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ON THE EXISTENCE OF FELLER SEMIGROUPS WITH DIRICHLET CONDITION

Dedicated to Professor Tosinobu Muramatu on his 60th birthday

By

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Abstract. This paper is devoted to the functional analytic approach to the problem of construction of Feller semigroups with Dirichlet boundary condition in the *characteristic* case. Intuitively, our result may be stated as follows: One can construct a Feller semigroup corresponding to such a diffusion phenomenon that a Markovian particle moves continuously in the state space until it "dies" at which time it reaches the set where the absorption phenomenon occurs.

Key words and phrases. Feller semigroups, Dirichlet condition, characteristic case.

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Introduction and Results

This paper is devoted to the functional analytic approach to the problem of construction of Feller semigroups with Dirichlet boundary condition. The problem of construction of such Feller semigroups has never before, to the author's knowledge, been studied in the *characteristic* case. In this paper, we consider the characteristic case and solve from the viewpoint of functional analysis the problem of construction of Markov processes with Dirichlet condition, which we formulate precisely. For detailed study of the elliptic or non-characteristic case, the reader might refer to Bony-Courrège-Priouret [BCP] and Cancelier [C].

Let D be a bounded domain of Euclidean space $\boldsymbol{R}^N$, with C^∞ boundary ∂D; its closure $\bar{D}=D\cup\partial D$ is an N-dimensional, compact C^∞ manifold with boundary. Let A be a second-order, *degenerate* elliptic differential operator with real coefficients such that

$$Au(x)=\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x)+\sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x_i}(x)+c(x)u(x),$$

where:

1) $a^{ij}\in C^\infty(\boldsymbol{R}^N)$, $a^{ij}=a^{ji}$ and

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j\geqq 0, \qquad x\in\boldsymbol{R}^N,\ \xi\in\boldsymbol{R}^N .$$

2) $b^i\in C^\infty(\boldsymbol{R}^N)$.

3) $c\in C^\infty(\boldsymbol{R}^N)$ and $c\leqq 0$ on $\bar{D}$.

Following Fichera [F], we introduce a function $b(x')$ on the boundary ∂D by the formula:

$$b(x')=\sum_{i=1}^{N}\Big(b^i(x')-\sum_{j=1}^{N}\frac{\partial a^{ij}}{\partial x_j}(x')\Big)n_i ,$$

where $\boldsymbol{n}=(n_1, n_2, \cdots, n_N)$ is the unit interior normal to the boundary ∂D. We divide the boundary ∂D into the following four disjoint subsets:

$$\Sigma_3=\{x'\in\partial D;\ \sum_{i,j=1}^{N} a^{ij}(x')n_i n_j>0\},$$

$$\Sigma_2=\{x'\in\partial D\,;\ \sum_{i,j=1}^{N} a^{ij}(x')n_i n_j=0,\ b(x')<0\},$$

$$\Sigma_1=\{x'\in\partial D\,;\ \sum_{i,j=1}^{N} a^{ij}(x')n_i n_j=0,\ b(x')>0\},$$

$$\Sigma_0=\{x'\in\partial D\,;\ \sum_{i,j=1}^{N} a^{ij}(x')n_i n_j=0,\ b(x')=0\}.$$

Our fundamental hypothesis for the operator A is the following (cf. Figure 1):

(*H*) *Each set* Σ_i *consists of a finite number of connected* hypersurfaces.
It is worth pointing out (cf. [OR], [SV]) that one may impose a boundary condition only on the set

$$M=\Sigma_2\cup\Sigma_3\,.$$

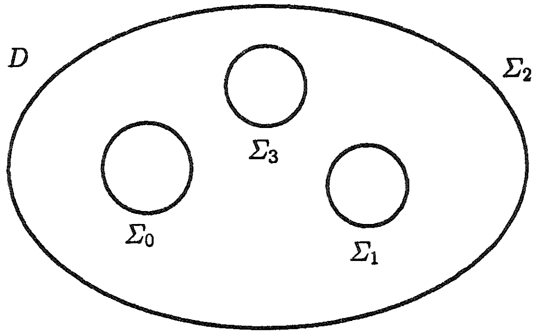

Figure 1.

Let $C(\bar{D})$ be the space of real-valued, continuous functions f on $\bar{D}$. We equip the space $C(\bar{D})$ with the topology of uniform convergence on the whole $\bar{D}$; hence it is a Banach space with the maximum norm

$$\|f\|=\max_{x\in\bar{D}}|f(x)|\,.$$

Now we introduce a subspace of $C(\bar{D})$:

$$C_0(\bar{D}\setminus M)=\{u\in C(\bar{D})\,;\ u=0 \text{ on } \Sigma_2\cup\Sigma_3\}.$$

The space $C_0(\bar{D}\setminus M)$ is a closed subspace of $C(\bar{D})$; hence it is a Banach space.

A strongly continuous semigroup $\{T_t\}_{t\geq 0}$ on the space $C_0(\bar{D}\setminus M)$ is called a *Feller semigroup* on $\bar{D}\setminus M$ if it is non-negative and contractive on $C_0(\bar{D}\setminus M)$:

$$f\in C_0(\bar{D}\setminus M),\ 0\leqq f\leqq 1 \text{ on } \bar{D}\setminus M \Longrightarrow 0\leqq T_t f\leqq 1 \text{ on } \bar{D}\setminus M\,.$$

It is known (cf. [Ta, Chapter 9]) that if T_t is a Feller semigroup on $\bar{D}\setminus M$, then there exists a unique Markov transition function p_t on $\bar{D}\setminus M$ such that

$$T_t f(x) = \int_{\bar{D}\setminus M} p_t(x, dy) f(y), \qquad f \in C_0(\bar{D}\setminus M).$$

Furthermore, the function p_t is the transition function of some strong *Markov process*; hence the value $p_t(x, E)$ expresses the transition probability that a Markovian particle starting at position x will be found in the set E at time t.

The next theorem asserts that there exists a Feller semigroup on $\bar{D}$ corresponding to such a diffusion phenomenon that a Markovian particle moves continuously in the state space $\bar{D}$ until it "dies" at which time it reaches the set $\Sigma_2 \cup \Sigma_3$.

THEOREM 1. *Assume that the operator A satisfies hypothesis (H)*:

(H) *Each set Σ_i consists of a finite number of connected* hypersurfaces. *We define a linear operator $\mathcal{A}$ from the space $C_0(\bar{D}\setminus M)$ into itself as follows.*

(1) *The domain $D(\mathcal{A})$ of $\mathcal{A}$ is the space*

$$D(\mathcal{A}) = \{u \in C^2(\bar{D});\ u = Au = 0 \ \text{on}\ \Sigma_2 \cup \Sigma_3\}.$$

(2) $\mathcal{A}u = Au,\ u \in D(\mathcal{A})$.

Then the operator $\mathcal{A}$ is closable in the space $C_0(\bar{D}\setminus M)$, and its minimal closed extension $\bar{\mathcal{A}}$ is the infinitesimal generator of some Feller semigroup $\{T_t\}_{t\geq 0}$ on $\bar{D}\setminus M$.

Theorem 1 is proved by Bony-Courrège-Priouret [BCP] in the elliptic case (cf. [BCP, Théorème XVI]) and then by Cancelier [C] in the non-characteristic case: $\partial D = \Sigma_3$ (cf. [C, Théorème 7.2]).

By a version of the Hille-Yosida theorem in semigroup theory, the proof of Theorem 1 is reduced to the study of the Dirichlet problem in the theory of partial differential equations. The essential step in the proof is the following existence and uniqueness theorem for the Dirichlet problem in the framework of Hölder spaces:

THEOREM 2. *Assume that hypothesis (H) is satisfied and that*

$$c < 0 \quad \text{on}\ \bar{D},$$

and

$$c^* = \sum_{i,j=1}^{N} \frac{\partial^2 a^{ij}}{\partial x_i \partial x_j} - \sum_{i=1}^{N} \frac{\partial b^i}{\partial x_i} + c < 0 \qquad \text{on}\ \bar{D}.$$

Then, for each integer $m \geq 2$, one can find a constant $\lambda = \lambda(m) > 0$ such that, for any function f in the space $C^{2m+2+2\theta}(\bar{D})$, $0 < \theta < 1$, there exists a unique solution $u \in C^{m+\theta}(\bar{D})$ of the Dirichlet problem:

$$(*)\qquad \begin{cases} (A-\lambda)u=f & \text{in } D, \\ u=0 & \text{on } \Sigma_2\cup\Sigma_3. \end{cases}$$

Moreover, the solution u satisfies the inequality

$$\|u\|_{C^{m+\theta}(\bar{D})}\leqq C_{m+\theta}(\lambda)\|f\|_{C^{2m+2+2\theta}(\bar{D})},$$

where $C_{m+\theta}(\lambda)>0$ *is a constant independent of* f.

Theorem 2 is an improvement of Theorem 1.8.2 of Oleĭnik-Radkevič [OR]. We remark that Theorem 2 is proved by Cancelier [C] in the non-characteristic case: $\partial D=\Sigma_3$ (cf. [C, Théorème 4.5]).

The rest of this paper is organized as follows.

Section 1 provides a brief description of the basic definitions and results about Feller semigroups, which forms a functional analytic background for the proof of Theorem 1. Our proof of Theorem 1 is based on a Feller semigroup version of the Hille-Yosida theorem (Theorem 1.4) in terms of the maximum principle.

In Section 2, we study the Dirichlet problem

$$(D)\qquad \begin{cases} Au=f & \text{in } D, \\ u=g & \text{on } \Sigma_2\cup\Sigma_3, \end{cases}$$

in the framework of spaces of bounded measurable functions, and prove existence and uniqueness theorems for problem (D) (Theorem 2.3 and Theorem 2.6), by using a method of *elliptic regularization* as in Oleĭnik-Radkevič [OR] and also as in Cancelier [C]. It is hypothesis (H) that makes it possible to develop the basic machinery of Oleĭnik-Radkevič [OR] with a minimum of bother and the principal ideas can be presented more concretely and explicitly.

In Section 3, we prove regularity theorems (Theorem 3.1 and Theorem 3.5) for the weak solutions of problem (D) constructed in Section 2 in the framework of Hölder spaces. In the proof, *uniform estimates* for approximate solutions of problem (D) play an essential role (Lemma 3.4 and Lemma 3.7). Theorem 2 follows from these theorems by a well-known interpolation argument.

The final Section 4 is devoted to the proof of Theorem 1. We verify all the conditions of the generation theorem of Feller semigroups (Theorem 1.4) in Section 1.

The author would like to express his hearty thanks to the referee whose helpful criticisms of the manuscript resulted in a number of improvements.

1. Theory of Feller Semigroups

This section provides a brief description of the basic definitions and results about Feller semigroups, which forms a functional analytic background for the proof of Theorem 1.

1.1. Markov Transition Functions and Feller Semigroups

Let (K, ρ) be a locally compact, separable metric space and $\mathcal{B}$ the σ-algebra of all Borel sets in K.

