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ON STABILITY OF A CERTAIN MINIMAL SUBMANIFOLD IN $SU(3)/SO(3)$

By

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§1. Introduction.

Let M be a compact irreducible symmetric space. It is known that the first conjugate locus $F_p(M)$ of M with respect to $p\in M$ has a stratification. We denote by $F_p^0(M)$ the maximal dimensional strata. H. Tasaki proved the following theorem:

THEOREM ([8]). *For any point p in M, $F_p^0(M)$ is a noncompact minimal submanifold of M. If M is a compact connected simple Lie group, then $F_p^0(M)$ is stable.*

If M is of rank one, then $F_o^0(M)$ is stable. These results are obtained by Berger [1].

In this paper we shall study on stability of a noncompact minimal submanifold $F_p^0(M)$ in the compact irreducible symmetric space $M=SU(3)/SO(3)$.

In general, a noncompact minimal submanifold F in a Riemannian manifold M is said to be *stable* if the second variation of the volume of F is nonnegative for every variation of compact support.

The purpose of this paper is to prove the following theorem:

THEOREM. *If M is $SU(3)/SO(3)$, then $F_p^0(M)$ is stable.*

In §2 we explain the structure of $F_o^0(M)$ when M is simpley connected which is obtained by T. Sakai and M. Takeuchi. In §3 we shall give the proof of the theorem.

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§ 2. Preliminaries.

1. Let (G, K) be a compact symmetric pair and θ be the involutive automorphism of G associated with (G, K). Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively. We denote also by θ the induced involutive automorphism of $\mathfrak{g}$. Take a bi-invariant Riemannian meric $\langle\,,\rangle$ on G and denote also by $\langle\,,\rangle$ the induced G-invariant Riemannian metric on $M=G/K$. Then M is a compact symmetric space with respect to $\langle\,,\rangle$. Let π denote the natural projection from G to M. Put $o=\pi(e)$, where e is the identity element of G. Since K lies between

$$G_\theta=\{g\in G\,;\,\theta(g)=g\}$$

and its identity component, we have

$$\mathfrak{k}=\{X\in\mathfrak{g}\,;\,\theta X=X\}.$$

Put

$$\mathfrak{m}=\{X\in\mathfrak{g}\,;\,\theta X=-X\}.$$

Since θ is an involutive automorphism, we have a direct sum decomposition of $\mathfrak{g}$:

$$\mathfrak{g}=\mathfrak{k}+\mathfrak{m}\,.$$

Take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$ and a maximal abelian subalgebra $\mathfrak{t}$ in $\mathfrak{g}$ containing $\mathfrak{a}$. Then the complexification $\mathfrak{t}^C$ of $\mathfrak{t}$ is a Cartan subalgebra of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$. For an element $\alpha\in\mathfrak{t}$, put

$$\mathfrak{g}_\alpha=\{X\in\mathfrak{g}^C\,;\,[H, X]=2\pi\sqrt{-1}\langle\alpha, H\rangle X \quad \text{for each } H\in\mathfrak{t}\}.$$

An element $\alpha\in\mathfrak{t}-\{0\}$ is called a root if $\mathfrak{g}_\alpha\neq\{0\}$. We denote by $\Sigma(G)$ the set of all roots. We have a direct sum decomposition of $\mathfrak{g}^C$:

$$\mathfrak{g}^C=\mathfrak{t}^C+\sum_{\alpha\in\Sigma(G)}\mathfrak{g}_\alpha\,.$$

For an element $\gamma\in\mathfrak{a}$, put

$$\tilde{\mathfrak{g}}_\gamma^C=\{X\in\mathfrak{g}^C\,;\,[H, X]=2\pi\sqrt{-1}\langle\gamma, H\rangle X \ \text{ for each } H\in\mathfrak{a}\}.$$

An element of $\gamma\in\mathfrak{a}-\{0\}$ is called a restricted root if $\tilde{\mathfrak{g}}_\gamma^C\neq\{0\}$. We denote by $\Sigma(G, K)$ the set of all restricted roots. We denote by $^-$ the orthogonal projection from $\mathfrak{t}$ to $\mathfrak{a}$. We have

$$\tilde{\mathfrak{g}}_\gamma^C=\sum_{\bar{\alpha}=\gamma}\mathfrak{g}_\alpha\,,$$

$$\Sigma(G, K)=\overline{\Sigma(G)-\Sigma_0(G)}\,,$$

where $\Sigma_0(G)=\Sigma(G)\cap\mathfrak{k}$.

Choose lexicographic orderings $>$ on $\mathfrak{t}$ and $\mathfrak{a}$ such that

$$\alpha\in\Sigma(G), \qquad \bar{\alpha}\geqq 0 \Longrightarrow \alpha\geqq 0 .$$

We denote by $\Sigma^+(G)$ the set of positive roots and by $\Sigma^+(G, K)$ the set of positive restricted roots. We put

$$\mathfrak{k}_\gamma=\mathfrak{k}\cap(\mathfrak{g}_\gamma^C+\mathfrak{g}_{-\gamma}^C), \qquad \mathfrak{m}_\gamma=\mathfrak{m}\cap(\mathfrak{g}_\gamma^C+\mathfrak{g}_{-\gamma}^C)$$

for each $\gamma\in\Sigma^+(G, K)$ and

$$\mathfrak{k}_0=\{X\in\mathfrak{k}\,;\ [\mathfrak{a}, X]=\{0\}\}.$$

Then we have the following lemma:

LEMMA 1 ([7], Lemma 1.1). *We have the orthogonal direct sum decompositions*

$$\mathfrak{k}=\mathfrak{k}_0+\sum_{\gamma\in\Sigma^+(G,K)}\mathfrak{k}_\gamma, \qquad \mathfrak{m}=\mathfrak{a}+\sum_{\gamma\in\Sigma^+(G,K)}\mathfrak{m}_\gamma .$$

We can choose $S_\alpha\in\mathfrak{k}$ and $T_\alpha\in\mathfrak{m}$ for each $\alpha\in\Sigma^+(G)-\Sigma_0(G)$ in such a way that:

(1) *For each $\gamma\in\Sigma^+(G, K)$, the sets $\{S_\alpha\,;\ \alpha\in\Sigma^+(G)-\Sigma_0(G),\ \bar{\alpha}=\gamma\}$ and $\{T_\alpha\,;\ \alpha\in\Sigma^+(G)-\Sigma_0(G),\ \bar{\alpha}=\gamma\}$ are orthonormal basis of $\mathfrak{k}_\gamma$ and $\mathfrak{m}_\gamma$ respectively;*

(2) *For each $\alpha\in\Sigma^+(G)-\Sigma_0(G)$ and each $H\in\mathfrak{a}$, we have*

$$[H, S_\alpha]=2\pi\langle\alpha, H\rangle T_\alpha , \qquad [H, T_\alpha]=-2\pi\langle\alpha, H\rangle S_\alpha ,$$

$$Ad(\exp H)S_\alpha=(\cos 2\pi\langle\alpha, H\rangle)S_\alpha+(\sin 2\pi\langle\alpha, H\rangle)T_\alpha ,$$

$$Ad(\exp H)T_\alpha=-(\sin 2\pi\langle\alpha, H\rangle)S_\alpha+(\cos 2\pi\langle\alpha, H\rangle)T_\alpha ;$$

