 \text{ and } \Pi_1=\{\alpha_1\},$$

$$G/K\cong SU(3)/S(U(1)\times U(2))\cong P_2C \quad \text{if } G=A_2 \text{ and } \Pi_1=\{\alpha_2\},$$

$$G/K\cong Sp(2)/(U(1)\times Sp(1))\cong P_3C \quad \text{if } G=B_2 \text{ and } \Pi_1=\{\alpha_1\},$$

$$G/K\cong SO(5)/(U(1)\times SO(3))\cong Q_3(C) \quad \text{if } G=B_2 \text{ and } \Pi_1=\{\alpha_2\}.$$

It is clear that the first case is isomorphic to the second one in the above. And we note that if $\Pi_1=\{\alpha_1, \alpha_2\}$ then $K=T$ is a maximal torus of G, i.e., it is the case of Corollary 3. Also if $\Pi=\phi$ then $K=G$, i.e., G/K consists of a one point $\{o\}$.

Then we state the main theorem.

THEOREM 4. *Let G be a compact simply connected simple Lie group with* rank $G=2$. *Let $\Pi=\{\alpha_1, \alpha_2\}$ be a fundamental system of roots relative to a maximal torus T of G. We put $\Pi_1=\{\alpha_i\}$ $(i=1, 2)$. We denote by K the analytic subgroup of G with maximal rank which corresponds to Π_1. Let (ρ, V) be an irreducible complex representation of K with highest weight $\hat{\rho}=n_1\varpi_1+n_2\varpi_2$, where $\{\varpi_1, \varpi_2\}$ is the system of fundamental weights. Then the Hodge number $h^{0,p}(\mathrm{End}(E_\rho))$ is as follows:*

(I) *If G is of type A_2 and $\Pi_1=\{\alpha_2\}$ then*

$$h^{0,p}(\mathrm{End}(E_\rho))=\begin{cases} 1, & if\ p=0, \\ \dfrac{1}{2}\sum_{k=1}^{n_2}(2k+1)(k+2)(k-1), & if\ p=1, \\ 0, & if\ p\geqq 2. \end{cases}$$

(II) *If G is of type B_2 and $\Pi_1=\{\alpha_1\}$ then*

$$h^{0,p}(\mathrm{End}(E_\rho))=\begin{cases} 1, & if\ p=0, \\ \dfrac{1}{3}\sum_{k=1}^{n_1}(2k+3)(2k+1)(2k-1), & if\ p=1, \\ 0, & if\ p\geqq 2. \end{cases}$$

(III) *If G is of type B_2 and $\Pi_1=\{\alpha_2\}$ then*

$$h^{0,p}(\mathrm{End}(E_\rho))=\begin{cases} 1, & \text{if } p=0, \\ \frac{1}{2}\sum_{k=1}^{n_2}(2k-1)(k+2)(k-1), & \text{if } p=1, \\ 0, & \text{if } p\geqq 2. \end{cases}$$

(IV) *If G is of type G_2 and $\Pi_1=\{\alpha_1\}$ then*

$$h^{0,p}(\mathrm{End}(E_\rho))=\begin{cases} 1, & \text{if } p=0, \\ 7, & \text{if } p=1 \text{ and } n_1=2, \\ 21, & \text{if } p=1 \text{ and } n_1\geqq 3, \\ \frac{1}{40}\sum_{k=5}^{n_1}(2k+1)(k+5)(k+2)(k-1)(k-4), & \text{if } p=2, \\ 0, & \text{otherwise}. \end{cases}$$

(V) *If G is of type G_2 and $\Pi_1=\{\alpha_2\}$ then*

$$h^{0,p}(\mathrm{End}(E_\rho))=\begin{cases} 1, & \text{if } p=0, \\ 2261, & \text{if } p=1, \\ \frac{1}{24}\sum_{k=2}^{n_2}(21k-2)(18k-1)(13k-1)(8k-1)(3k-1)k, & \\ & \text{if } p=2, \\ 0, & \text{if } p\geqq 3. \end{cases}$$

THEOREM 5. *Under the same assumption of Theorem 4, the Hodge number $h^{0,p}(\mathfrak{sl}(E_\rho))$ is equal to*

$$h^{0,p}(\mathfrak{sl}(E_\rho))=\begin{cases} 0, & \text{if } p=0, \\ h^{0,p}(\mathrm{End}(E_\rho)), & \text{if } p\geqq 1, \end{cases}$$

for every cases (I)$\sim$(V) in Theorem 4.