A function $p_t(x, E)$, defined for all $t \geqq 0$, $x \in K$ and $E \in \mathcal{B}$, is called a (temporally homogeneous) *Markov transition function* on K if it satisfies the following four conditions:

(a) $p_t(x, \cdot)$ is a non-negative measure on $\mathcal{B}$ and $p_t(x, K) \leqq 1$ for each $t \geqq 0$ and each $x \in K$.

(b) $p_t(\cdot, E)$ is a Borel measurable function for each $t \geqq 0$ and each $E \in \mathcal{B}$.

(c) $p_0(x, \{x\}) = 1$ for each $x \in K$.

(d) (The Chapmen-Kolmogorov equation) For any $t, s \geqq 0$, $x \in K$ and any $E \in \mathcal{B}$, we have

$$p_{t+s}(x, E) = \int_K p_t(x, dy) p_s(y, E). \tag{1.1}$$

The value $p_t(x, E)$ expresses the transition probability that a physical particle starting at position x will be found in the set E at time t. Equation (1.1) expresses the idea that a particle "start afresh"; this property is called the *Markov property*.

We add a point ∂ to K as the point at infinity if K is not compact, and as an isolated point if K is compact; so the space $K_\partial = K \cup \{\partial\}$ is compact.

Let $C(K)$ be the space of real-valued, bounded continuous functions on K. The space $C(K)$ is a Banach space with the supremum norm

$$\|f\| = \sup_{x \in K} |f(x)|.$$

We say that a function $f \in C(K)$ converges to zero as $x \to \partial$ if, for each $\varepsilon > 0$, there exists a compact subset E of K such that

$$|f(x)| < \varepsilon, \qquad x \in K \setminus E,$$

and write $\lim_{x \to \partial} f(x) = 0$. We let

$$C_0(K) = \left\{ f \in C(K)\, ;\ \lim_{x \to \partial} f(x) = 0 \right\}.$$

The space $C_0(K)$ may be identified with the subspace of $C(K_\partial)$ which consists of all functions f satisfying $f(\partial)=0$:

$$C_0(K)=\{f\in C(K_\partial)\,;\, f(\partial)=0\}.$$

A Markov transition function p_t is called a C_0-*function* if we have

$$f\in C_0(K)\Longrightarrow T_t f=\int_K p_t(\cdot,\, dy)f(y)\in C_0(K).$$

A Markov transition function p_t on K is said to be *uniformly stochastically continuous* on K if the following condition is satisfied: For each $\varepsilon>0$ and each compact $E\subset K$, we have

$$\lim_{t\downarrow 0}\sup_{x\in E}\,[1-p_t(x,\, U_\varepsilon(x))]=0\,,$$

where $U_\varepsilon(x)=\{y\in K\,;\,\rho(x,\, y)<\varepsilon\}$ is an ε-neighborhood of x.

Then we have the following (cf. [Ta, Theorem 9.2.3]):

THEOREM 1.1. *Let p_t be a C_0-transition function on K. Then the associated operators $\{T_t\}_{t\geqq 0}$, defined by the formula*

$$T_t f(x)=\int_K p_t(x,\, dy)f(y),\qquad f\in C_0(K), \tag{1.2}$$

is strongly continuous in t on $C_0(K)$ if and only if p_t is uniformly stochastically continuous on K and satisfies the following condition (L):

(L) *For each $s>0$ and each compact $E\subset K$, we have*

$$\lim_{x\to\partial}\sup_{0\leqq t\leqq s} p_t(x,\, E)=0\,.$$

A family $\{T_t\}_{t\geqq 0}$ of bounded linear operators acting on $C_0(K)$ is called a *Feller semigroup* on K if it satisfies the following three conditions:

(i) $T_{t+s}=T_t\cdot T_s$, $t,\, s\geqq 0$; $T_0=I=$the identity.

(ii) The family $\{T_t\}$ is strongly continuous in t for $t\geqq 0$:

$$\lim_{s\downarrow 0}\|T_{t+s}f-T_t f\|=0\,,\qquad f\in C_0(K)\,.$$

(iii) The family $\{T_t\}$ is non-negative and contractive on $C_0(K)$:

$$f\in C_0(K),\quad 0\leqq f\leqq 1 \quad\text{on } K \Longrightarrow 0\leqq T_t f\leqq 1 \quad\text{on } K.$$

The next theorem gives a characterization of Feller semigroups in terms of Markov transition functions (cf. [Ta, Theorem 9.2.6]):

THEOREM 1.2. *If p_t is a uniformly stochastically continuous C_0-transition function on K and satisfies condition (L), then its associated operators $\{T_t\}_{t\geqq 0}$*

form a Feller semigroup on K.

Conversely, if $\{T_t\}_{t\geqq 0}$ *is a Feller semigroup on K, then there exists a uniformly stochastically continuous* C_0*-transition* p_t *on K, satisfying condition (L), such that formula* (1.2) *holds.*

1.2. Generation Theorems of Feller Semigroups

If $\{T_t\}_{t\geqq 0}$ is a Feller semigroup on K, then we define its *infinitesimal generator* $\mathfrak{A}$ by the formula

$$\mathfrak{A}u=\lim_{t\downarrow 0}\frac{T_t u-u}{t}, \tag{1.3}$$

provided that the limit (1.3) exists in the space $C_0(K)$.

The next theorem is a version of the Hille-Yosida theorem adapted to the present context (cf. [Ta, Theorem 9.3.1 and Corollary 9.3.2]):

THEOREM 1.3. (i) *Let* $\{T_t\}_{t\geqq 0}$ *be a Feller semigroup on K and* $\mathfrak{A}$ *its infinitesimal generator. Then we have the following:*

(a) *The domain* $D(\mathfrak{A})$ *is everywhere dense in the space* $C_0(K)$.

(b) *For each* $\alpha>0$, *the equation* $(\alpha I-\mathfrak{A})u=f$ *has a unique solution u in* $D(\mathfrak{A})$ *for any* $f\in C_0(K)$. *Hence, for each* $\alpha>0$, *the Green operator* $(\alpha I-\mathfrak{A})^{-1}: C_0(K)\to C_0(K)$ *can be defined by the formula*

$$u=(\alpha I-\mathfrak{A})^{-1}f, \qquad f\in C_0(K).$$

(c) *For each* $\alpha>0$, *the operator* $(\alpha I-\mathfrak{A})^{-1}$ *is non-negative on the space* $C_0(K)$:

$$f\in C_0(K),\quad f\geqq 0 \quad \text{on } K \Longrightarrow (\alpha I-\mathfrak{A})^{-1}f\geqq 0 \quad \text{on } K.$$

(d) *For each* $\alpha>0$, *the operator* $(\alpha I-\mathfrak{A})^{-1}$ *is bounded on the space* $C_0(K)$ *with norm*

$$\|(\alpha I-\mathfrak{A})^{-1}\|\leqq\frac{1}{\alpha}.$$

(ii) *Conversely, if* $\mathfrak{A}$ *is a linear operator from the space* $C_0(K)$ *into itself satisfying condition* (a) *and if there is a constant* $\alpha_0\geqq 0$ *such that, for all* $\alpha>\alpha_0$, *conditions* (b) *through* (d) *are satisfied, then* $\mathfrak{A}$ *is the infinitesimal generator of some Feller semigroup* $\{T_t\}_{t\geqq 0}$ *on K.*

We conclude this section by giving useful criteria in terms of the *maximum principle* (cf. [BCP, Théorème de Hille-Yosida-Ray]; [Ta, Theorem 9.3.3 and Corollary 9.3.4]):

THEOREM 1.4. *Let K be a locally compact metric space and let B be a linear operator from the space $C_0(K)$ into itself. We assume that:*

(α) *The domain $D(B)$ of B is everywhere dense in the space $C_0(K)$.*

(β) *If $u \in D(B)$ and $\sup_K u > 0$, then there exists a point x of K such that*

$$\begin{cases} u(x) = \sup_K u\,, \\ Bu(x) \leqq 0\,. \end{cases}$$

(γ) *For some $\alpha_0 \geqq 0$, the range $R(\alpha_0 I - B)$ of $\alpha_0 I - B$ is everywhere dense in the space $C_0(K)$.*

Then the operator B is closable in the space $C_0(K)$, and its minimal closed extension $\bar{B}$ is the infinitesimal generator of some Feller semigroup $\{T_t\}_{t \geqq 0}$ on K.

2. The Dirichlet Problem—(1)—

In this section, we shall study the Dirichlet problem in the framework of spaces of bounded measurable functions, and prove existence and uniqueness theorems for problem (D), by using a method of *elliptic regularization* as in Oleĭnik-Radkevič [OR] and also as in Cancelier [C].

2.1. Function Spaces

First we recall the basic definitions and facts about the function spaces which will be used in subsequent sections.

If Ω is an open subset of Euclidean space $\boldsymbol{R}^n$, we let

$L^\infty(\Omega)$ = the space of equivalence classes of essentially bounded, Lebesgue measurable functions u on Ω.

The space $L^\infty(\Omega)$ is a Banach space with the norm

$$\|u\|_\infty = \operatorname{ess\,sup}_{x \in \Omega} |u(x)|\,.$$

If k is a positive integer, we let

$W^{k,\infty}(\Omega)$ = the space of equivalence classes of functions $u \in L^\infty(\Omega)$ all of whose derivatives $\partial^\alpha u$, $|\alpha| \leqq k$, in the sense of distributions are in $L^\infty(\Omega)$.

The space $W^{k,\infty}(\Omega)$ is a Banach space with the norm

$$\|u\|_{k,\infty} = \sum_{|\alpha| \leqq k} \|\partial^\alpha u\|_\infty\,.$$

Let $0 < \theta < 1$. A function u defined on Ω is said to be *Hölder continuous* with exponent θ if the quantity

$$[u]_{\theta;\Omega}=\sup_{\substack{x,y\in\Omega\\ x\neq y}}\frac{|u(x)-u(y)|}{|x-y|^{\theta}}$$

is finite. We say that u is *locally Hölder continuous* with exponent θ if it is Hölder continuous with exponent θ on compact subsets of Ω.

We let

$C^{\theta}(\Omega)$=the space of functions in $C(\Omega)$ which are locally Hölder continuous with exponent θ on Ω.

If k is a positive integer, we let

$C^{k+\theta}(\Omega)$=the space of functions in $C^k(\Omega)$ all of whose k-th order derivatives are locally Hölder continuous with exponents θ on Ω.

Now assume that Ω is *bounded*. We let

$C(\bar{\Omega})$=the space of functions in $C(\Omega)$ having continuous extensions to the closure $\bar{\Omega}$ of Ω.

If k is a positive integer, we let

$C^k(\bar{\Omega})$=the space of functions in $C^k(\Omega)$ all of whose derivatives of order $\leq k$ have continuous extensions to $\bar{\Omega}$.

The space $C^k(\bar{\Omega})$ is a Banach space with the norm

$$\|u\|_{C^k(\bar{\Omega})}=\max_{|\alpha|\leq k}\sup_{x\in\Omega}|\partial^{\alpha}u(x)|.$$

Further we let

$C^{\theta}(\bar{\Omega})$=the space of functions in $C(\bar{\Omega})$ which are Hölder continuous with exponent θ on Ω.