(3) *For each $\alpha\in\Sigma^+(G)-\Sigma_0(G)$, we have*

$$[S_\alpha, T_\alpha]=2\pi\bar{\alpha} .$$

2. From now on we assume that M is irreducible. Then $\Sigma(G, K)$ is irreducible and there exists a unique highest root δ in $\Sigma^+(G, K)$. Let r be the rank of M and $\Pi(G, K)=\{\gamma_i\}_{1\leq i\leq r}$ be the fundamental root system of $\Sigma(G,K)$. Put

$$S=\{H\in\mathfrak{a}\,;\ \langle H, \delta\rangle=1/2,\ \langle H, \gamma_i\rangle\geqq 0 \ \text{ for } 1\leqq i\leqq r\},$$

$$S^0=\{H\in\mathfrak{a}\,;\ \langle H, \delta\rangle=1/2,\ \langle H, \gamma_i\rangle>0 \ \text{ for } 1\leqq i\leqq r\},$$

$$F_p(M)=gK\,\mathrm{Exp}\,S ,$$

$$F_p^0(M)=gK\,\mathrm{Exp}\,S^0 \ \text{ for } p=\pi(g)\in M, \ \text{ where } g\in G,$$

$$m_H=-2\pi\sum_{\alpha\in\Sigma^+(G)-\Sigma_0(G),\ \bar{\alpha}\neq\delta}(\cot 2\pi\langle\alpha, H\rangle)\bar{\alpha} .$$

Then $F_p(M)$ is the first conjugate locus of M with respect to a point p (see [2], Chap. VII, § 3). The vector $(k \exp H)_* m_H$ is the mean curvature vector of K Exp H at $k \exp H$ for each $H \in S^0$. Let $(\boldsymbol{R}\bar{\delta})^\perp$ denote the orthogonal complement of $\boldsymbol{R}\bar{\delta}$ in $\mathfrak{a}$. The submanifold $F_p^0(M)$ is open and dense in $F_p(M)$. H. Tasaki proved the following theorem:

THEOREM 1 ([8]). *For each point p in M, $F_p^0(M)$ is a noncompact minimal submanifold of M. Furthermore, if M is a compact connected simple Lie group, then $F_p^0(M)$ is stable.*

For each $X \in \mathfrak{g}$, we define a vector field $X^* \in \mathfrak{X}(M)$ by

$$X_p^* = \frac{d}{dt} \exp tX \cdot p |_{t=0} .$$

We denote by $\bar{\nabla}$ the covariant derivative of M. We have

$$g_*(\bar{\nabla}_{X^*} Y^*) = \bar{\nabla}_{(Ad(g)X)^*}(Ad(g)Y)^* , \tag{2.1}$$

for g in G and X, Y in $\mathfrak{g}$, and

$$(\bar{\nabla}_{X^*} Y^*)_o = \begin{cases} 0 & \text{for } X \in \mathfrak{m} \text{ and } Y \in \mathfrak{m} , \\ -[X, Y] & \text{for } X \in \mathfrak{m} \text{ and } Y \in \mathfrak{k} , \\ 0 & \text{for } X \in \mathfrak{k} \text{ and } Y \in \mathfrak{k} , \\ 0 & \text{for } X \in \mathfrak{k} \text{ and } Y \in \mathfrak{m} \end{cases} \tag{2.2}$$

under the identification of $\mathfrak{m}$ with the tangent space $T_o(M)$ of M at the origin o. Let $m(\gamma)$ denote the multiplicity of $\gamma \in \Sigma(G, K)$. Then we obtain the following relations:

$$\begin{aligned} T_{\mathrm{Exp}\, H}(\mathrm{Exp}\, S^0) &= (\exp H)_*(\boldsymbol{R}\bar{\delta})^\perp , \\ T_{k\, \mathrm{Exp}\, H}(K\, \mathrm{Exp}\, H) &= (k \exp H)_* \sum_{\gamma \in \Sigma(G, K) - \{\bar{\delta}\}} \mathfrak{m}_\gamma , \\ T_{k\, \mathrm{Exp}\, H}(F_o^0(M)) &= (k \exp H)_*\Big(\sum_{\gamma \in \Sigma(G, K) - \{\bar{\delta}\}} \mathfrak{m}_\gamma + (\boldsymbol{R}\bar{\delta})^\perp\Big), \\ N_{k\, \mathrm{Exp}\, H}(F_o^0(M)) &= (k \exp H)_*(\boldsymbol{R}\bar{\delta} + \mathfrak{m}_{\bar{\delta}}), \\ \mathrm{codim}\,(F_o^0(M)) &= 1 + m(\bar{\delta}), \\ N_{k\, \mathrm{Exp}\, H}(K\, \mathrm{Exp}\, H) &= (k \exp H)_*(\mathfrak{a} + \mathfrak{m}_{\bar{\delta}}), \end{aligned} \tag{2.3}$$

for $H \in S^0$ and $k \in K$ (see [8]).

Let A, B and R denote the shape operator, the second fundamental form of $F_o^0(M) \subset M$ and the Riemannian curvature tensor of M, respectively. We define symmetric linear transformations $\bar{R}_{k\,\mathrm{Exp}\,H}$ and $\tilde{A}_{k\,\mathrm{Exp}\,H}$ on the normal space $N_{k\,\mathrm{Exp}\,H}(F_0^0(M))$ at $k\,\mathrm{Exp}\,H$, where $k\in K$ and $H\in S^0$, as follows:

$$\bar{R}_{k\,\mathrm{Exp}\,H}(v)=\sum(R(e_i,\,v)e_i)^{\perp}\,,$$

$$\tilde{A}_{k\,\mathrm{Exp}\,H}(v)=\sum B(e_i,\,A^v e_i)\,,$$

for each $v\in N_{k\,\mathrm{Exp}\,H}(F_0^0(M))$, where $\{e_i\}$ is an orthonormal basis of the tangent space $T_{k\,\mathrm{Exp}\,H}(F_0^0(M))$. Let $N(F_0^0(M))$ denote the normal bundle of $F_0^0(M)$ and $\Gamma(N(F_0^0(M)))$ denote the vector space of all C^∞ sections of $N(F_0^0(M))$. Put

$$\Gamma_0(N(F_0^0(M)))=\{V\in\Gamma(N(F_0^0(M)))\,;\ V \text{ has a compact support}\}.$$

Let $J=\Delta+\bar{R}-\tilde{A}$ denote the Jacobi operator, where Δ is the negative of the rough Laplacian of the normal connection of $N(F_0^0(M))$.

Then $F_0^0(M)$ is stable if and only if the following inequality holds (see [5]):

$$\int_F (JV,\,V)dv_{F_0^0(M)}\geqq 0 \qquad \text{for each } V\in\Gamma_0(N(F_0^0(M)))\,.$$

Identifying $\boldsymbol{R}\bar{\delta}+\mathfrak{m}_{\bar{\delta}}$ with $N_{k\,\mathrm{Exp}\,H}(F_0^0(M))$ by linear isometry $(k\exp H)_*$, we can consider $\bar{R}_{k\,\mathrm{Exp}\,H}$ and $\tilde{A}_{k\,\mathrm{Exp}\,H}$ as the symmetric linear transformations on $\boldsymbol{R}\bar{\delta}+\mathfrak{m}_{\bar{\delta}}$. Then we have the following theorem:

THEOREM 2. *As a linear operator on $\boldsymbol{R}\bar{\delta}+\mathfrak{m}_{\bar{\delta}}$, the symmetric linear transformation $\bar{R}_{k\,\mathrm{Exp}\,H}-\tilde{A}_{k\,\mathrm{Exp}\,H}$ is of the following form:*

$$\bar{R}_{k\,\mathrm{Exp}\,H}-\tilde{A}_{k\,\mathrm{Exp}}=\Big[-\frac{4\pi^2}{\|\bar{\delta}\|^2}\sum_{\alpha\in\Sigma^+(G)-\Sigma_0(G),\,\bar{\alpha}\neq\bar{\delta}}\frac{(\bar{\alpha},\,\bar{\delta})^2}{\sin^2 2\pi\langle\alpha,\,H\rangle}\Big]id\,.$$

PROOF. For the sake of brevity, we denote $\bar{R}_{k\,\mathrm{Exp}\,H}$ by $\bar{R}$, $\tilde{A}_{k\,\mathrm{Exp}\,H}$ by $\tilde{A}$, and $\sum_{\alpha\in\Sigma^+(G)-\Sigma_0(G),\,\bar{\alpha}\neq\bar{\delta}}$ by $\sum'_\alpha$. Let $\{H_i\}$ be an orthonormal basis of $(\boldsymbol{R}\bar{\delta})^{\perp}$. Then

$$\{(k\exp H)_*H_i\}\cup\{(k\exp H)_*T_\alpha\,;\ \alpha\in\Sigma^+(G)-\Sigma_0(G),\ \bar{\alpha}\neq\bar{\delta}\}$$

forms an orthonormal basis of $T_{k\,\mathrm{Exp}\,H}(F_0^0(M))$.