We shall prove these theorems in the next section. We state some consequences of these results here. If $h^{0,0}(\mathfrak{sl}(E_\rho))=h^{0,2}(\mathfrak{sl}(E_\rho))=0$, then we can identify $H^{0,1}(G/K, \mathrm{End}(E_\rho))$ with the tangent space at the homogeneous structures of the moduli space of E_ρ ([6], [9] and [7, Chapter VII]). Then we get the following corollary.

COROLLARY 6. *Under the same assumption of theorems above, the dimension of the moduli space of irreducible Einstein-Hermitian structures is as follows:*
(I) *If G is of type A_2 and $\Pi_1=\{\alpha_2\}$ then*

$$\frac{1}{2}\sum_{k=1}^{n_2}(2k+1)(k+2)(k-1).$$

(II) *If G is of type B_2 and $\Pi_1=\{\alpha_1\}$ then*

$$\frac{1}{3}\sum_{k=1}^{n_1}(2k+3)(2k+1)(2k-1).$$

(III) *If G is of type B_2 and $\Pi_1=\{\alpha_2\}$ then*

$$\frac{1}{2}\sum_{k=1}^{n_2}(2k-1)(k+2)(k-1).$$

(IV) *If G is of type G_2 and $\Pi_1=\{\alpha_1\}$ then*

$$\begin{cases} 0, & \text{for } n_1=0 \text{ or } 1,\\ 7, & \text{for } n_1=2,\\ 21, & \text{for } n_1=3 \text{ or } 4.\end{cases}$$

(V) *If G is of type G_2 and $\Pi_1=\{\alpha_2\}$ then*

$$\begin{cases} 0, & \text{for } n_2=0,\\ 261, & \text{for } n_2=1.\end{cases}$$

Because of Theorem 5, we see that if G is of classical type, then $h^{0,0}(\mathfrak{sl}(E_\rho))=h^{0,2}(\mathfrak{sl}(E_\rho))=0$. Moreover we note that if $\Pi_1=\{\alpha_i\}$ then rank $E_\rho=n_i+1$ in theorems and corollary above. Also under the same condition, we note that the dimension of the moduli space of E_ρ depends only on n_i. Then we get Theorem 1 from Corollaries 3 and 6.

4. Proof of Theorems

In this sections, we prove Theorems 4 and 5. The two theorems 4 and 5 are proved at the same time. First we note equations (1) and (2) in § 2. So $\mathrm{End}(E_\rho)$ and $\mathfrak{sl}(E_\rho)$ are defined by representations $(\rho\otimes\rho^*, V\otimes V^*)$ and $(\rho\otimes\rho^*, \mathfrak{sl}(V))$, respectively. Because of Lemma 2, $\rho\otimes\rho^*$ is trivial on the center of K. The semisimple part of K is of type A_1 in these cases. Thus we can apply the Clebsch-Gordan theorem to the representation $\rho\otimes\rho^*$. Therefore we see that if $\Pi_1=\{\alpha_i\}$ and the highest weight of ρ is given by $\hat{\rho}=n_1\varpi+n_2\varpi$, then the highest weight which corresponds to each irreducible component of $(\rho\otimes\rho^*, \mathrm{End}(V))$ are given by

(4) $$n_i\alpha_i,\ (n_i-1)\alpha_i,\ (n_i-2)\alpha_i,\ \cdots,\ \alpha_i,\ 0.$$

Also under the same assumption the highest weight which corresponds to each irreducible component of $(\rho\otimes\rho^*, \mathfrak{sl}(V))$ are given by

(5) $$n_i\alpha_i,\ (n_i-1)\alpha_i,\ (n_i-2)\alpha_i,\ \cdots,\ 2\alpha_i,\ \alpha_i.$$

Let (ρ_k, V_k) be the complex irreducible representation of K with highest weight $\hat{\rho}_k = k\alpha_i$ and we put $E_{\rho_k} = G\times_{\rho_k} V_k$. Then (4) and (5) imply the following, respectively:

$$H^{0,p}(G/K, \mathrm{End}(E_\rho)) = \bigoplus_{k=0}^{n2} H^{0,p}(G/K, E_{\rho_k}), \tag{6}$$

$$H^{0,p}(G/K, \mathfrak{sl}(E_\rho)) = \bigoplus_{k=1}^{n2} H^{0,p}(G/K, E_{\rho_k}). \tag{7}$$

Next we compute the cohomology one by one. We denote by δ the half of the sum of the positive roots, i. e.,

$$\delta = \varpi_1 + \varpi_2$$

in these cases. And we denote by S_α the reflection with respect to $\alpha \in \Delta$. We use the tables of root systems in [4] for the following.