If k is a positive integer, we let

$C^{k+\theta}(\bar{\Omega})$=the space of functions in $C^k(\bar{\Omega})$ all of whose k-th order derivatives are Hölder continuous with exponent θ on Ω.

The space $C^{k+\theta}(\bar{\Omega})$ is a Banach space with the norm

$$\|u\|_{C^{k+\theta}(\bar{\Omega})}=\|u\|_{C^k(\bar{\Omega})}+\max_{|\alpha|=k}[\partial^{\alpha}u]_{\theta;\Omega}.$$

If M is an n-dimensional compact C^{∞} manifold without boundary and m is a non-negative integer, then the spaces $W^{m,\infty}(M)$ and $C^{m+\theta}(M)$ are defined respectively to be locally the spaces $W^{m,\infty}(\boldsymbol{R}^n)$ and $C^{m+\theta}(\boldsymbol{R}^n)$, upon using local coordinate systems flattening out M, together with a partition of unity. The norms of the spaces $W^{m,\infty}(M)$ and $C^{m+\theta}(M)$ will be denoted by $\|\cdot\|_{m,\infty}$ and $\|\cdot\|_{C^{m+\theta}(M)}$, respectively.

We recall the following results (cf. [Tr]):

I) If k is a positive integer, then we have

$$W^{k,\infty}(M)=\left\{\varphi\in C^{k-1}(M);\ \max_{|\alpha|\leq k-1}\sup_{\substack{x,y\in M\\ x\neq y}}\frac{|\partial^\alpha\varphi(x)-\partial^\alpha\varphi(y)|}{|x-y|}<\infty\right\},$$

where $|x-y|$ is the geodesic distance between x and y with respect to the Riemannian metric of M.

II) The space $C^{k+\theta}(M)$ is a *real interpolation space* between the spaces $W^{k,\infty}(M)$ and $W^{k+1,\infty}(M)$; more precisely we have

$$\begin{aligned}C^{k+\theta}(M)&=(W^{k,\infty}(M),\ W^{k+1,\infty}(M))_{\theta,\infty}\\&=\left\{u\in W^{k,\infty}(M);\ \sup_{t>0}\frac{K(t,u)}{t^\theta}<\infty\right\},\end{aligned}$$

where

$$K(t,u)=\inf_{u=u_0+u_1}(\|u_0\|_{k,\infty}+t\|u_1\|_{k+1,\infty}).$$

2.2. Formulation of the Dirichlet Problem

Let D be a bounded domain of Euclidean space $\boldsymbol{R}^N$ with C^∞ boundary ∂D. Its closure $\bar{D}=D\cup\partial D$ is an N-dimensional, compact C^∞ manifold with boundary.

We let

$$Au(x)=\sum_{i,j=1}^N a^{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}(x)+\sum_{i=1}^N b^i(x)\frac{\partial u}{\partial x_i}(x)+c(x)u(x)$$

be a second-order, *degenerate* elliptic differential operator with real coefficients such that:

1) $a^{ij}\in C^\infty(\boldsymbol{R}^N)$, $a^{ij}=a^{ji}$ and

$$\sum_{i,j=1}^N a^{ij}(x)\xi_i\xi_j\geqq 0,\qquad x\in\boldsymbol{R}^N,\ \xi\in\boldsymbol{R}^N.$$

2) $b^i\in C^\infty(\boldsymbol{R}^N)$.

3) $c\in C^\infty(\boldsymbol{R}^N)$ and $c\leqq 0$ on $\bar{D}$.

Following Fichera [F], we introduce a function $b(x')$ on the boundary ∂D by the formula:

$$b(x')=\sum_{i=1}^N\left(b^i(x')-\sum_{i=1}^N\frac{\partial a^{ij}}{\partial x_j}(x')\right)n_i,$$

where $\boldsymbol{n}=(n_1, n_2, \cdots, n_N)$ is the unit interior normal to the boundary ∂D. The function b will be called the *Fichera function* for the operator A. It is easy to verify that the Fichera function b is invariantly defined on the characteristic set:

$$\Sigma^0=\left\{x'\in\partial D\,;\ \sum_{i,j=1}^{N}a^{ij}(x')n_in_j=0\right\}.$$

Let A^* be the formal adjoint operator for A:

$$A^*v(x)=\sum_{i,j=1}^{N}a^{ij}(x)\frac{\partial^2 v}{\partial x_i\partial x_j}(x)+\sum_{i=1}^{N}\left(2\sum_{j=1}^{N}\frac{\partial a^{ij}}{\partial x_j}(x)-b^i(x)\right)\frac{\partial v}{\partial x_i}(x)$$
$$+\left(\sum_{i,j=1}^{N}\frac{\partial^2 a^{ij}}{\partial x_i\partial x_j}(x)-\sum_{i=1}^{N}\frac{\partial b^i}{\partial x_i}(x)+c(x)\right)v(x).$$

It is easy to see that the Fichera function b^* for the operator A^* is given by

$$b^*(x')=-b(x')=-\sum_{i=1}^{N}\left(b^i(x')-\sum_{j=1}^{N}\frac{\partial a^{ij}}{\partial x_j}(x')\right)n_i.$$

In order to formulate precisely the Dirichlet problem for the operator A, we divide the boundary ∂D into the following four disjoint subsets:

$$\Sigma_3=\partial D\setminus\Sigma^0=\left\{x'\in\partial D\,;\ \sum_{i,j=1}^{N}a^{ij}(x')n_in_j>0\right\},$$

$$\Sigma_2=\left\{x'\in\partial D\,;\ \sum_{i,j=1}^{N}a^{ij}(x')n_in_j=0,\ b(x')<0\right\},$$

$$\Sigma_1=\left\{x'\in\partial D\,;\ \sum_{i,j=1}^{N}a^{ij}(x')n_in_j=0,\ b(x')>0\right\},$$

$$\Sigma_0=\left\{x'\in\partial D\,;\ \sum_{i,j=1}^{N}a^{ij}(x')n_in_j=0,\ b(x')=0\right\}.$$

We remark that the sets Σ_3, Σ_2, Σ_1 and Σ_0 are all invariantly defined.

Our fundamental hypothesis for the operator A is the following:

(*H*) *Each set Σ_i consists of a finite number of connected* hypersurfaces.
This hypothesis makes it possible to develop the basic machinery of Oleĭnik-Radkevič [OR] with a minimum of bother and the principal ideas can be presented more concretely and explicitly.

We shall consider the following Dirichlet problem: For given bounded measurable functions f and g defined in D and on $\Sigma_2\cup\Sigma_3$, respectively, find a bounded measurable function u in D such that

$$(D)\qquad \begin{cases} Au=f & \text{in } D,\\ u=g & \text{on } \Sigma_2\cup\Sigma_3.\end{cases}$$

Now we give the precise definition of a weak solution of problem (D):

DEFINITION 2.1. A bounded measurable function u in D is called a *weak solution* of problem (D) if, for any function $v \in C^2(\bar{D})$ satisfying $v=0$ on $\Sigma_1 \cup \Sigma_3$, we have

$$\iint_D u \cdot A^* v\,dx = \iint_D f v\,dx - \int_{\Sigma_3} g \frac{\partial v}{\partial \nu} d\sigma + \int_{\Sigma_2} b g v\,d\sigma \,, \tag{2.1}$$

where $\partial/\partial\nu$ is the conormal derivative associated with the operator A:

$$\frac{\partial}{\partial \nu} = \sum_{i=1}^{N} a^{ij} n_j \frac{\partial}{\partial x_i},$$

and b is the Fichera function and $d\sigma$ is the surface element of ∂D.

Our definition of a weak solution may be justified by using the following Green formula for the operators A and A^* (cf. [OR, formula (1.1.14)]):

THEOREM 2.2. *For all functions u and v in $C^2(\bar{D})$, we have*

$$\iint_D (Au \cdot v - u \cdot A^* v)\,dx = -\int_{\Sigma_3} \left(\frac{\partial u}{\partial \nu} v - u \frac{\partial v}{\partial \nu}\right) d\sigma - \int_{\partial D \setminus \Sigma_0} b u v\,d\sigma \,. \tag{2.2}$$

2.3. Existence Theorem for Problem (D)

First we prove the following *existence* theorem for problem (D) (cf. [OR, Theorem 1.5.1]):

THEOREM 2.3. *Assume that hypothesis (H) is satisfied and that*

$$c < 0 \quad \text{on } \bar{D} \,. \tag{2.3}$$

Then, for any $f \in L^\infty(D)$ and any $g \in L^\infty(\Sigma_2 \cup \Sigma_3)$, there exists a weak solution $u \in L^\infty(D)$ of the Dirichlet problem:

$$\begin{cases} Au = f & \text{in } D \,, \\ u = g & \text{on } \Sigma_2 \cup \Sigma_3 \,. \end{cases} \tag{D}$$

Furthermore, the solution u satisfies the inequality

$$\operatorname{ess\,sup}_D |u| \leq \max\left(\frac{1}{c_0} \operatorname{ess\,sup}_D |f| \,,\ \operatorname{ess\,sup}_{\Sigma_2 \cup \Sigma_3} |g|\right), \tag{2.4}$$

where

$$c_0 = \min_{\bar{D}} (-c) > 0 \,.$$

PROOF. I) First we construct approximate solutions of problems (D) by making good use of a method of *elliptic regularization*, just as in Oleĭnik-Radkevič [OR].

Let f be an arbitrary function in the space $L^\infty(D)$, and choose a sequence $\{f_n\}_{n=1}^\infty$ in the space $C^\theta(\bar{D})$ $(0<\theta<1)$ such that

(2.5a) $$\max_{\bar{D}} |f_n| \leqq \text{ess sup}_D |f|,$$

(2.5b) $$f_n \longrightarrow f \quad \text{in } L^2(D) \text{ as } n \to \infty,$$

and also a sequence $\{g_n\}_{n=1}^\infty$ in the space $C^{2+\theta}(\bar{D})$ $(0<\theta<1)$ such that

(2.6a) $$\max_{\partial D} |g_n| \leqq \text{ess sup}_{\Sigma_2 \cup \Sigma_3} |g|,$$

(2.6b) $$g_n \longrightarrow g \quad \text{in } L^2(\Sigma_2 \cup \Sigma_3) \text{ as } n \to \infty.$$

This can be done by using regularizations (mollifiers) of f and g.