We shall show that $\bar{R}$ and A are scalar operators. We define a closed subgroup K_H of K for $H\in S^0$ as follows:

$$K_H=\{k\in K\,;\ k\,\mathrm{Exp}\,H=\mathrm{Exp}\,H\}. \tag{2.4}$$

Let $\mathfrak{k}_H$ denote the Lie algebra of K_H. Then $\mathfrak{k}_H=\mathfrak{k}_0+\mathfrak{k}_{\bar{\delta}}$. The group K_H acts on the normal space $N_{\mathrm{Exp}\,H}(K\,\mathrm{Exp}\,H)$ naturally. Identifying $\mathfrak{a}+\mathfrak{m}_{\bar{\delta}}$ with $N_{\mathrm{Exp}\,H}(K\,\mathrm{Exp}\,H)$ by linear isometry $(\exp H)_*$, we can consider that K_H acts on $\mathfrak{a}+\mathfrak{m}_{\bar{\delta}}$. Let $\rho_H(k)$ be the action of $k\in K_H$ on $\mathfrak{a}+\mathfrak{m}_{\bar{\delta}}$. Since $(\bar{\delta},\,H)=1/2$ for each $H\in S^0$, we get

(2.5) $$\rho_H(k)=s\circ Ad(k)\circ s\,,$$

where $s=id$ on $\mathfrak{a}$ and $s=-id$ on $\mathfrak{m}_{\bar\delta}$. In particular, ρ_H is equivalent to the adjoint representation of K_H on $\mathfrak{a}+\mathfrak{m}_{\bar\delta}$. Put

$$M_{\bar\delta}=\operatorname{Exp}(\boldsymbol{R}\bar\delta+\mathfrak{m}_{\bar\delta})\,.$$

The manifold $M_{\bar\delta}$ is a maximal dimensional totally geodesic sphere in M of constant curvature κ, where κ is the maximum of the sectional curvatures of M. The manifold $M_{\bar\delta}$ is called the Helgason sphere of M. Then the pair $([\mathfrak{m}_{\bar\delta}, \mathfrak{m}_{\bar\delta}]+\mathfrak{k}_{\bar\delta}, \boldsymbol{R}\bar\delta+\mathfrak{m}_{\bar\delta})$ is the symmetric pair of $M_{\bar\delta}$ and $ad(\mathfrak{k}_{\bar\delta}+[\mathfrak{m}_{\bar\delta}, \mathfrak{m}_{\bar\delta}])|(\boldsymbol{R}\bar\delta+\mathfrak{m}_{\bar\delta})=\mathfrak{so}(\boldsymbol{R}\bar\delta+\mathfrak{m}_{\bar\delta})$ (see [2], Chap. VII, § 11).

It is well-known that the natural representation of $\mathfrak{so}(n)$ on $\boldsymbol{R}^n$ is irreducible. Since $\mathfrak{k}_{\bar\delta}+[\mathfrak{m}_{\bar\delta}, \mathfrak{m}_{\bar\delta}]\subset\mathfrak{k}_0+\mathfrak{k}_{\bar\delta}$, the symmetric linear transformations $\bar R$ and $\tilde A$ are scalar operators.

Since M is symmetric, we have

$$\begin{aligned}\langle\bar R(\bar\delta), \bar\delta\rangle&=\textstyle\sum'_\alpha\langle(R(T_\alpha, \bar\delta)T_\alpha)^\perp, \bar\delta\rangle+\sum_i\langle(R(H_i, \bar\delta)H_i)^\perp, \bar\delta\rangle\\ &=-(\textstyle\sum'_\alpha\langle[[T_\alpha, \bar\delta], T_\alpha]^\perp, \bar\delta\rangle+\sum_i\langle[[H_i, \bar\delta], H_i]^\perp, \bar\delta\rangle)\\ &=-\textstyle\sum'_\alpha\langle[[T_\alpha, \bar\delta], T_\alpha]^\perp, \bar\delta\rangle\,,\end{aligned}$$

where we denote by $\perp$ the orthogonal projection from $\mathfrak{m}$ to $\boldsymbol{R}\bar\delta+\mathfrak{m}_{\bar\delta}$. Thus we have

$$\bar R=-\frac{4\pi^2}{\|\bar\delta\|^2}\textstyle\sum'_\alpha\langle\bar\alpha, \bar\delta\rangle^2 id\,.$$

From (2.1), (2.2), we have

(2.6) $$\begin{aligned}(\exp H)_*^{-1}B((\exp H)_*T_\alpha, (\exp H)_*H_i)&=\frac{-1}{\sin 2\pi\langle\alpha, H\rangle}(\exp H)_*^{-1}(\bar\nabla_{H_i^*}S_\alpha^*)^\perp\\ &=2\pi\cot 2\pi\langle\alpha, H\rangle T_\alpha^\perp=0\,.\end{aligned}$$

Since $\operatorname{Exp} S^0$ is totally geodesic, we have

(2.7) $$(\exp H)_*^{-1}B((\exp H)_*H_i, (\exp H)_*H_j)=0\,.$$

The following is proved in [8]:

$$(\exp H)_*^{-1}B((\exp H)_*T_\alpha, (\exp H)_*T_\beta)=(\cot 2\pi\langle\beta, H\rangle)[T_\alpha, S_\beta]^\perp$$

Using the above equation, we have

$$\begin{aligned}\langle\tilde A(\bar\delta), \bar\delta\rangle&=\textstyle\sum'_\alpha\sum'_\beta(\cot^2 2\pi(\beta, H\rangle)\langle[T_\alpha, S_\beta], \bar\delta\rangle^2\\ &=\textstyle\sum'_\alpha(\cot^2 2\pi\langle\alpha, H\rangle)(2\pi\langle\alpha, \bar\delta\rangle)^2\end{aligned}$$

Thus we have

$$\tilde{A}=\frac{4\pi^2}{\|\bar{\delta}\|^2}\Sigma'_\alpha(\cot^2 2\pi\langle\alpha, H\rangle)\langle\bar{\alpha}, \bar{\delta}\rangle^2 id.$$

Q.E.D.

3. From now on, we assume that M is simply connected. Put

$$K_1=\{k\in K;\ \mathrm{Exp}\,Ad(k)H=\mathrm{Exp}\,H \text{ for each } H\in S^0\}.$$

Then clearly K_1 is a closed subgroup of K. Let $\mathfrak{k}_1$ be the Lie algebra of K_1. Then $\mathfrak{k}_1=\mathfrak{k}_0+\mathfrak{k}_{\bar{\delta}}$.

PROPOSITION 1 ([4], Lemma 3.10, [6], p. 52, Cor. 1). *K_H (defined by (2.4)) is a closed subgroup of K which is independent of the choice of $H\in S^0$ and consequently equal to K_1.*

PROPOSITION 2 ([4], Prop. 3.11). *We denote by $\Phi: K/K_1\times S^0\to M$ the mapping defined by $\Phi(kK_1, H)=\mathrm{Exp}\,Ad(k)H$. Then we have the following:*

(1) *Φ is a differentiable mapping into M whose image is $F_\delta^0(M)$.*

(2) *Φ is an injective mapping.*

(3) *Φ is everywhere regular.*

(4) *$F_\delta^0(M)$ is an embedded submanifold of M, i.e., the topology on $F_\delta^0(M)$ induced by Φ coincides with the relative topology of M.*

From Proposition 1 and (2.5), ρ_H is independent of the choice of $H\in S^0$. From now on, ρ is to stand for ρ_H.