Case (I) In this case G is of type A_2 and $\Pi_1 = \{\alpha_2\}$. Then we see

$$\begin{aligned}\delta + \hat{\rho}_k &= \delta + k\alpha_2 \\ &= -(k-1)\varpi_1 + (2k+1)\varpi_2.\end{aligned}$$

From this, we see that

$$\delta + \hat{\rho} \text{ is } \begin{cases} \text{regular} & \text{if } k \neq 1, \\ \text{singular} & \text{if } k = 1 \end{cases}$$

in the sense of [3]. Also we see that

$$\text{the index of } \delta + \hat{\rho}_k = \begin{cases} 0, & \text{if } k = 0, \\ 1, & \text{if } k \neq 0, 1. \end{cases}$$

By the way, we have

$$S_{\alpha_1}(\delta + \hat{\rho}_k) = (k-1)\varpi_1 + (k+2)\varpi_2.$$

This implies that $S_{\alpha_1}(\delta + \hat{\rho}_k)$ is contained in the fundamental Weyl chamber. We put

$$\lambda_k = (k-2)\varpi_1 + (k+1)\varpi_2$$

and V_{λ_k} denotes the complex irreducible representation space of G with highest weight λ_k. By means of Bott's generalized Borel-Weil theorem ([3, Theorem IV′]), we get

$$H^{0,p}(G/K, E_{\rho_k}) \cong \begin{cases} V_{\lambda_k}, & \text{for } k \geqq 1 \text{ and } p = 1, \\ \{0\}, & \text{for } k \geqq 1 \text{ and } p \neq 1, \end{cases} \tag{8}$$

and

$$(9)\qquad H^{0,p}(G/K,\ E_{\rho_k})\cong\begin{cases} \boldsymbol{C}, & \text{for } k=0 \text{ and } p=0,\\ \{0\}, & \text{for } k=0 \text{ and } p\geqq 1.\end{cases}$$

as complex G-spaces. We have

$$H^{0,p}(G/K,\ \mathrm{End}(E_\rho))\cong\begin{cases} \boldsymbol{C}, & \text{for } p=0,\\ \bigoplus_{k=1}^{n_2} V_{\lambda_k}, & \text{for } p=1,\\ \{0\}, & \text{for } p\geqq 2,\end{cases}$$

from equations (6), (8) and (9) and

$$H^{0,p}(G/K,\ \mathfrak{sl}(E_\rho))\cong\begin{cases} \bigoplus_{k=1}^{n_2} V_{\lambda_k}, & \text{for } p=1,\\ \{0\}, & \text{for } p\neq 1,\end{cases}$$

from equations (7), (8) and (9) as complex G-spaces. We can compute $h^{0,p}(\mathrm{End}(E_\rho))$ and $h^{0,p}(\mathfrak{sl}(E_\rho))$ by Weyl's dimension formula for a complex irreducible representation space. Then we obtain theorems in the case (I).

Case (II) In this case G is of type B_2 and $\Pi_1=\{\alpha_1\}$. Then we see

$$\begin{aligned}\delta+\hat{\rho}_k&=\delta+k\alpha_2\\ &=(2k+1)\varpi_1-(2k-1)\varpi_2.\end{aligned}$$

From this, we see $\delta+\hat{\rho}_k$ is regular for any k. Also we see

$$\text{the index of } \delta+\hat{\rho}_k=\begin{cases} 0, & \text{for } k=0,\\ 1, & \text{for } k\neq 0.\end{cases}$$