Now let $u_{\varepsilon,n}$ be a solution of the Dirichlet problem for the *elliptic* operators $A_\varepsilon = \varepsilon\Delta + A$ $(\varepsilon>0)$:

$(D_{\varepsilon,n})$ $$\begin{cases} A_\varepsilon u_{\varepsilon,n} = f_n & \text{in } D, \\ u_{\varepsilon,n} = g_n & \text{on } \partial D, \end{cases}$$

where $\Delta = \sum_{i=1}^N \partial^2/\partial x_i^2$ is the usual Laplacian. We know (cf. [GT]) that such a solution $u_{\varepsilon,n}$ of problem $(D_{\varepsilon,n})$ exists and is unique in the space $C^{2+\theta}(\bar{D})$. Thus, applying the maximum principle (cf. Theorem A.2) to the elliptic operators A_ε, we obtain from inequalities (2.5a) and (2.6a) that

(2.7) $$\begin{aligned} \sup_D |u_{\varepsilon,n}| &\leqq \max\left(\frac{1}{c_0} \max_{\bar{D}} |f_n|, \max_{\partial D} |g_n|\right) \\ &\leqq \max\left(\frac{1}{c_0} \text{ess sup}_D |f|, \text{ess sup}_{\Sigma_2 \cup \Sigma_3} |g|\right). \end{aligned}$$

II) Next we show that the limit function u_n of $u_{\varepsilon,n}$ when $\varepsilon \downarrow 0$ is a weak solution of the Dirichlet problem for the operator A:

(D_n) $$\begin{cases} A u_n = f_n & \text{in } D, \\ u_n = g_n & \text{on } \Sigma_2 \cup \Sigma_3. \end{cases}$$

If we let

$$z_{\varepsilon,n} = u_{\varepsilon,n} - g_n,$$

then it follows that $z_{\varepsilon,n} \in C^{2+\theta}(\bar{D})$ and satisfies:

$$\begin{cases} A_\varepsilon z_{\varepsilon,n} = f_n - A_\varepsilon g_n & \text{in } D, \\ z_{\varepsilon,n} = 0 & \text{on } \partial D. \end{cases}$$

II-1) In order to estimate the $z_{\varepsilon,n}$, we need the following lemma (cf. [OR, Lemmas 1.5.1 and 1.8.3]):

LEMMA 2.4. *Let* $f\in C^{\theta}(\bar{D})$ $(0<\theta<1)$ *and let* $u_{\varepsilon}\in C^{2+\theta}(\bar{D})$ *be a unique solution of the Dirichlet problem for the elliptic operators* $A_{\varepsilon}=\varepsilon\Delta+A$ $(\varepsilon>0)$:

$$(D_{\varepsilon})\qquad \begin{cases} A_{\varepsilon}u_{\varepsilon}=f & \text{in } D\,, \\ u_{\varepsilon}=0 & \text{on } \partial D\,. \end{cases}$$

If hypothesis (*H*) *and condition* (2.3) *are satisfied, then the solution* u_{ε} *satisfies the estimates*

$$\max_{\Sigma_3}|\operatorname{grad} u_{\varepsilon}|\leqq M\|f\|_{C(\bar{D})}\,, \tag{2.8a}$$

$$\max_{\Sigma_2}|\operatorname{grad} u_{\varepsilon}|\leqq M\|f\|_{C(\bar{D})}\,, \tag{2.8b}$$

$$\max_{\Sigma_0}|\operatorname{grad} u_{\varepsilon}|\leqq \frac{C}{\sqrt{\varepsilon}}\|f\|_{C(\bar{D})}\,, \tag{2.8c}$$

where $M>0$ *and* $C>0$ *are constants independent of* $\varepsilon>0$.

PROOF. Let x_0' be an arbitrary point of the set $\Sigma_3\cup\Sigma_2\cup\Sigma_0$. We choose a local coordinate system $(y_1, y_2, \cdots, y_N)$ in a tubular neighborhood $\mathcal{U}$ of x_0' such that:

$$\begin{cases} x_0'=0\,, \\ D=\{y_N>0\}\,, \\ \partial D=\{y_N=0\}\,, \end{cases}$$

and assume that, in terms of this coordinate system, the operator $A_{\varepsilon}=\varepsilon\Delta+A$ is of the form

$$\begin{aligned} A_{\varepsilon}=\varepsilon\Big(\sum_{i,j=1}^{N}\mu^{ij}\frac{\partial^2}{\partial y_i\partial y_j}+\sum_{i=1}^{N}\nu^i\frac{\partial}{\partial y_i}\Big)+\alpha^{NN}\frac{\partial^2}{\partial y_N^2} \\ +2\sum_{j=1}^{N-1}\alpha^{Nj}\frac{\partial^2}{\partial y_N\partial y_j}+\sum_{i,j=1}^{N-1}\alpha^{ij}\frac{\partial^2}{\partial y_i\partial y_j}+\beta^N\frac{\partial}{\partial y_N}+\sum_{i=1}^{N-1}\beta^i\frac{\partial}{\partial y_i}+c. \end{aligned} \tag{2.9}$$

We remark that:

(a) $\alpha^{NN}(0)>0$ if $x_0'\in\Sigma_3$.
(b) $\alpha^{NN}(0)=0$ and $\beta^N(0)<0$ if $x_0'\in\Sigma_2$.
(c) $\alpha^{NN}(0)=0$ and $\beta^N(0)=0$ if $x_0'\in\Sigma_0$.

In order to prove estimate (2.8), it suffices to prove that

$$\Big|\frac{\partial u_{\varepsilon}}{\partial y_N}(0)\Big|\leqq M\|f\|_{C(\bar{D})}\,, \tag{2.8a$'$}$$

$$\Big|\frac{\partial u_{\varepsilon}}{\partial y_N}(0)\Big|\leqq M\|f\|_{C(\bar{D})}\,, \tag{2.8b$'$}$$

(2.8c′) $$\left|\frac{\partial u_\varepsilon}{\partial y_N}(0)\right| \leqq \frac{C}{\sqrt{\varepsilon}} \|f\|_{C(\bar{D})} ,$$

since $u_\varepsilon=0$ on ∂D and hence $\partial u_\varepsilon/\partial y_j=0$ on ∂D, for $1\leqq j\leqq N-1$.

(a) First we prove estimate (2.8a′): We let

$$b_k(y', y_N)=\exp[-ky_N]-1 , \qquad y=(y', y_N)\in \mathcal{U} ,$$

where $k>0$ is a large constant to be chosen later on. Then it follows from formula (2.9) that

$$A_\varepsilon(b_k)=\varepsilon(\mu^{NN}k^2-\nu^N k)+\alpha^{NN}k^2-\beta^N k+cb_k \qquad \text{in } \mathcal{U}.$$

Thus, since $\alpha^{NN}(0)>0$, we have for k sufficiently large

$$A_\varepsilon(b_k)\geqq \alpha_0 k^2 \qquad \text{in } \mathcal{U},$$

with some constant $\alpha_0>0$.

We let

$$\varphi_\pm(y)=mb_k(y)\pm u_\varepsilon(y),$$

where $m=m(k)>0$ is a constant given by

$$m=\frac{1}{k^2}\frac{\|f\|_{C(\bar{D})}}{\alpha_0}+\frac{\|u_\varepsilon\|_{C(\bar{D})}}{\min_{\bar{D}\setminus\mathcal{U}}(-b_k)} .$$

Then it is easy to verify that

$$\varphi_\pm|_{\bar{D}\setminus\mathcal{U}}\leqq\left(1+\frac{b_k}{\min_{\bar{D}\setminus\mathcal{U}}(-b_k)}\right)\|u_\varepsilon\|_{C(\bar{D})}\leqq 0 ,$$

and

$$\varphi_\pm|_{\Sigma_3}=0 .$$

But we have

$$\begin{aligned} A_\varepsilon(\varphi_\pm)&=mA_\varepsilon(b_k)\pm f \\ &\geqq m\alpha_0 k^2\pm f \\ &=\|f\|_{C(\bar{D})}\pm f+\left(\frac{\|u_\varepsilon\|_{C(\bar{D})}}{\min_{\bar{D}\setminus\mathcal{U}}(-b_k)}\right)\alpha_0 k^2 \\ &\geqq\left(\frac{\|u_\varepsilon\|_{C(\bar{D})}}{\min_{\bar{D}\setminus\mathcal{U}}(-b_k)}\right)\alpha_0 k^2 \\ &>0 \qquad \text{in } \mathcal{U} . \end{aligned}$$

Thus, applying the maximum principle (cf. Theorem A.1) to the functions $\varphi_\pm$, we obtain that

$$\varphi_\pm\leqq 0 \qquad \text{in } \mathcal{U}.$$

Hence it follows that

$$\pm\frac{\partial u_\varepsilon}{\partial y_N}(0)-mk=\frac{\partial\varphi_\pm}{\partial y_N}(0)\leqq 0\,.$$

This proves that for *all* sufficiently large k

$$\left|\frac{\partial u_\varepsilon}{\partial y_N}(0)\right|\leqq mk=\Big(\frac{1}{\alpha_0 k}\Big)\|f\|_{C(\bar D)}+\Big(\frac{k}{\min_{\bar D\setminus\mathcal{U}}(-b_k)}\Big)\|u_\varepsilon\|_{C(\bar D)}\,. \tag{2.10}$$

On the other hand, applying the maximum principle to the functions u_ε, we obtain (cf. estimate (2.7)) that

$$\|u_\varepsilon\|_{C(\bar D)}\leqq\frac{1}{c_0}\|f\|_{C(\bar D)}\,. \tag{2.11}$$

Therefore, the desired estimate (2.8a′) follows by combining estimates (2.10) and (2.11).

(b) Next we prove estimate (2.8b′): Since $\beta^N(0)<0$, it follows that if k is sufficiently large, we have for some constant $\beta_0>0$

$$\begin{aligned}A_\varepsilon(b_k)&=\varepsilon(\mu^{NN}k^2-\nu^N k)+\alpha^{NN}k^2-\beta^N k+cb_k\\ &\geqq\beta_0 k \qquad \text{in } \mathcal{U}.\end{aligned}$$

We let

$$\phi_\pm(y)=lb_k(y)\pm u_\varepsilon(y)\,,$$

where $l=l(k)>0$ is a constant given by

$$l=\frac{1}{k}\frac{\|f\|_{C(\bar D)}}{\beta_0}+\frac{\|u_\varepsilon\|_{C(\bar D)}}{\min_{\bar D\setminus\mathcal{U}}(-b_k)}\,.$$

Then, just as in case (a), it follows that

$$\phi_\pm|_{\bar D\setminus\mathcal{U}}\leqq 0\,,$$

$$\phi_\pm|_{\Sigma_2}=0\,,$$

and

$$\begin{aligned}A_\varepsilon(\phi_\pm)&=lA_\varepsilon(b_k)\pm f\\ &\geqq\Big(\frac{\|u_\varepsilon\|_{C(\bar D)}}{\min_{\bar D\setminus\mathcal{U}}(-b_k)}\Big)\beta_0 k\\ &>0 \qquad \text{in } \mathcal{U}.\end{aligned}$$

Thus, applying again the maximum principle to the functions $\phi_\pm$, we obtain that

$$\phi_\pm\leqq 0 \qquad \text{in } \mathcal{U}.$$

Hence we have (just as in case (a))

$$\left|\frac{\partial u_\varepsilon}{\partial y_N}(0)\right| \leqq lk = \frac{1}{\beta_0}\|f\|_{C(\bar{D})} + \left(\frac{k}{\min_{\bar{D}\setminus\mathcal{U}}(-b_k)}\right)\|u_\varepsilon\|_{C(\bar{D})}$$

$$\leqq \left(\frac{1}{\beta_0} + \frac{k}{\min_{\bar{D}\setminus\mathcal{U}}(-b_k)}\frac{1}{c_0}\right)\|f\|_{C(\bar{D})}.$$

This proves estimate (2.8b′).