LEMMA 2.

(1) *The space $\boldsymbol{R}\bar{\delta}+\mathfrak{m}_{\bar{\delta}}$ is invariant under the action of K_1.*

(2) *The group K_1 acts trivially on $(\boldsymbol{R}\bar{\delta})^\perp$.*

PROOF.

(1) $k_*N_{\mathrm{Exp}\,H}(F_\delta^0(M))\subset N_{\mathrm{Exp}\,H}(F_\delta^0(M))$ for each $k\in K_1$. Hence we obtain (1) using (2.3).

(2) Let $k\in K_1$, $H\in S^0$ and $X\in(\boldsymbol{R}\bar{\delta})^\perp$. For sufficiently small t, $H+tX$ is in S^0. Using Proposition 1, we get

$$\begin{aligned}k_*(\exp H)_*X&=\frac{d}{dt}k\,\mathrm{Exp}(H+tX)|_{t=0}\\&=\frac{d}{dt}\mathrm{Exp}(H+tX)|_{t=0}\\&=(\exp H)_*X.\end{aligned}$$

Hence $\rho(k)X=X$. Q.E.D.

4. The restriction $N(F_0^0(M))|K\,\mathrm{Exp}\,H$ of the vector bundle $N(F_0^0(M))$ on $K\,\mathrm{Exp}\,H$ is a homogeneous vector bundle isomorphic to $K\times_\rho N_{\mathrm{Exp}\,H}(F_0^0(M))$. A section of this vector bundle is identified with an element of

$$\begin{aligned}&C^\infty(K;\boldsymbol{R}\bar{\delta}+\mathfrak{m}_{\bar{\delta}})_{K_1}\\&=\{f\in C^\infty(K;\boldsymbol{R}\bar{\delta}+\mathfrak{m}_{\bar{\delta}});\ f(ku)=\rho(u^{-1})f(k)\ \text{for}\ k\in K,\ u\in K_1\}\end{aligned}$$

by the following mapping:

$$\Gamma(K\times_\rho N_{\mathrm{Exp}\,H}(F_0^0(M)))\longrightarrow C^\infty(K;\boldsymbol{R}\bar{\delta}+\mathfrak{m}_{\bar{\delta}})_{K_1};\ V\mapsto V^\natural, \tag{2.8}$$

where $V^\natural(k)=(k\exp H)_*^{-1}V_{k\,\mathrm{Exp}\,H}$ for each $k\in K$. From Proposition 1 and Lemma 2(1), we obtain the following bundle isomorphism:

$$\begin{array}{ccc}K\times_\rho N_{\mathrm{Exp}\,H}(F_0^0(M)) & \longrightarrow & K\times_\rho N_{\mathrm{Exp}\,H'}(F_0^0(M))\\ \downarrow & & \downarrow\\ K\,\mathrm{Exp}\,H & \longrightarrow & K\,\mathrm{Exp}\,H';\\ [(k,v)] & \longmapsto & [k,\exp(H'-H)_*v]\\ \downarrow & & \downarrow\\ k\,\mathrm{Exp}\,H & \longmapsto & k\,\mathrm{Exp}\,H',\end{array}$$

where $H, H'\in S^0$. We remark that the above bundle isomorphism is independent of representation of K-orbit $K\,\mathrm{Exp}\,H$ by Proposition 2(2). We may identify $\Gamma(K\times_\rho N_{\mathrm{Exp}\,H}(F_0^0(M)))$ with $\Gamma(K\times_\rho N_{\mathrm{Exp}\,H'}(F_0^0(M)))$ by the bundle isomorphism above:

$$\begin{aligned}\Gamma(K\times_\rho N_{\mathrm{Exp}\,H}(F_0^0(M)))&\longleftrightarrow\Gamma(K\times_\rho N_{\mathrm{Exp}\,H'}(F_0^0(M)))\\ V&\longleftrightarrow V',\end{aligned}\tag{2.9}$$

where $V'_{k\,\mathrm{Exp}\,H'}=k_*\exp(H'-H)_*k_*^{-1}V_{k\,\mathrm{Exp}\,H}$ for each $k\in K$. Then we can consider $\Gamma(K\times_\rho N_{\mathrm{Exp}\,H}(F_0^0(M)))$ as a subspace of $\Gamma(N(F_0^0(M)))$ by (2.9) and the above remark. Let $V^\natural$ denote an element of $\Gamma(N(F_0^0(M)))$ corresponding to $V\in\Gamma(K\times_\rho N_{\mathrm{Exp}\,H}(F_0^0(M)))$. Then the following relations hold in correspondence (2.9):

$$V^\natural=V'^\natural\quad\text{on}\ C^\infty(K;\boldsymbol{R}\bar{\delta}+\mathfrak{m}_{\bar{\delta}})_{K_1},$$

$$V^\natural=V'^\natural\quad\text{on}\ \Gamma(N(F_0^0(M))).$$

Define a mapping J_H from $\Gamma(K\times_\rho N_{\mathrm{Exp}\,H}(F_0^0(M)))$ to itself by

$$J_HV=(JV^\natural)|K\,\mathrm{Exp}\,H\ \text{for each}\ V\in\Gamma(K\times_\rho N_{\mathrm{Exp}\,H}(F_0^0(M))).$$

We can consider J_H as a mapping from $C^\infty(K; \boldsymbol{R}\bar{\delta}+\mathfrak{m}_{\bar{\delta}})_{K_1}$ to itself by using (2.8). We shall prove the following theorem:

THEOREM 3. *The mapping* $J_H: C^\infty(K; \boldsymbol{R}\bar{\delta}+\mathfrak{m}_{\bar{\delta}})_{K_1} \to C^\infty(K; \boldsymbol{R}\bar{\delta}+\mathfrak{m}_{\bar{\delta}})_{K_1}$ *is given by the following equation:*

$$J_H = -\sum_{\alpha\in\Sigma^+(G)-\Sigma_0(G),\,\bar{\alpha}\neq\bar{\delta}} \frac{1}{\sin^2 2\pi\langle\alpha, H\rangle}\tilde{S}_\alpha^2$$

$$-\frac{4\pi^2}{\|\bar{\delta}\|^2}\sum_{\alpha\in\Sigma^+(G)-\Sigma_0(G),\,\bar{\alpha}\neq\bar{\delta}} \frac{\langle\alpha, \delta\rangle^2}{\sin^2 2\pi\langle\alpha, H\rangle},$$

where $\tilde{S}_\alpha$ *denote the left invariant vector field on* K *such that* $(\tilde{S}_\alpha)_e = S_\alpha$.

PROOF. For the sake of brevity, we denote $\sum_{\alpha\in\Sigma^+(G)-\Sigma_0(G),\bar{\alpha}\neq\bar{\delta}}$ by $\sum'_\alpha$. Let $\tilde{Z}$ denote the left invariant vector field on G such that $\tilde{Z}_e = Z \in \mathfrak{g}$. Let $v \in T_x(M)$, $W \in \mathfrak{X}(M)$ and $x = \pi(g)$ for some $g \in G$. We take an element $Z \in \mathfrak{g}$ satisfying $d\pi(\tilde{Z}_g) = v$ and write $Z = X + Y$ for some $X \in \mathfrak{m}$ and $Y \in \mathfrak{k}$. Then the following equation holds (see [3]):

$$(2.10)\qquad \bar{\nabla}_v W = \frac{d}{dt}(\exp(-tAd(g)Z))_* W_{g\exp tZK|t=0}$$

$$-\frac{d}{dt}(\exp(-tAd(g)Y))_* W_{gK|t=0}\,.$$

Let $\Delta: \Gamma(N(F_0^0(M))) \to \Gamma(N(F_0^0(M)))$ be the negative of the rough Laplacian of the normal connection of $N(F_0^0(M))$. Define a mapping

$$\Delta_H: \Gamma(K\times_\rho N_{\mathrm{Exp}\,H}(F_0^0(M))) \longrightarrow \Gamma(K\times_\rho N_{\mathrm{Exp}\,H}(F_0^0(M)))$$

by

$$\Delta_H V = (\Delta V^{\shortparallel})|K\,\mathrm{Exp}\,H \qquad \text{for each } V \in \Gamma(K\times_\rho N_{\mathrm{Exp}\,H}(F_0^0(M)))\,.$$