By the way, we have

$$S_{\alpha_2}(\delta+\hat{\rho}_k)=2\varpi_1+(2k-1)\varpi_2.$$

This implies that $S_{\alpha_2}(\delta+\hat{\rho}_K)$ is contained in the fundamental Weyl chamber. In the same way as in the case (I), we get

$$H^{0,p}(G/K,\ \mathrm{End}(E_\rho))\cong\begin{cases} \boldsymbol{C}, & \text{for } p=0,\\ \bigoplus_{k=1}^{n_1} V_{\lambda_k}, & \text{for } p=1,\\ \{0\}, & \text{for } p\geqq 2,\end{cases}$$

$$H^{0,p}(G/K,\ \mathfrak{sl}(E_\rho))\cong\begin{cases} \bigoplus_{k=1}^{n_1} V_{\lambda_k}, & \text{for } p=1,\\ \{0\}, & \text{for } p\neq 1\end{cases}$$

as complex G-spaces. Here

$$\lambda_k=(2k-1)\varpi_1+(2k-2)\varpi_2$$

and V_{λ_k} is the irreducible complex representation space of G which corresponds to λ_k. We get $h^{0,p}(\mathrm{End}(E_\rho))$ and $h^{0,p}(\mathfrak{sl}(E_\rho))$ by Weyl's dimension formula as before.

Case (III) In this case G is of type B_2 and $\Pi_1=\{\alpha_2\}$. Then we see

$$\begin{aligned}\delta+\hat{\rho}_k&=\delta+k\alpha_2\\&=-(k-1)\varpi_1+(2k+1)\varpi_2.\end{aligned}$$

From this, we see

$$\delta+\hat{\rho}_k \text{ is } \begin{cases}\text{regular}, & \text{if } k\neq 1,\\ \text{singular}, & \text{if } k=1,\end{cases}$$

and

$$\text{the index of } \delta+\hat{\rho}_k=\begin{cases}0, & \text{for } k=0,\\ 1, & \text{for } k\neq 0,\ 1.\end{cases}$$

By the way, we have

$$S_{\alpha_1}(\delta+\hat{\rho}_k)=(k-1)\varpi_1+3\varpi_2.$$

This implies that $S_{\alpha_1}(\delta+\hat{\rho}_k)$ is in the fundamental Weyl chamber. As before, we get

$$H^{0,p}(G/K, \mathrm{End}(E_\rho))\cong\begin{cases}\boldsymbol{C}, & \text{for } p=0,\\ \bigoplus_{k=1}^{n_2} V_{\lambda_k}, & \text{for } p=1,\\ \{0\}, & \text{for } p\geqq 2,\end{cases}$$

$$H^{0,p}(G/K, \mathfrak{sl}(E_\rho))\cong\begin{cases}\bigoplus_{k=1}^{n_2} V_{\lambda_k}, & \text{for } p=1,\\ \{0\}, & \text{for } p\neq 1\end{cases}$$

as complex G-spaces. Here

$$\lambda_k=(k-2)\varpi_1+2\varpi_2$$

and V_{λ_k} is the irreducible complex representation space of G which corresponds to λ_k. We get $h^{0,p}(\mathrm{End}(E_\rho))$ and $h^{0,p}(\mathfrak{sl}(E_\rho))$ by Weyl's dimension formula as before.

Case (IV) In this case G is of type G_2 and $\Pi_1=\{\alpha_1\}$. Then we see

$$\begin{aligned}\delta+\hat{\rho}_k&=\delta+k\alpha_2\\&=(2k+1)\varpi_1-(k-1)\varpi_2.\end{aligned}$$

From this, we see that

$$\delta+\hat{\rho}_k \text{ is } \begin{cases}\text{regular}, & \text{if } k\neq 1 \text{ and } 4,\\ \text{singular}, & \text{if } k=1 \text{ or } 4,\end{cases}$$

and

$$\text{the index of } \delta+\hat{\rho}_k=\begin{cases} 0, & \text{for } k=0, \\ 1, & \text{for } k=1, 2, 3 \text{ or } 4, \\ 2, & \text{for } k\geqq 5. \end{cases}$$

By the way we have,

$$S_{\alpha_2}(\delta+\hat{\rho}_k)=-(k-4)\varpi_1+(k-1)\varpi_2$$

and

$$S_{\alpha_1}\circ S_{\alpha_2}(\delta+\hat{\rho}_k)=(k-4)\varpi_1+3\varpi_2.$$

This implies that $S_{\alpha_2}(\delta+\hat{\rho}_k)$ is contained in the fundamental Weyl chamber if $k=2, 3$ and $S_{\alpha_1}\circ S_{\alpha_2}(\delta+\hat{\rho}_k)$ is contained in the fundamental Weyl chamber if $k\geqq 5$. As before, we get