(c) Finally we prove estimate (2.8c′): We take a function $\psi_\varepsilon \in C^2(\boldsymbol{R}^{N-1})$ such that

$$\psi_\varepsilon(y') = \begin{cases} \sqrt{\varepsilon} & \text{if } |y'| \leqq \delta, \\ \sqrt{\varepsilon}\left(1 - \dfrac{(|y'|^2-\delta^2)^3}{27\delta^6}\right) & \text{if } \delta \leqq |y'| \leqq 2\delta, \end{cases}$$

where $\delta > 0$ is a small constant to be chosen later on. It is easy to verify the following:

(1) $$|\psi_\varepsilon| \leqq \sqrt{\varepsilon} \quad \text{on } \boldsymbol{R}^{N-1}.$$

(2) $$\left|\frac{\partial \psi_\varepsilon}{\partial y_j}\right| \leqq \frac{4}{\delta}\sqrt{\varepsilon} \quad \text{on } \boldsymbol{R}^{N-1}.$$

(3) $$\left|\frac{\partial^2 \psi_\varepsilon}{\partial y_j \partial y_k}\right| \leqq \frac{32}{3\delta^2}\sqrt{\varepsilon} \quad \text{on } \boldsymbol{R}^{N-1}.$$

Let $Q_{\delta,\varepsilon}$ be a subdomain of D defined by

$$Q_{\delta,\varepsilon} = \{y = (y', y_N) \in \boldsymbol{R}^N;\ |y'| < 2\delta,\ 0 < y_N < \psi_\varepsilon(y')\}.$$

Here we choose a constant $\delta > 0$ so small that the domain $Q_{\delta,\varepsilon}$ is contained in a tubular neighborhood $\mathcal{U}$ of $x_0' = (0, 0)$. In the domain $Q_{\delta,\varepsilon}$, we consider a function

$$w(y) = K_0(e^{-z(y)} - 1),$$

where

$$z(y) = \frac{K_1}{\sqrt{\varepsilon}}(y_N + \sqrt{\varepsilon} - \psi_\varepsilon(y')).$$

Here $K_0 > 0$ and $K_1 > 0$ are large constants to be chosen later on.

Then we have the following:

CLAIM 1. *$A_\varepsilon(w) \geqq c_0 K_0$ in the domain $Q_{\delta,\varepsilon}$ if $K_1 > 0$ is sufficiently large (independently of K_0) and if $\varepsilon > 0$ is sufficiently small. Here recall that*

$$c_0 = \min_{\bar{D}}(-c) > 0.$$

PROOF. Since the matrix (μ^{ij}) is positive definite and the matrix (α^{ij}) is non-negative definite, it is easy to see that

(2.12)
$$A_\varepsilon(w)=K_0K_1^2e^{-z(y)}\Big(\mu^{NN}-2\sum_{j=1}^{N-1}\mu^{Nj}\frac{\partial\phi_\varepsilon}{\partial y_j}+\sum_{i,j=1}^{N-1}\mu^{ij}\frac{\partial\phi_\varepsilon}{\partial y_i}\frac{\partial\phi_\varepsilon}{\partial y_j}\Big)$$
$$+K_0K_1e^{-z(y)}\sqrt{\varepsilon}\sum_{i,j=1}^{N-1}\mu^{ij}\frac{\partial^2\phi_\varepsilon}{\partial y_i\partial y_j}+K_0K_1e^{-z(y)}\sqrt{\varepsilon}\Big(-\nu^N+\sum_{i=1}^{N-1}\nu^i\frac{\partial\phi_\varepsilon}{\partial y_i}\Big)$$
$$+K_0K_1^2e^{-z(y)}\frac{1}{\varepsilon}\Big(\alpha^{NN}-2\sum_{j=1}^{N-1}\alpha^{Nj}\frac{\partial\phi_\varepsilon}{\partial y_j}+\sum_{i,j=1}^{N-1}\alpha^{ij}\frac{\partial\phi_\varepsilon}{\partial y_i}\frac{\partial\phi_\varepsilon}{\partial y_j}\Big)$$
$$+K_0K_1e^{-z(y)}\frac{1}{\sqrt{\varepsilon}}\sum_{i,j=1}^{N-1}\alpha^{ij}\frac{\partial^2\phi_\varepsilon}{\partial y_i\partial y_j}+K_0K_1e^{-z(y)}\frac{1}{\sqrt{\varepsilon}}\Big(-\beta^N+\sum_{i=1}^{N-1}\beta^i\frac{\partial\phi_\varepsilon}{\partial y_i}\Big)$$
$$+cK_0e^{-z(y)}-cK_0$$
$$\geqq K_0\Big[K_1^2\Big(\mu^{NN}-2\sqrt{\varepsilon}\sum_{j=1}^{N-1}\mu^{Nj}\frac{1}{\sqrt{\varepsilon}}\frac{\partial\phi_\varepsilon}{\partial y_j}\Big)$$
$$+K_1\varepsilon\sum_{i,j=1}^{N-1}\mu^{ij}\frac{1}{\sqrt{\varepsilon}}\frac{\partial^2\phi_\varepsilon}{\partial y_i\partial y_j}+K_1\sqrt{\varepsilon}\Big(-\nu^N+\sqrt{\varepsilon}\sum_{i=1}^{N-1}\nu^i\frac{1}{\sqrt{\varepsilon}}\frac{\partial\phi_\varepsilon}{\partial y_i}\Big)$$
$$+K_1\sum_{i,j=1}^{N-1}\alpha^{ij}\frac{1}{\sqrt{\varepsilon}}\frac{\partial^2\phi_\varepsilon}{\partial y_i\partial y_j}-K_1\frac{\beta_N}{\sqrt{\varepsilon}}$$
$$+K_1\sum_{i=1}^{N-1}\beta^i\frac{1}{\sqrt{\varepsilon}}\frac{\partial\phi_\varepsilon}{\partial y_i}+c\Big]e^{-z(y)}+c_0K_0\,.$$

But we find that

$$\beta^N=O(\sqrt{\varepsilon})\qquad\text{in } Q_{\delta,\varepsilon},$$

since $\beta^N=0$ on Σ_0.

Therefore, we obtain from inequality (2.12) that

$$A_\varepsilon(w)\geqq c_0K_0\qquad\text{in } Q_{\delta,\varepsilon}$$

if $K_1>0$ is sufficiently large (independently of K_0) and if $\varepsilon>0$ is sufficiently small.

CLAIM 2. *$A_\varepsilon(w\pm u_\varepsilon)>0$ in the domain $Q_{\delta,\varepsilon}$ if $K_0>0$ is sufficiently large.*

PROOF. By Claim 1, if follows that

$$A_\varepsilon(w\pm u_\varepsilon)=A_\varepsilon(w)\pm f\geqq c_0K_0\pm f>0\qquad\text{in } \partial Q_{\delta,\varepsilon}$$

if $K_0>0$ is so large that

$$K_0>\frac{\|f\|_{C(\bar{D})}}{c_0}\,. \tag{2.13}$$

CLAIM 3. *$w\pm u_\varepsilon\leqq 0$ on the boundary $\partial Q_{\delta,\varepsilon}$ if $K_0>0$ is sufficiently large.*

PROOF. First, since we have for $|y'|\leqq 2\delta$

$$\begin{cases} u_\varepsilon(y', 0)=0\,, \\ w(y', 0)=K_0(e^{-(K_1/\sqrt{\varepsilon})(\sqrt{\varepsilon}-\psi_\varepsilon(y'))}-1)\leqq 0\,, \end{cases}$$

it follows that on the set $\partial Q_{\delta,\varepsilon}\cap\{y_N=0\}$

$$w(y', 0)\pm u_\varepsilon(y', 0)\leqq 0\,.$$

Next we recall that

$$\|u_\varepsilon\|_{C(\bar{D})}\leqq\frac{1}{c_0}\|f\|_{C(\bar{D})}\,.$$

Hence it follows that on the set $\partial Q_{\delta,\varepsilon}\cap\{y_N=\psi(y')\}$

$$\begin{aligned} w(y', \psi(y'))\pm u_\varepsilon(y', \psi(y'))&=K_0(e^{-K_1}-1)\pm u_\varepsilon(y', \psi(y')) \\ &\leqq K_0(e^{-K_1}-1)+\|u_\varepsilon\|_{C(\bar{D})} \\ &\leqq K_0(e^{-K_1}-1)+\frac{1}{c_0}\|f\|_{C(\bar{D})} \\ &\leqq 0\,, \end{aligned}$$

if $K_0>0$ is so large that

$$K_0>\frac{1}{c_0(1-e^{-K_1})}\|f\|_{C(\bar{D})}\,. \tag{2.14}$$

By virtue of Claims 2 and 3, we can apply the maximum principle (Theorem A.1) to the functions $w\pm u_\varepsilon$, we obtain that

$$w\pm u_\varepsilon\leqq 0 \quad \text{in } Q_{\delta,\varepsilon}\,.$$

Hence it follows that

$$\pm\frac{\partial u_\varepsilon}{\partial y_N}(0)-\frac{K_0K_1}{\sqrt{\varepsilon}}=\frac{\partial}{\partial y_N}(w\pm u_\varepsilon)(0)\leqq 0\,,$$

so that

$$\left|\frac{\partial u_\varepsilon}{\partial y_N}(0)\right|\leqq\frac{K_0K_1}{\sqrt{\varepsilon}}.$$

In view of inequalities (2.13) and (2.14), this proves estimate (2.8c′).

The proof of Lemma 2.4 is now complete.

II-2) Now, applying Lemma 2.4 to the functions $z_{\varepsilon,n}$ (n being fixed), we obtain that

$$\max_{\Sigma_2\cup\Sigma_2}|\operatorname{grad} z_{\varepsilon,n}|\leqq M_n(\|f_n\|_{C(\bar{D})}+\|g_n\|_{C^2(\bar{D})}),$$

$$\max_{\Sigma_0}|\operatorname{grad} z_{\varepsilon,n}|\leqq\frac{C_n}{\sqrt{\varepsilon}}(\|f_n\|_{C(\bar{D})}+\|g_n\|_{C^2(\bar{D})}),$$

and hence that

(2.15a) $$\max_{\Sigma_2\cup\Sigma_3}|\operatorname{grad} u_{\varepsilon,n}|\leqq M'_n(\|f_n\|_{C(\bar{D})}+\|g_n\|_{C^2(\bar{D})}),$$

(2.15b) $$\max_{\Sigma_0}|\operatorname{grad} u_{\varepsilon,n}|\leqq \frac{C'_n}{\sqrt{\varepsilon}}(\|f_n\|_{C(\bar{D})}+\|g_n\|_{C^2(\bar{D})}),$$

since $u_{\varepsilon,n}=z_{\varepsilon,n}-g_n$ and $g_n\in C^{2+\theta}(\bar{D})$. Here $M_n>0$, $M'_n>0$, $C_n>0$ and $C'_n>0$ are constants independent of ε.