We consider Δ_H as a mapping from $C^\infty(K; \boldsymbol{R}\bar{\delta}+\mathfrak{m}_{\bar{\delta}})_{K_1}$ to itself by using (2.8). We shall prove the following equation:

$$(2.11)\qquad \Delta_H = -\sum\nolimits'_\alpha \frac{1}{\sin^2 2\pi\langle\alpha, H\rangle}\tilde{S}_\alpha^2\,.$$

Then the proof will be finished by using Theorem 2. We consider the homogeneous vector bundle $K\times T_{\mathrm{Exp}\,H}(\mathrm{Exp}\,S^0)$ on $K\,\mathrm{Exp}\,H$. Then we get by Lemma 2(2),

$$\Gamma(K\times T_{\mathrm{Exp}\,H}(\mathrm{Exp}\,S^0))$$

$$=\{V\in\Gamma(N(K\,\mathrm{Exp}\,H));\ V_{k\,\mathrm{Exp}\,H}\in k_*T_{\mathrm{Exp}\,H}(\mathrm{Exp}\,S^0) \text{ for each } k\in K\}\,.$$

We consider $\Gamma(K\times T_{\mathrm{Exp}\,H}(\mathrm{Exp}\,S^0))$ as a subspace of $\mathfrak{X}(F_0^0(M))$ in the way above.

Let $X^{\natural}$ denote an element of $\mathfrak{X}(F_0^0(M))$ corresponding to $X\in\Gamma(K\times T_{\mathrm{Exp}\,H}(\mathrm{Exp}\,S^0))$. Let H_i be an orthonormal basis of $(\boldsymbol{R}\bar{\delta})^{\perp}$ and define $\bar{H}_i\in\Gamma(K\times T_{\mathrm{Exp}\,H}(\mathrm{Exp}\,S^0))$ by

$$(\bar{H}_i)_{k\,\mathrm{Exp}\,H}=(k\exp H)_*H_i\,.$$

Then, by (2.10), we get

$$\begin{aligned}(2.12)\qquad (\bar{\nabla}_{\bar{H}_i^{\natural}}\bar{H}_i^{\natural})(\mathrm{Exp}\,H)&=\frac{d}{dt}(\exp(-t)Ad(\exp H)H_i)_*(\bar{H}_i^{\natural})_{\exp H\,\mathrm{Exp}\,tH_i|t=0}\\&=\frac{d}{dt}(\exp(-tH_i))_*(\bar{H}_i^{\natural})_{\mathrm{Exp}(H+tH_i)|t=0}\\&=\frac{d}{dt}(\exp(-tH_i))_*(\exp(tH_i+H))_*H_{i|t=0}\\&=0\,.\end{aligned}$$

Let ∇ be the covariant derivative of $F_0^0(M)$. The following is proved in [8]:

$$\begin{aligned}&\Sigma'_{\alpha}\frac{1}{\sin^2 2\pi\langle\alpha,\,H\rangle}(\nabla_{S_\alpha^*}S_\alpha^*)(\mathrm{Exp}\,H)\\&=(\exp H)_*m_H\in(\exp H)_*(\boldsymbol{R}\bar{\delta})^{\perp}\,.\end{aligned}$$

For sufficiently small t, $tm_H+H\in S^0$. Using this fact and (2.10), we get

$$\begin{aligned}(2.13)\qquad &\Sigma'_{\alpha}\frac{1}{\sin^2 2\pi\langle\alpha,\,H\rangle}(\bar{\nabla}_{\nabla_{S_\alpha^*}S_\alpha^*}V^{\natural})(\mathrm{Exp}\,H)\\&=\frac{d}{dt}(\exp(-tm_H))_*V^{\natural}_{\exp tm_H\,\mathrm{Exp}\,H|t=0}\\&=0\,,\qquad\text{for each } V\in\Gamma(K\times_\rho N_{\mathrm{Exp}\,H}(F_0^0(M)))\,.\end{aligned}$$

Similarly we can prove

$$(\bar{\nabla}_{\bar{H}_i^{\natural}}\bar{\nabla}_{\bar{H}_i^{\natural}}V^{\natural})(\mathrm{Exp}\,H)=0\,.$$

Using (2.6), (2.7) and the above equation, we have

$$(2.14)\qquad (\nabla^{\perp}_{\bar{H}}\nabla^{\perp}_{\bar{H}}V^{\natural})(\mathrm{Exp}\,H)=0\,.$$

Using (2.12), (2.13) and (2.14), we obtain

$$(\Delta_H V^{\natural})(\mathrm{Exp}\,H)=-\Sigma'_{\alpha}\frac{1}{\sin^2 2\pi\langle\alpha,\,H\rangle}(\nabla^{\perp}_{S_\alpha^*}\nabla^{\perp}_{S_\alpha^*}V^{\natural})(\mathrm{Exp}\,H)\,.$$

By (2.10), we have

$$\begin{aligned}(\bar{\nabla}_{S_\alpha^*}V^{\natural})(\exp tS_\alpha\,\mathrm{Exp}\,H)&=\frac{d}{ds}(\exp(-sS_\alpha))_*V^{\natural}_{\exp tS_\alpha\exp sS_\alpha\,\mathrm{Exp}\,H|s=0}\\&-\frac{d}{ds}(\exp(-s)Ad(\exp tS_\alpha\exp H)(\cos 2\pi\langle\alpha,\,H\rangle)S_\alpha)_*V^{\natural}_{\exp tS_\alpha\,\mathrm{Exp}\,H|s=0}\,.\end{aligned}$$

By taking the normal components to $F_0^0(M)$ of both sides in the above equation, we get

$$(\nabla^{\perp}_{\tilde{S}^*_\alpha} V^{\natural})(\exp tS_\alpha \operatorname{Exp} H)=\frac{d}{ds}(\exp(-sS_\alpha))_* V^{\natural}_{\exp tS_\alpha \exp sS_\alpha \operatorname{Exp} H \mid s=0}\,.$$

Hence we get

$$(\overline{\nabla}_{S^*_\alpha}\nabla^{\perp}_{\tilde{S}^*_\alpha} V^{\natural})(\operatorname{Exp} H)=\frac{d}{dt}(\exp(-tS_\alpha))_*(\nabla^{\perp}_{\tilde{S}^*_\alpha} V^{\natural})(\exp tS_\alpha \operatorname{Exp} H)_{\mid t=0}$$

$$-\frac{d}{dt}(\exp(-t)Ad(\exp H)(\cos 2\pi\langle\alpha,\ H\rangle)S_\alpha)_*(\nabla^{\perp}_{\tilde{S}^*_\alpha} V^{\natural})(\operatorname{Exp} H)_{\mid t=0}$$

$$=\frac{\partial^2}{\partial s\partial t}(\exp(-tS_\alpha))_*(\exp(-sS_\alpha))_* V_{\exp tS_\alpha \exp sS_\alpha \operatorname{Exp} H \mid t=s=0}$$

$$-\frac{\partial^2}{\partial t\partial s}(\exp(-t)Ad(\exp H)(\cos 2\pi\langle\alpha,\ H\rangle)S_\alpha \exp(-s)S_\alpha)_* V_{\exp sS_\alpha \operatorname{Exp} H \mid t=s=0}.$$

If we take the normal components to $F_0^0(M)$, we have

$$(\nabla^{\perp}_{\tilde{S}^*_\alpha}\nabla^{\perp}_{\tilde{S}^*_\alpha} V^{\natural})(\operatorname{Exp} H)$$

$$=\frac{\partial^2}{\partial t\partial s}(\exp(-tS_\alpha))_*(\exp(-sS_\alpha))_* V_{\exp tS_\alpha \exp sS_\alpha \operatorname{Exp} H \mid t=s=0}\,.$$

Put $f=V^{\natural}$. Then we get

$$(\Delta_H f)(e)=-\Sigma'_\alpha \frac{1}{\sin^2 2\pi\langle\alpha,\ H\rangle}(\tilde{S}^2_\alpha f)(e).$$

Hence we get (2.11). Thus the proof is finished. Q.E.D.