$$H^{0,p}(G/K, \mathrm{End}(E_\rho))\cong\begin{cases} \boldsymbol{C}, & \text{if } p=0, \\ \{0\}, & \text{if } p=1 \text{ and } n_1=0, 1, \\ V_{\lambda_2}, & \text{if } p=1 \text{ and } n_1=2, \\ V_{\lambda_2}\oplus V_{\lambda_3}, & \text{if } p=1 \text{ and } n_1\geqq 3, \\ \bigoplus_{k=5}^{n_1} V_{\lambda_k}, & \text{if } p=2, \\ \{0\}, & \text{if } p\geqq 3, \end{cases}$$

$$H^{0,p}(G/K, \mathfrak{sl}(E_\rho))\cong\begin{cases} \{0\}, & \text{if } p=0, \\ \{0\}, & \text{if } p=1 \text{ and } n_1=0, 1, \\ V_{\lambda_2}, & \text{if } p=1 \text{ and } n_1=2, \\ V_{\lambda_2}\oplus V_{\lambda_3}, & \text{if } p=1 \text{ and } n_1\geqq 3, \\ \bigoplus_{k=5}^{n_1} V_{\lambda_k}, & \text{if } p=2, \\ \{0\}, & \text{if } p\geqq 3 \end{cases}$$

as complex G-spaces. Here

$$\lambda_k=\begin{cases} -(k-3)\varpi_1+(k-2)\varpi_2, & \text{for } k=2, 3, \\ (k-5)\varpi_1+2\varpi_2, & \text{for } k\geqq 5, \end{cases}$$

And V_{λ_k} is the irreducible complex representation space of G which corresponds to λ_k. We can compute $h^{0,p}(\mathrm{End}(E_\rho))$ and $h^{0,p}(\mathfrak{sl}(E_\rho))$ by Weyl's dimension formula as before.

Case (V) In this case G is of type G_2 and $\Pi_1=\{\alpha_2\}$. Then we see

$$\begin{aligned}\delta+\hat{\rho}_k&=\delta+k\alpha_2\\ &=-(3k-1)\varpi_1+(2k+1)\varpi_2.\end{aligned}$$

From this, we see that

$$\delta+\hat{\rho}_k \text{ is } \begin{cases} \text{regular}, & \text{if } k\neq 2, \\ \text{singular}, & \text{if } k=2. \end{cases}$$

and

$$\text{the index of } \delta+\hat{\rho}_k=\begin{cases} 0, & \text{for } k=0, \\ 1, & \text{for } k=1, \\ 2, & \text{for } k\geqq 2. \end{cases}$$

By the way, we have

$$S_{\alpha_1}(\delta+\hat{\rho}_k)=(3k-1)\varpi_1+5k\varpi_2.$$

This implies that $S_{\alpha_1}(\delta+\hat{\rho}_k)$ is contained in the fundamental Weyl chamber. As before, we get

$$H^{0,p}(G/K, \operatorname{End}(E_\rho))\cong\begin{cases} \boldsymbol{C}, & \text{for } p=0, \\ \{0\}, & \text{for } p=1 \text{ and } n_2=0, \\ V_{\lambda_1}, & \text{for } p=1 \text{ and } n_2\geqq 1, \\ \bigoplus_{k=2}^{n_2} V_{\lambda_k}, & \text{for } p=2, \\ \{0\}, & \text{for } p\geqq 3, \end{cases}$$

$$H^{0,p}(G/K, \mathfrak{sl}(E_\rho))\cong\begin{cases} \{0\}, & \text{for } p=0, \\ \{0\}, & \text{for } p=1 \text{ and } n_2=0, \\ V_{\lambda_1}, & \text{for } p=1 \text{ and } n_2\geqq 1, \\ \bigoplus_{k=2}^{n_2} V_{\lambda_k}, & \text{for } p=2, \\ \{0\}, & \text{for } p\geqq 3 \end{cases}$$

as complex G-spaces. Here

$$\lambda_k=(3k-2)\varpi_1+(5k-1)\varpi_2,$$

and V_{λ_k} is the irreducible complex representation space of G which corresponds to λ_k. We can compute $h^{0,p}(\operatorname{End}(E_\rho))$ and $h^{0,p}(\mathfrak{sl}(E_\rho))$ by Weyl's dimension formula as before.

Now we complete the proof of Theorems 4 and 5.

REMARK. We only write the dimensions in the statement of Theorems and Corollary, but we get the cohomology groups and the tangent spaces of moduli spaces as the representation space of G. Indeed, we have determined the highest weight of each irreducible component in the proof.

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