Then, applying Green's formula (2.2) to the operators $A_\varepsilon=\varepsilon\Delta+A$ and $A^*_\varepsilon=\varepsilon\Delta+A^*$, we find that for all $v\in C^2(\bar{D})$ satisfying $v=0$ on $\Sigma_1\cup\Sigma_3$

(2.16) $$\begin{aligned}\iint_D f_n v\,dx&=\iint_D A_\varepsilon u_{\varepsilon,n}\cdot v\,dx\\ &=\iint_D u_{\varepsilon,n}\cdot A^*_\varepsilon v\,dx-\int_{\Sigma_3}\left(\frac{\partial u_{\varepsilon,n}}{\partial\nu}v-u_{\varepsilon,n}\frac{\partial v}{\partial\nu}\right)d\sigma\\ &\quad-\int_{\partial D\setminus\Sigma_0} bu_{\varepsilon,n}v\,d\sigma-\varepsilon\left(\int_{\partial D}\left(\frac{\partial u_{\varepsilon,n}}{\partial \boldsymbol{n}}v-u_{\varepsilon,n}\frac{\partial v}{\partial \boldsymbol{n}}\right)d\sigma\right)\\ &=\varepsilon\iint_D u_{\varepsilon,n}\cdot\Delta v\,dx+\iint_D u_{\varepsilon,n}\cdot A^*v\,dx+\int_{\Sigma_3} g_n\frac{\partial v}{\partial\nu}d\sigma\\ &\quad-\int_{\Sigma_2} bg_n v\,d\sigma+\varepsilon\int_{\partial D} g_n\frac{\partial v}{\partial \boldsymbol{n}}d\sigma-\varepsilon\int_{\Sigma_0\cup\Sigma_2}\frac{\partial u_{\varepsilon,n}}{\partial \boldsymbol{n}}v\,d\sigma\,.\end{aligned}$$

But we recall (cf. [Y, Chapter V, Section 2, Theorem 1]) that the unit ball in the Hilbert space $L^2(D)$ is *sequentially weakly compact*. Hence, by estimate (2.7), one can find a subsequence $\{u_{\varepsilon_k,n}\}_{k=1}^\infty$ which converges weakly to some function u_n in $L^2(D)$ as $\varepsilon_k\downarrow 0$. Thus, we can let $\varepsilon_k\downarrow 0$ in formula (2.16) to obtain that for all $v\in C^2(\bar{D})$ satisfying $v=0$ on $\Sigma_1\cup\Sigma_3$

(2.17) $$\iint_D f_n v\,dx=\iint_D u_n\cdot A^*v\,dx+\int_{\Sigma_3} g_n\frac{\partial v}{\partial\nu}d\sigma-\int_{\Sigma_2} bg_n v\,d\sigma\,.$$

Indeed, by estimate (2.15), it follows that the last term of the right-hand side of formula (2.16) tends to zero as $\varepsilon_k\downarrow 0$.

On the other hand, it is easy to verify that the set

$$K=\left\{w\in L^2(D)\,;\,\operatorname{ess\,sup}_D|w|\leqq\max\left(\frac{1}{c_0}\operatorname{ess\,sup}_D|f|,\ \operatorname{ess\,sup}_{\Sigma_2\cup\Sigma_3}|g|\right)\right\}$$

is convex and strongly closed in the space $L^2(D)$. Thus it follows from an application of Mazur's theorem (cf. [Y, Chapter V, Section 1, Theorem 11]) that the set K is *weakly closed* in $L^2(D)$. This proves that $u_n\in K$:

(2.18) $$\operatorname{ess\,sup}_D|u_n|\leqq\max\left(\frac{1}{c_0}\operatorname{ess\,sup}_D|f|,\ \operatorname{ess\,sup}_{\Sigma_2\cup\Sigma_3}|g|\right),$$

since $u_{\varepsilon,n}\in K$ for $\varepsilon>0$.

Therefore, we have proved that the function u_n is a weak solution of problem (D_n) and satisfies estimate (2.18).

III) Finally we show that the limit function u of u_n when $n\to\infty$ is a weak solution of problem (D):

$$(D)\qquad \begin{cases} Au=f & \text{in } D, \\ u=g & \text{on } \Sigma_2\cup\Sigma_3. \end{cases}$$

By estimate (2.18), it follows that the sequence $\{u_n\}_{n=1}^{\infty}$ is weakly compact in the space $L^2(D)$. Hence one can find a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ which converges weakly to some function u in $L^2(D)$ as $n_k\to\infty$.

Therefore, letting $n_k\to\infty$ in formula (2.17), we obtain from assertions (2.5b) and (2.6b) that for all $v\in C^2(\bar{D})$ satisfying $v=0$ on $\Sigma_1\cup\Sigma_3$

$$\iint_D fv\,dx=\iint_D u\cdot A^*v\,dx+\int_{\Sigma_3} g\frac{\partial v}{\partial \nu}d\sigma-\int_{\Sigma_2} bgv\,d\sigma.$$

Furthermore, since $u_{n_k}\in K$, it follows from an application of Mazur's theorem that $u\in K$, that is, the function u satisfies inequality (2.4).

The proof of Theorem 2.3 is now complete.

REMARK 2.5. It can be shown (cf. [OR, Theorem 1.5.2]) that if g is a function in the space $C(\Sigma_2\cup\Sigma_3)$, then the weak solution u constructed in Theorem 2.3 assumes the given boundary values g on the set $\Sigma_2\cup\Sigma_3$.

2.4, Uniqueness Theorem for Problem (D)

Next we prove the following *uniqueness* theorem for problem (D) (cf. [OR, Theorem 1.6.1]):

THEOREM 2.6. *Assume that hypothesis (H) is satisfied, and that*

$$c^*=\sum_{i,j=1}^{N}\frac{\partial^2 a^{ij}}{\partial x_i\partial x_j}-\sum_{i=1}^{N}\frac{\partial b^i}{\partial x_i}+c<0 \quad \text{on } \bar{D}. \tag{2.19}$$

Then any homogeneous solution $u\in L^\infty(D)$ of problem (D) is equal to zero almost everywhere in D, that is, if we have for any function $v\in C^2(\bar{D})$ satisfying $v=0$ on $\Sigma_1\cup\Sigma_3$

$$\iint_D u\cdot A^*v\,dx=0, \tag{2.20}$$

then the solution u is equal to zero almost everywhere in D.

PROOF. I) We modify the domain D and the operator A^* so that the set $\Sigma_0\cup\Sigma_2$ is of type Σ_2 or of type Σ_3.

By hypothesis (H), one can choose a bounded domain Ω with C^∞ boundary $\partial\Omega$ such that (cf. Figure 2)

$$\begin{cases} D\cup\Sigma_0\cup\Sigma_2\subset\Omega\,, \\ \Sigma_1\cup\Sigma_3\subset\partial\Omega\,, \end{cases}$$

and one may assume that

$$c^*<0 \quad \text{on } \bar{\Omega}\,. \tag{2.19'}$$

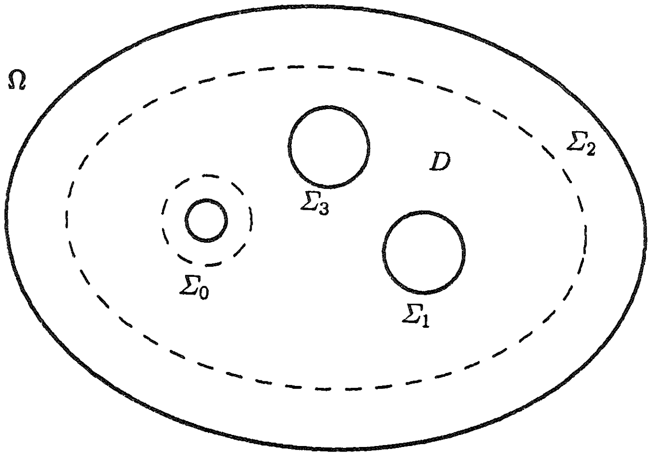

Figure 2.

Now we take a function $a\in C^\infty(\bar{\Omega})$ such that

$$\begin{cases} a=0 & \text{in } D, \\ a>0 & \text{in } \bar{\Omega}\setminus\bar{D}\,, \end{cases}$$

and consider the Dirichlet problem for the *elliptic* operators $\varepsilon\Delta+A^*+a\Delta$ $(\varepsilon>0)$:

$$(\tilde{D}_\varepsilon^*) \qquad \begin{cases} (\varepsilon\Delta+A^*+a\Delta)v_\varepsilon=\varphi & \text{in } \Omega\,, \\ v_\varepsilon=0 & \text{on } \partial\Omega\,, \end{cases}$$

where $\Delta=\sum_{i=1}^N \partial^2/\partial x_i^2$ is the usual Laplacian. We remark that:

(i) The Fichera function $\tilde{b}^*$ for the operator $A^*+a\Delta$ is equal to $-b$ on Σ_1, and so

$$\tilde{b}^*(x')<0 \quad \text{on } \Sigma_1\,.$$

(ii) $\sum_{i,j=1}^N a^{ij}(x')n_i n_j+a(x')\sum_{i=1}^N n_i^2>0$ on $\partial\Omega\setminus\Sigma_1$.

In other words, the boundary $\partial\Omega$ is of type Σ_2 or of type Σ_3 for the operator $A^*+a\Delta$.

Let φ be an arbitrary function in the space $C_0^\infty(D)$. Then we know (cf.