Put

$$C^\infty(F_0^0(M))_K$$
$$=\{\varphi\in C^\infty(F_0^0(M))\,;\ \varphi(k \operatorname{Exp} H)=\varphi(\operatorname{Exp} H) \text{ for } k\in K,\ H\in S^0\}.$$

By Proposition 2(2), an element f of $C^\infty(S^0)$ is extended to an element $f^{\natural}$ of $C^\infty(F_0^0(M))_K$ in a natural manner. Namely we put $f^{\natural}(k \operatorname{Exp} H)=f(H)$ for each $k\in K$ and $H\in S^0$. Put

$$C_0^\infty(S^0)=\{f\in C^\infty(S^0)\,;\ f \text{ has a compact support}\}.$$

By Proposition 2(2), we extend $g\in C^\infty(K \operatorname{Exp} H_0)(H_0\in S^0)$ to $g^{\natural}\in C^\infty(F_0^0(M))$ by $g^{\natural}(k \operatorname{Exp} H)=g(\operatorname{Exp} H_0)$ for $k\in K$ and $H\in S^0$. Then

$$\|V^{\natural}\|=\|V\|^{\natural} \qquad \text{for each } V\in\Gamma(K\times_\rho N_{\operatorname{Exp} H_0}(F_0^0(M)))\,.$$

We denote by *grad* and $grad_{F_0^0(M)}$ the gradient on S^0 and $F_0^0(M)$ respectively. Then we shall show the following lemma:

LEMMA 3.

$$\|grad_{F_0^0(M)} f^{\natural}\| = \|grad f\|^{\natural} \qquad \textit{for each } f \in C^\infty(S^0).$$

PROOF. Let $\{H_i\}$ be an orthonormal basis of $(\boldsymbol{R}\tilde{\delta})^{\perp}$. Theh $\{(\exp H)_* H_i\}$ is an orthonormal basis of $T_{\mathrm{Exp}\, H}(\mathrm{Exp}\, S^0)$. We have

$$\begin{aligned}(S_\alpha^* f^{\natural})(\mathrm{Exp}\, H) &= \frac{d}{dt} f^{\natural}(\exp tS_\alpha \,\mathrm{Exp}\, H)_{|t=0} \\ &= \frac{d}{dt} f(H)_{|t=0} \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}((\exp H)_* H_i) f^{\natural} &= \frac{d}{dt} f^{\natural}(\mathrm{Exp}(tH_i + H))_{|t=0} \\ &= \frac{d}{dt} f(H + tH_i)_{|t=0} \\ &= (H_i f)(H).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\|grad_{F_0^0(M)} f^{\natural}\|^2(\mathrm{Exp}\, H) &= \sum \{(H_i f)(H)\}^2 \\ &= \|grad f\|^2(H).\end{aligned}$$

Since $f^{\natural}(k\,\mathrm{Exp}\, H) = f^{\natural}(\mathrm{Exp}\, H)$, we get

$$\begin{aligned}\|grad_{F_0^0(M)} f^{\natural}\|(k\,\mathrm{Exp}\, H) &= \|grad f\|(H) \\ &= \|grad f\|^{\natural}(k\,\mathrm{Exp}\, H).\end{aligned}$$

Q.E.D.

§3. Proof of theorem.

In this section, we put $G = SU(3)$, $K = SO(3)$, $\theta(g) = \bar{g}$ for each $g \in G$ and $\langle\,,\rangle$ = the negative of the Killing form. We put

$$\mathfrak{a} = \mathfrak{t} = \left\{ \sqrt{-1}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix};\ x, y, z \in \boldsymbol{R},\ x+y+z=0 \right\},$$

and introduce a lexicographic order in $\mathfrak{t}$ defined by

$$E_{11} > E_{22} > E_{33}.$$

Then we have

$$\Sigma^{+}(G)=\Sigma^{+}(G, K)=\left\{\alpha_{ij}=\frac{\sqrt{-1}}{12\pi}(E_{ii}-E_{jj})\,;\, i<j\right\}$$

$$=\{\alpha_{12},\ \alpha_{23},\ \alpha_{13}=\tilde{\delta}\},$$

$$\Sigma_{0}(G)=\emptyset,$$

$$S^{0}=\left\{\frac{1}{2}\pi\sqrt{-1}\begin{bmatrix} x+1 & 0 & 0 \\ 0 & -2x & 0 \\ 0 & 0 & x-1 \end{bmatrix};\ -\frac{1}{3}<x<\frac{1}{3}\right\},$$

We put

$$S_{13}=\frac{1}{2\sqrt{3}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},\quad T_{13}=\frac{\sqrt{-1}}{2\sqrt{3}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$S_{23}=\frac{1}{2\sqrt{3}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},\quad T_{23}=\frac{\sqrt{-1}}{2\sqrt{3}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$S_{12}=\frac{1}{2\sqrt{3}}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad T_{12}=\frac{\sqrt{-1}}{2\sqrt{3}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $S_{ij}=S_{\alpha_{ij}}$, $T_{ij}=T_{\alpha_{ij}}$. By a straightforward calculation we get the folowing lemma:

LEMMA 4. (1) *For $H, H'\in S^{0}$, $k\in K$ and $\alpha, \beta\in\Sigma^{+}(G)-\{\tilde{\delta}\}$,*

$$\frac{\langle (S_{\alpha}^{*})_{k\,\mathrm{Exp}\,H},\ (S_{\beta}^{*})_{k\,\mathrm{Exp}\,H}\rangle}{\langle (S_{\alpha}^{*})_{k\,\mathrm{Exp}\,H'},\ (S_{\beta}^{*})_{k\,\mathrm{Exp}\,H'}\rangle}$$

is independent of the choice $k\in K$.

(2) $\sin^{2}2\pi\langle\alpha, H\rangle$ *for $H\in S^{0}$ is independent of the choice of $\alpha\in\Sigma^{+}(G)-\{\tilde{\delta}\}$.*

Let C be the negative of the Casimir differential operator of K relative to $\langle\,,\rangle$. Then using Theorem 3 and Lemma 4(2), we get

$$J_{H}=\frac{1}{\cos^{2}\frac{3}{2}\pi x}\left(C-\frac{1}{2}\right), \tag{3.1}$$

$$\text{for } H=\frac{1}{2}\pi\sqrt{-1}\begin{bmatrix} x+1 & 0 & 0 \\ 0 & -2x & 0 \\ 0 & 0 & x-1 \end{bmatrix}\in S^{0}\quad\left(-\frac{1}{3}<x<\frac{1}{3}\right).$$

Let $dv_{F_0^0(M)}$, dv_{S^0} and $dv_{K \operatorname{Exp} H_0}$ denote the volume elements of $F_0^0((M)$, S^0 and $K \operatorname{Exp} H_0$, respectively. Then we shall show the following lemma:

LEMMA 5.

$$\int_{F_0^0(M)} f^{\natural} g^{\natural} dv_{F_0^0(M)}$$

$$=\frac{\int_{K \operatorname{Exp} H_0} g\, dv_{K \operatorname{Exp} H_0}}{\left|\prod_{\alpha\in\Sigma^+(G)-\{\delta\}} \sin 2\pi\langle\alpha, H_0\rangle\right|}\int_{S^0} f(H)\left|\prod_{\alpha\in\Sigma^+(G)-\{\delta\}} \sin 2\pi\langle\alpha, H\rangle\right| dv_{S^0}$$

for each $f\in C_0^\infty(S^0)$ *and* $g\in C^\infty(K \operatorname{Exp} H_0)$.