[GT]) that problem $(\tilde{D}_\varepsilon^*)$ has a unique solution v_ε in the space $C^\infty(\bar{\Omega})$ and that

$$\max_{\bar{\Omega}} |v_\varepsilon| \leqq \frac{1}{c_0^*} \max_{\bar{D}} |\varphi|, \tag{2.21}$$

where

$$c_0^* = \min_{\bar{D}} (-c^*) > 0 .$$

Since $v_\varepsilon \in C^\infty(\bar{\Omega})$ and $v_\varepsilon = 0$ on $\Sigma_1 \cup \Sigma_3$ and since $a=0$ in D, it follows from an application of Green's formula (2.2) and condition (2.20) that

$$\begin{aligned} \iint_D u\varphi dx &= \iint_D u \cdot \varepsilon \Delta v_\varepsilon dx + \iint_D u \cdot A^* v_\varepsilon dx + \iint_D u \cdot a \Delta v_\varepsilon dx \\ &= \iint_D u \cdot \varepsilon \Delta v_\varepsilon dx . \end{aligned} \tag{2.22}$$

We choose a sequence $\{u_n\}_{n=1}^\infty$ in the space $C_0^\infty(D)$ such that

$$u_n \longrightarrow u \quad \text{in } L^2(D).$$

Then we have by Schwarz's inequality

$$\begin{aligned} \left| \iint_D u \cdot \varepsilon \Delta v_\varepsilon dx \right| &= \left| \iint_D (u-u_n) \varepsilon \Delta v_\varepsilon dx + \iint_D u_n \cdot \varepsilon \Delta v_\varepsilon dx \right| \\ &\leqq \left| \iint_D (u-u_n) \varepsilon \Delta v_\varepsilon dx \right| + \varepsilon \left| \iint_D \Delta u_n \cdot v_\varepsilon dx \right| \\ &\leqq \left(\iint_D \varepsilon^2 (\Delta v_\varepsilon)^2 dx \right)^{1/2} \| u-u_n \|_{L^2(D)} \\ &\quad + \varepsilon \max_{\bar{\Omega}} |v_\varepsilon| \iint_D |\Delta u_n| dx . \end{aligned} \tag{2.23}$$

II) In order to estimate the first term on the last inequality, we need the following lemma due to Oleĭnik-Radkevič ([OR, Lemma 1.6.1]):

LEMMA 2.7. *Let $f \in C^\theta(\bar{D})$ $(0<\theta<1)$ and let $v_\varepsilon \in C^{2+\theta}(\bar{D})$ be a unique solution of the Dirichlet problem for the elliptic operators $\varepsilon\Delta + A$ $(\varepsilon>0)$:*

$$(D_\varepsilon) \qquad \begin{cases} (\varepsilon\Delta + A) v_\varepsilon = f & \text{in } D, \\ v_\varepsilon = 0 & \text{on } \partial D. \end{cases}$$

Assume that condition (2.3) *is satisfied and that for some constant $C>0$ independent of ε*

$$\max_{\partial D} |\operatorname{grad} v_\varepsilon| \leqq \frac{C}{\sqrt{\varepsilon}} .$$

Then we have the estimate

$$\iint_D \varepsilon^2(\Delta v_\varepsilon)^2 dx \leqq C',$$

with some constant $C'>0$ *independent of* ε.

III) Since the boundary $\partial\Omega$ is of type Σ_2 or of type Σ_3 for the operator $A^*+a\Delta$, it follows from an application of Lemma 2.4 that

$$\max_{\partial\Omega} |\operatorname{grad} v_\varepsilon| \leqq M^* \|\varphi\|_{C(\bar{D})},$$

where $M^*>0$ is a constant independent of ε. Hence, applying Lemma 2.7 to the operator $A^*+a\Delta$, we obtain that

$$\iint_\Omega \varepsilon^2(\Delta v_\varepsilon)^2 dx \leqq C^*, \tag{2.24}$$

where $C^*>0$ is a constant independent of ε.

Therefore, combining estimates (2.23), (2.24) and (2.21), we find that

$$\left|\iint_D u\cdot\varepsilon\Delta v_\varepsilon dx\right| \leqq \sqrt{C^*}\|u-u_n\|_{L^2(D)} + \varepsilon\frac{\max_{\bar{D}}|\varphi|}{c_0^*}\iint_D |\Delta u_n| dx,$$

so that

$$\lim_{\varepsilon\downarrow 0}\iint_D u\cdot\varepsilon\Delta v_\varepsilon dx = 0,$$

since $u_n\to u$ in $L^2(D)$. Hence, combining this fact with formula (2.22), we have

$$\iint_D u\varphi dx = 0.$$

This proves that $u=0$ a.e. in D, since $\varphi\in C_0^\infty(D)$ is arbitrary.

The proof of Theorem 2.6 is complete.

3. The Dirichlet Problem—(2)—

In this section, we prove regularity theorems for the weak solutions of problem (D) constructed in Theorem 2.3 in the framework of the spaces $W^{m,\infty}(D)$ and $C^{m+\theta}(\bar{D})$ where $m\geqq 1$.

3.1. Lipschitz Continuity for Weak Solutions

First we prove a regularity theorem for problem (D) in the space $W^{1,\infty}(D)$ (cf. [OR, Theorem 1.8.1]; [C, Théoréme 4.4]), which gives a sufficient condition for the Lipschitz continuity for weak solutions of the Dirichlet problem.

THEOREM 3.1. *Assume that hypothesis* (H) *is satisfied and that condition* (2.3) *is satisfied. Then one can find a constant* $\lambda>0$ *such that, for any function* *f in the space* $W^{1,\infty}(D)$, *there exists a weak solution* $u \in W^{1,\infty}(D)$ *of the Dirichlet problem*:

$$(*) \qquad \begin{cases} (A-\lambda)u=f & \text{in } D, \\ u=0 & \text{on } \Sigma_2 \cup \Sigma_3. \end{cases}$$

Moreover, the solution u satisfies the inequality

$$(3.1) \qquad \|u\|_{1,\infty} \leqq C_1(\lambda) \|f\|_{1,\infty},$$

where $C_1(\lambda)>0$ *is a constant independent of f.*

PROOF. I) We modify the domain D and the operator A so that the set $\Sigma_0 \cup \Sigma_1$ is of type Σ_2 or of type Σ_3.

By hypothesis (H), one can choose a bounded domain Ω with C^∞ boundary $\partial\Omega$ such that (cf. Figure 3)

$$\begin{cases} D \cup \Sigma_0 \cup \Sigma_1 \subset \Omega, \\ \Sigma_2 \cup \Sigma_3 \subset \partial\Omega, \end{cases}$$

and one may assume that

$$(2.3') \qquad c<0 \quad \text{on } \bar{\Omega}.$$

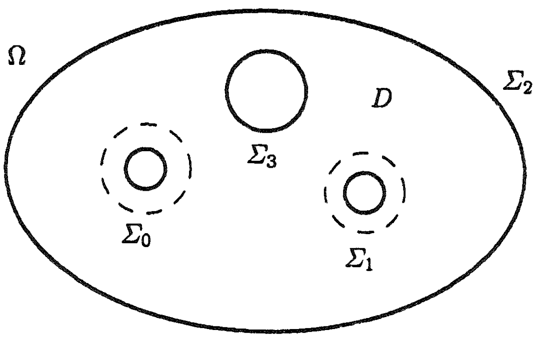

Figure 3.

Now we take a function $a \in C^\infty(\bar{\Omega})$ such that

$$\begin{cases} a=0 & \text{in } D, \\ a>0 & \text{in } \bar{\Omega} \setminus \bar{D}, \end{cases}$$

and consider the Dirichlet problem for the *elliptic* operators $\varepsilon\Delta + A + a\Delta - \lambda$ $(\varepsilon>0)$:

$$(\tilde{D}_\varepsilon) \qquad \begin{cases} (\varepsilon\Delta + A + a\Delta - \lambda)u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta = \sum_{i=1}^N \partial^2/\partial x_i^2$ is the usual Laplacian and $\lambda>0$. We remark that:

(i) The Fichera function $\tilde{b}$ for the operator $A+a\Delta-\lambda$ is equal to b on Σ_2, and so

$$\tilde{b}(x')<0 \quad \text{on } \Sigma_2.$$

(ii) $\sum_{i,j=1}^N a^{ij}(x')n_i n_j + a(x')\sum_{i=1}^N n_i^2>0$ on $\partial\Omega\setminus\Sigma_2$.

In other words, the boundary $\partial\Omega$ is of type Σ_2 or of type Σ_3 for the operators $A+a\Delta-\lambda$, $\lambda>0$.

II) First let f be an arbitrary function in the space $C^{1+\theta}(\bar{D})$, $0<\theta<1$. We show that there exists a weak solution $u\in W^{1,\infty}(D)$ of problem (*) which satisfies inequality (3.1).

One may assume that

$$f\in C^{1+\theta}(\bar{\Omega}),$$

and that

$$\|f\|_{C^1(\bar{\Omega})}\leqq\|f\|_{C^1(\bar{D})}. \tag{3.2}$$

Then we know (cf. [GT]) that problem $(\tilde{D}_\varepsilon)$ has a unique solution u_ε in the space $C^{3+\theta}(\bar{\Omega})$ and that

$$\max_{\bar{\Omega}}|u_\varepsilon|\leqq\frac{1}{\lambda}\max_{\bar{\Omega}}|f|,$$

since $(\varepsilon\Delta+A+a\Delta-\lambda)1=c-\lambda\leqq-\lambda$ on $\bar{\Omega}$.

II-1) We show that there exists a subsequence $\{u_{\varepsilon_k}\}$ which converges uniformly in $\bar{\Omega}$ to a function $u\in W^{1,\infty}(\Omega)$, as $\varepsilon_k\downarrow 0$.

II-1a) To do so, if $\varphi\in C^1(\bar{D})$, we define a continuous function $B_A(\varphi,\varphi)$ on $\bar{D}$ by the formula

$$B_A(\varphi,\varphi)(x)=2\sum_{i,j=1}^N a^{ij}(x)\frac{\partial\varphi}{\partial x_i}(x)\frac{\partial\varphi}{\partial x_j}(x)-c(x)\cdot\varphi(x)^2,\quad x\in\bar{D},$$

where

$$A=\sum_{i,j=1}^N a^{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^N b^i(x)\frac{\partial}{\partial x_i}+c(x).$$

We remark that the function $B_A(\varphi,\varphi)$ is *non-negative* on $\bar{D}$ for all $\varphi\in C^1(\bar{D})$.

The next result may be proved just as in the proof of Théorème 4.1 of Cancelier [C].

LEMMA 3.2. *If $\varphi\in C^\infty(\bar{D})$, we let*

$$p_1(x)=\sum_{j=1}^N\left|\frac{\partial\varphi}{\partial x_j}(x)\right|^2,\quad x\in\bar{D},$$

and

$$R_1(x)=Ap_1(x)-\sum_{j=1}^{N}B_A\Big(\frac{\partial\varphi}{\partial x_j},\frac{\partial\varphi}{\partial x_j}\Big)(x),\quad x\in\bar{D}.$$

Then, for each $\eta>0$, *there exist constants* $\beta_0>0$ *and* $\beta_1>0$ *such that we have for all* $\varphi\in C^\infty(\bar{D})$

$$|R_1(x)|\leqq\eta\sum_{j=1}^{N}B_A\Big(\frac{\partial\varphi}{\partial x_j},\frac{\partial\varphi}{\partial x_j}\Big)(x)+\beta_0\|\varphi\|^2_{C(\bar{D})}$$
$$+\beta_1\|\varphi\|^2_{C^1(\bar{D})}+\frac{1}{2}\|A\varphi\|^2_{C^1(\bar{D})},\quad x\in\bar{D}.$$

REMARK 3.3. The constants β_0 and β_1 are *uniform* for the operators $A+\varepsilon\Delta-\lambda I$, $0\leqq\varepsilon\leqq 1$, $\lambda\geqq 0$.

II-1b) The proof that $u\in W^{1,\infty}(D)$ is based on the following lemma (cf. [OR, Lemma 1.8.1]):

LEMMA 3.4. *Assume that hypothesis (H) is satisfied with* $\partial D=\Sigma_2\cup\Sigma_3$ *and that condition* (2.3) *is satisfied. Then one can find a constant* $\lambda>0$ *such that if* f *is a function in the space* $C^{1+\theta}(\bar{D})$, *then the unique solution* $u_\varepsilon\in C^{3+\theta}(\bar{D})$ *of the Dirichlet problem*

$$\begin{cases}(A+\varepsilon\Delta-\lambda)u_\varepsilon=f & \text{in } D,\\ u_\varepsilon=0 & \text{on } \partial D\end{cases}$$

satisfies the estimate

$$\|u_\varepsilon\|_{C^1(\bar{D})}\leqq C_1(\lambda)\|f\|_{C^1(\bar{D})},\tag{3.3}$$

where $C_1(\lambda)>0$ *is a constant independent of* $\varepsilon>0$.