PROOF. Put

$$g_{\alpha\beta}(k \operatorname{Exp} H)=\langle (S_\alpha^*)_{k \operatorname{Exp} H}, (S_\beta^*)_{k \operatorname{Exp} H}\rangle,$$

for $k\in K$, $H\in S^0$ and $\alpha, \beta\in\Sigma^+(G)-\{\delta\}$. By the change of variable by the mapping Φ, we replace the integration on $F_0^0(M)$ with the integration on $S^0\times K \operatorname{Exp} H_0$. By Lemma 4(1),

$$\int_{F_0^0(M)} f^{\natural} g^{\natural} dv_{F_0^0(M)}$$

$$=\int_{S^0\times K \operatorname{Exp} H_0} f(H) g(k \operatorname{Exp} H_0)\frac{\sqrt{det g_{\alpha\beta}(k \operatorname{Exp} H)}}{\sqrt{det g_{\alpha\beta}(k \operatorname{Exp} H_0)}} dv_{K \operatorname{Exp} H_0}\times dv_{S^0}$$

$$=\frac{\int_{K \operatorname{Exp} H_0} g\, dv_{K \operatorname{Exp} H_0}}{\left|\prod_{\alpha\in\Sigma^+(G)-\{\delta\}} \sin 2\pi\langle\alpha, H_0\rangle\right|}\int_{S^0} f(H)\left|\prod_{\alpha\in\Sigma^+(G)-\{\delta\}} \sin 2\pi\langle\alpha, H\rangle\right| dv_{S^0}.$$

Q.E.D.

Let $V\in\Gamma(K\times_\rho N_{\operatorname{Exp} H_0}(F_0^0(M)))$. We assume that there exists $\varphi\in C^\infty(S^0)$ such that $JV^{\natural}=\varphi^{\natural} V^{\natural}$. Then

$$J(f^{\natural} V^{\natural})=(\Delta_{F_0^0(M)} f^{\natural})V^{\natural}+f^{\natural}\varphi^{\natural} V^{\natural},$$

for each $f\in C_0^\infty(S^0)$, where $\Delta_{F_0^0(M)}$ is the negative of the Laplace operator of $F_0^0(M)$. Since $C^\infty(F_0^0(M))_K$ is invariant under $\Delta_{F_0^0(M)}$, we get by Lemma 3 and Lemma 5

$$\int_{F_0^0(M)} \langle J(f^{\natural} V^{\natural}), f^{\natural} V^{\natural}\rangle dv_{F_0^0(M)}$$

$$=\frac{\int_{K \operatorname{Exp} H_0} \|V\|^2 dv_{K \operatorname{Exp} H_0}}{\left|\prod_{\alpha\in\Sigma^+(G)-\{\delta\}} \sin 2\pi\langle\alpha, H_0\rangle\right|}\int_{S^0}(\|grad f\|^2+f^2\varphi)\left|\prod_{\alpha\in\Sigma^+(G)-\{\delta\}} \sin 2\pi\langle\alpha, H\rangle\right| dv_{S^0}.$$

LEMMA 6. *Let* $V_1, V_2 \in \Gamma(K \times_\rho N_{\mathrm{Exp}\, H_0}(F_0^0(M)))$ *and* $\varphi_1, \varphi_2 \in C^\infty(S^0)$. *If* $JV_i^{\natural} = \varphi_i^{\natural} V_i^{\natural}$ $(i=1, 2)$ *and* $\varphi_1 < \varphi_2$, *then*

$$\int_{K\,\mathrm{Exp}\, H_0} \langle V_1, V_2 \rangle dv_{K\,\mathrm{Exp}\, H_0} = 0 .$$

PROOF. For each $f \in C_0^\infty(S^0)$, $f \geqq 0$, $f \not\equiv 0$, we get

$$\int_{F_0^0(M)} \langle J(f^{\natural} V_1^{\natural}), f^{\natural} V_2^{\natural} \rangle dv_{F_0^0(M)} = \int_{F_0^0(M)} \langle f^{\natural} V_1^{\natural}, J(f^{\natural} V_2^{\natural}) \rangle dv_{F_0^0(M)} .$$

We calculate the equation above by using Lemma 5,

$$\int_{S^0} (\varphi_2 - \varphi_1) f \left| \prod_{\alpha \in \Sigma^+(G) - \{\delta\}} \sin 2\pi \langle \alpha, K \rangle \right| dv_{S^0} \int_{K\,\mathrm{Exp}\, H_0} \langle V_1, V_2 \rangle dv_{K\,\mathrm{Exp}\, H_0}$$
$$= 0 .$$

Hence the lemma holds. Q.E.D.

THEOREM 4. *If* M *is* $SU(3)/SO(3)$, *then* $F_p^0(M)$ *is stable.*

PROOF. We may assume $p = o$. We put

$$\mathfrak{u} = \left\{ \begin{bmatrix} 0 & t & 0 \\ -t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; t \in \boldsymbol{R} \right\}$$

and

$$\alpha = \frac{\sqrt{-1}}{2\sqrt{3}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

$\mathfrak{u}$ is a maximal abelian subalgebra of $\mathfrak{k}$. We introduce a lexicographic ordering $<$ on $\mathfrak{u}$ such that $\alpha > 0$. Let $D(K)$ be the set of all equivalence classes of finite dimensional complex irreducible representations of K. It is well-known that $D(K)$ is identified with the following set:

$$\{m\alpha ;\ m = 0, 1, 2, \cdots\} .$$

Let $V(\lambda)$ be a representation space of an element of $\lambda \in D(K)$. Let $L^2(K \times_\rho N_{\mathrm{Exp}\, H}(F_0^0(M))^C)$ be the completion of $\Gamma(K \times_\rho N_{\mathrm{Exp}\, H}(F_0^0(M))^C)$ relative to the L^2-inner product for each $H \in S^0$. By virtue of the Peter-Weyl theorem, we get

$$L^2(K \times_\rho N_{\mathrm{Exp}\, H}(F_0^0(M))^C) = \sum_{\lambda \in D(K)} V(\lambda) \otimes \mathrm{Hom}_{K_1}(V(\lambda), (\boldsymbol{R}\bar{\delta} + \mathfrak{m}_{\bar{\delta}})^C)$$

We know that the negative of the Casimir operator C of K is a scalar operator $a_\lambda id$ on each $V(\lambda)\otimes\mathrm{Hom}_{K_1}(V(\lambda), (\boldsymbol{R}\bar{\delta}+\mathfrak{m}_{\bar{\delta}})^C)$ with $a_\lambda=4\pi^2(\lambda+\alpha, \lambda)$. Put

$$D(K)'=\{\lambda\in D(K)\,;\ \mathrm{Hom}_{K_1}(V(\lambda), (\boldsymbol{R}\bar{\delta}+\mathfrak{m}_{\bar{\delta}})^C)\neq\{0\}\}\,.$$

For a fixed $H_0\in S^0$, we denote φ_H the diffeomorphism from $K\,\mathrm{Exp}\,H_0$ onto $K\,\mathrm{Exp}\,H$ defined as follows:

$$\varphi_H: K\,\mathrm{Exp}\,H_0\to K\,\mathrm{Exp}\,H;\ k\,\mathrm{Exp}\,H_0\mapsto k\,\mathrm{Exp}\,H\ (k\in K)\,.$$

Let V be in $\Gamma_0(N(F_0^0(M)))^C$. Then $V|K\,\mathrm{Exp}\,H\in\Gamma(K\times_\rho N_{\mathrm{Exp}\,H}(F_0^0(M)))^C$ for each $H\in S^0$. Let $\{V_{\lambda,i}\}_{1\leq i\leq p(\lambda)}$ (where $p(\lambda)=\dim(V(\lambda)\otimes\mathrm{Hom}_{K_1}(V(\lambda), (\boldsymbol{R}\bar{\delta}+\mathfrak{m}_{\bar{\delta}})^C))$ be an orthonormal basis of $V(\lambda)\otimes\mathrm{Hom}_{K_1}(V(\lambda), (\boldsymbol{R}\delta+\mathfrak{m}^C))(\subset\Gamma(K\times_\rho N_{\mathrm{Exp}\,H_0}(F_0^0(M))^C)$. By (3.1), we have

$$J(V_{\lambda i}^{\shortparallel})=\frac{1}{\cos^2\frac{3}{2}\pi x}\left(a_\lambda-\frac{1}{2}\right)V_{\lambda,i}^{\shortparallel}\,.$$

Since $\{V_{\lambda,i}^{\shortparallel}\}$ forms an orthonormal base of $\Gamma(N(F_0^0(M))|K\,\mathrm{Exp}\,H)$ on each K-orbit $K\,\mathrm{Exp}\,H$, we can express

$$V=\sum_{\lambda\in D(K)'}\sum_{i=1}^{p(\lambda)}f_{\lambda,i}^{\shortparallel}V_{\lambda,i}^{\shortparallel}\,,\tag{3.2}$$

where $f_{\lambda,i}(H)=\int_{K\,\mathrm{Exp}\,H}\langle V|K\,\mathrm{Exp}\,H,\ V_{\lambda,i}^{\shortparallel}|K\,\mathrm{Exp}\,H\rangle dv_{K\,\mathrm{Exp}\,H}\in C_0^\infty(S^0)$.

The right-hand side of (3.2) is absolutely uniformly convergent to $V|K\,\mathrm{Exp}\,H$ on each K-orbit $K\,\mathrm{Exp}\,H$. We shall show the right-hand side of (3.2) is absolutely uniformly convergent to V on each compact subset of $F_0^0(M)$. We have

$$\begin{aligned}
f_{\lambda,i}(H)&=\int_{K\,\mathrm{Exp}\,H}\langle V|K\,\mathrm{Exp}\,H,\ V_{\lambda,i}^{\shortparallel}|K\,\mathrm{Exp}\,H\rangle dv_{K\,\mathrm{Exp}\,H}\\
&=\frac{1}{a_\lambda\cos^2\frac{3}{2}\pi x}\int_{K\,\mathrm{Exp}\,H}\langle V|K\,\mathrm{Exp}\,H,\ \Delta(V_{\lambda,i}^{\shortparallel}|K\,\mathrm{Exp}\,H)\rangle dv_{K\,\mathrm{Exp}\,H}\\
&=\frac{1}{a_\lambda\cos^2\frac{3}{2}\pi x}\int_{K\,\mathrm{Exp}\,H}\langle\Delta(V|K\,\mathrm{Exp}\,H),\ V_{\lambda,i}^{\shortparallel}|K\,\mathrm{Exp}\,H\rangle dv_{K\,\mathrm{Exp}\,H}\\
&=\frac{1}{a_\lambda^3\cos^6\frac{3}{2}\pi x}\int_{K\,\mathrm{Exp}\,H}\langle\Delta^3(V|\mathrm{Exp}\,H),\ V_{\lambda,i}^{\shortparallel}|K\,\mathrm{Exp}\,H\rangle dv_{K\,\mathrm{Exp}\,H}\,.
\end{aligned}$$

Thus, by using the Cauchy-Schwartz' inequality,

$$|f_{\lambda,i}(H)| \leqq \frac{1}{a_\lambda^3 \cos^6 \frac{3}{2}\pi x}\left(\int_{K \operatorname{Exp} H_0} \varphi_H^* \|\Delta_{K \operatorname{Exp} H}^3(V|K \operatorname{Exp} H)\|^2 dv_{K \operatorname{Exp} H_0}\right)^{1/2}.$$

Let D be any compact set in S^0. Put

$$E = \max_{H \in D}\left(\frac{1}{\cos^2 \frac{3}{2}\pi x}\int_{K \operatorname{Exp} H_0} \varphi_H^* \|\Delta_{K \operatorname{Exp} H}^3(V|K \operatorname{Exp} H)\|^2 dv_{K \operatorname{Exp} H_0}\right)^{1/2}.$$

Then

$$\|f_{\lambda,i} V_{\lambda,i}\| \leqq \frac{E\|V_{\lambda,i}\|}{a_\lambda^3}.$$

Hence it is sufficient to prove the following equation:

$$\lim_{\|\lambda\|\to\infty} \frac{\max\|V_{\lambda,i}\|}{a_\lambda^2} = 0.$$

Let $\{e_k\}_{1 \leqq k \leqq 1+m(\bar{\delta})}$ be an orthonormal basis of $\boldsymbol{R}\bar{\delta}+\mathfrak{m}_{\bar{\delta}}$. Put $d_\lambda = \dim V(\lambda)$. Then $d_\lambda = 2m+1$ for $\lambda = m\alpha$. Let $\rho(\lambda)$ be a representation of λ. We define $\rho(\lambda)_p^q$ as the following equation:

$$\rho(\lambda)_p^q(k) = \langle \rho(\lambda)(k) v_p, v_q \rangle \quad (k \in K),$$

where $\{v_p\}_{1 \leqq p \leqq d_\lambda}$ is a unitary frame of $V(\lambda)$. Then we express $V_{\lambda,i} = \sum a_{pq}^k \bar{\rho}(\lambda)_p^q e_k$ (for some $a_{pq}^k \in \boldsymbol{C}$, $d_\lambda = \sum |a_{pq}^k|^2$). By the Cauchy-Schwarts' inequality and the fact that each $|a_{pq}^k|^2 \leqq d_\lambda$,

$$\|V_{\lambda,i}\|^2 \leqq d_\lambda^2 (1+m(\bar{\delta})) \sum |a_{pq}^k|^2 |\rho(\lambda)_p^q|^2$$
$$\leqq d_\lambda^3 \sum |\rho(\lambda)_p^q|^2 = d_\lambda^4 (1+m(\bar{\delta})).$$

Thus we get

$$\frac{\max\|V_{\lambda,i}\|}{a_\lambda^2} \leqq \frac{(2m+1)^2 \sqrt{1+m(\bar{\delta})}}{\left\{\frac{1}{2}m(m+1)\right\}^2} \longrightarrow 0 \quad (\text{as } m \to \infty).$$

Hence the right-hand side of (3.2) is absolutely uniformly convergence on the compact subset. Thus, by Lemma 6, we have

$$\int_{F_0^0(M)} \langle JV, V \rangle dv_{F_0^0(M)}$$
$$= \sum_{\lambda \in D(K)'} \sum_{i=1}^{p(\lambda)} \int_{S^0} \left\{\|grad\, f_{\lambda,i}\|^2 + f_{\lambda,i}^2 \frac{a_\lambda - 1/2}{\cos^2 \frac{3}{2}\pi x}\right\} \left|\prod_{\alpha \in \Sigma^+(G) - \{\delta\}} \sin 2\pi \langle \alpha, H \rangle\right| dv_{S^0}$$
$$\times \int_{K \operatorname{Exp} H_0} \|V_{\lambda,i}\|^2 dv_{K \operatorname{Exp} H_0}.$$

Since $0 < a_\alpha < a_{2\alpha} = 1/2 < a_{3\alpha} \cdots$ and $\operatorname{Hom}_{K_1}(V(\alpha), (\boldsymbol{R}\bar{\delta}+\mathfrak{m}_{\bar{\delta}})^C) = \{0\}$, we get

$$\int_{F_o^0(M)} \langle JV, V\rangle dv_{F_o^0(M)} \geqq 0 \qquad \text{for each } V \in \Gamma(N(F_o^0(M)))^C .$$

Therefore $F_\alpha^0(M)$ is stable. Q.E.D.

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