PROOF. We remark (cf. estimate (2.11)) that the solution u_ε satisfies the estimate

$$\|u_\varepsilon\|_{C(\bar{D})}\leqq\frac{1}{\lambda}\|f\|_{C(\bar{D})},\tag{3.4}$$

since $(A+\varepsilon\Delta-\lambda)1=c-\lambda\leqq-\lambda$ on $\bar{D}$. Thus, to prove estimate (3.3), it suffices to show that

$$\max_{\bar{D}}|\text{grad}\,u_\varepsilon|\leqq M(\lambda)\|f\|_{C^1(\bar{D})},\tag{3.5}$$

where $M(\lambda)>0$ is a constant independent of $\varepsilon>0$.

We let

$$p_1^\varepsilon(x)=\sum_{j=1}^{N}\Big|\frac{\partial u_\varepsilon}{\partial x_j}(x)\Big|^2,\quad x\in\bar{D},$$

and assume that the function $p_1^\varepsilon(x)$ attains its positive maximum at a point x_0

of D. Then, since the matrix (a^{ij}) is non-negative definite, we obtain that

(3.6) $$(A+\varepsilon\Delta)p_1^\varepsilon(x_0)\leqq c(x_0)p_1^\varepsilon(x_0).$$

But it follows from an application of Lemma 3.2 with $\eta=1/2$ that

$$(A+\varepsilon\Delta-\lambda)p_1^\varepsilon(x)=\sum_{j=1}^{N}B_{A+\varepsilon\Delta-\lambda I}\left(\frac{\partial u_\varepsilon}{\partial x_j}, \frac{\partial u_\varepsilon}{\partial x_j}\right)(x)+R_1(x),$$

with

(3.7) $$|R_1(x)|\leqq\frac{1}{2}\sum_{j=1}^{N}B_{A+\varepsilon\Delta-\lambda}\left(\frac{\partial u_\varepsilon}{\partial x_j}, \frac{\partial u_\varepsilon}{\partial x_j}\right)(x)$$
$$+\beta_0\|u_\varepsilon\|^2_{C(\bar{D})}+\beta_1\|u_\varepsilon\|^2_{C^1(\bar{D})}+\frac{1}{2}\|f\|^2_{C^1(\bar{D})}.$$

Here we remark (cf. Remark 3.3) that the constants β_0 and β_1 are independent of $\varepsilon>0$ and $\lambda>0$.

Hence we obtain from inequalities (3.6), (3.7) and (3.4) that

$$\begin{aligned}
\lambda p_1^\varepsilon(x_0)&\leqq(\lambda-c(x_0))p_1^\varepsilon(x_0)\\
&\leqq(\lambda-A-\varepsilon\Delta)p_1^\varepsilon(x_0)\\
&=-\left((A+\varepsilon\Delta-\lambda)p_1^\varepsilon(x_0)-\sum_{j=1}^{N}B_{A+\varepsilon\Delta-\lambda I}\left(\frac{\partial u_\varepsilon}{\partial x_j}, \frac{\partial u_\varepsilon}{\partial x_j}\right)(x_0)\right)\\
&\quad-\sum_{j=1}^{N}B_{A+\varepsilon\Delta-\lambda I}\left(\frac{\partial u_\varepsilon}{\partial x_j}, \frac{\partial u_\varepsilon}{\partial x_j}\right)(x_0)\\
&\leqq-\frac{1}{2}\sum_{j=1}^{N}B_{A+\varepsilon\Delta-\lambda I}\left(\frac{\partial u_\varepsilon}{\partial x_j}, \frac{\partial u_\varepsilon}{\partial x_j}\right)(x_0)\\
&\quad+\beta_0\|u_\varepsilon\|^2_{C(\bar{D})}+\beta_1(\|u_\varepsilon\|^2_{C(\bar{D})}+p_1^\varepsilon(x_0))+\frac{1}{2}\|f\|^2_{C^1(\bar{D})}\\
&\leqq\left(\frac{\beta_0+\beta_1}{\lambda^2}\right)\|f\|^2_{C(\bar{D})}+\beta_1p_1^\varepsilon(x_0)+\frac{1}{2}\|f\|^2_{C^1(\bar{D})}.
\end{aligned}$$

This proves that

$$(\lambda-\beta_1)p_1^\varepsilon(x_0)\leqq\left(\frac{\beta_0+\beta_1}{\lambda^2}\right)\|f\|^2_{C(\bar{D})}+\frac{1}{2}\|f\|^2_{C^1(\bar{D})}.$$

Therefore, if $\lambda>0$ is so large that

$$\lambda>\beta_1,$$

then it follows that

$$p_1^\varepsilon(x_0)\leqq C(\lambda)\|f\|^2_{C^1(\bar{D})},$$

where $C(\lambda)>0$ is a constant *independent* of $\varepsilon>0$.

Thus we have proved that

(3.8) $$\max_{\bar{D}} p_1^\varepsilon \leqq C(\lambda)\|f\|^2_{C^1(\bar{D})} + \max_{\partial D} p_1^\varepsilon ,$$

or equivalently

(3.8′) $$\max_{\bar{D}} |\operatorname{grad} u_\varepsilon| \leqq M_1(\lambda)\|f\|_{C^1(\bar{D})} + \max_{\partial D} |\operatorname{grad} u_\varepsilon| .$$

On the other hand, it follows from an application of Lemma 2.4 that

(3.9) $$\max_{\partial D} |\operatorname{grad} u_\varepsilon| \leqq M_2(\lambda)\|f\|_{C(D)} ,$$

since $\partial D = \Sigma_2 \cup \Sigma_3$.

Therefore, the desired estimate (3.5) (and hence estimate (3.3)) follows by combining estimates (3.8′) and (3.9).

The proof of Lemma 3.4 is complete.

II-1c) Now it follows from an application of Lemma 3.4 with $A = A + a\Delta$ and inequality (3.2) that

(3.10) $$\|u_\varepsilon\|_{C^1(\Omega)} \leqq C_1(\lambda)\|f\|_{C^1(\bar{\Omega})} \leqq C_1(\lambda)\|f\|_{C^1(\bar{D})} .$$

This proves that the sequence $\{u_\varepsilon\}$ is uniformly bounded and equicontinuous on $\bar{\Omega}$. Hence, by virtue of the Ascoli-Arzelà theorem, one can choose a subsequence $\{u_{\varepsilon_k}\}$ which converges uniformly to a function $u \in C(\bar{\Omega})$, as $\varepsilon_k \downarrow 0$. Furthermore, since the unit ball in the Hilbert space $L^2(\Omega)$ is *sequentially weakly compact* (cf. [Y, Chapter V, Section 2, Theorem 1]), one may assume that the sequence $\{\partial_j u_{\varepsilon_k}\}$ converges weakly to a function ϕ_j in $L^2(\Omega)$, for each $1 \leqq j \leqq N$. Then we have

$$\partial_j u = \phi_j \in L^2(\Omega), \qquad 1 \leqq j \leqq N .$$

On the other hand, it is easy to verify that the set

$$K = \{v \in L^2(\Omega) ; \|v\|_\infty \leqq C_1(\lambda)\|f\|_{C^1(\bar{D})}\}$$

is convex and strongly closed in $L^2(\Omega)$. Thus it follows from an application of Mazur's theorem (cf. [Y, Chapter V, Section 1, Theorem 11]) that the set K is *weakly closed* in $L^2(\Omega)$. But we have

$$\begin{cases} \partial_j u_{\varepsilon_k} \in K , \\ \partial_j u_{\varepsilon_k} \longrightarrow \phi_j \quad \text{weakly in } L^2(\Omega) \text{ for each } 1 \leqq j \leqq N. \end{cases}$$

Hence we find that

$$\partial_j u = \phi_j \in K, \ 1 \leqq j \leqq N,$$

that is,

$$\|\partial_j u\|_\infty \leqq C_1(\lambda)\|f\|_{C^1(\bar{D})}, \ 1 \leqq j \leqq N .$$

Summing up, we have proved that

$$(3.11)\qquad \begin{cases} u\in W^{1,\infty}(\Omega), \\ \|u\|_{1,\infty}\leqq C_1(\lambda)\|f\|_{C^1(\bar{D})}, \end{cases}$$

where $C_1(\lambda)>0$ is a constant *independent* of f.

II-2) Finally we show that the function u is a weak solution of the Dirichlet problem:

$$(*)\qquad \begin{cases} (A-\lambda)u=f & \text{in } D, \\ u=0 & \text{on } \Sigma_2\cup\Sigma_3. \end{cases}$$

That is, we show that for all $v_1\in C^2(\bar{D})$ satisfying $v_1=0$ on $\Sigma_1\cup\Sigma_3$

$$(3.12)\qquad \iint_D fv_1dx=\iint_D u\cdot(A^*-\lambda)v_1dx\,.$$

II-2a) First, since u_ε is a solution of problem $(\tilde{D}_\varepsilon)$, we obtain from Green's formula (2.2) that for all $v\in C^2(\bar{\Omega})$ satisfying $v=0$ on $\partial\Omega\backslash\Sigma_2$

$$(3.13)\qquad \begin{aligned} \iint_\Omega fvdx &= \iint_\Omega \varepsilon\Delta u_\varepsilon\cdot vdx+\iint_\Omega a\Delta u_\varepsilon\cdot vdx \\ &\quad +\iint_\Omega (A-\lambda)u_\varepsilon\cdot vdx \\ &= \varepsilon\iint_\Omega u_\varepsilon\cdot\Delta vdx+\iint_\Omega u_\varepsilon\cdot\Delta(av)dx \\ &\quad +\iint_\Omega u_\varepsilon\cdot(A^*-\lambda)vdx-\varepsilon\int_{\Sigma_2} v\frac{\partial u_\varepsilon}{\partial \boldsymbol{n}}d\sigma, \end{aligned}$$

since $a=0$ on Σ_2 and hence $av=0$ on $\partial\Omega$.

But we recall that the subsequence $\{u_{\varepsilon_k}\}_{k=1}^\infty$ converges uniformly to the function $u\in W^{1,\infty}(\Omega)$, as $\varepsilon_k\downarrow 0$. Thus, letting $\varepsilon_k\downarrow 0$ in formula (3.13), we obtain that

$$(3.14)\qquad \iint_\Omega fvdx=\iint_\Omega u\cdot\Delta(av)dx+\iint_\Omega u\cdot(A^*-\lambda)vdx\,.$$

Indeed, by estimate (3.10), the last term of the right-hand side of formula (3.13) tends to zero as $\varepsilon_k\downarrow 0$.

II-2b) By hypothesis (H), we can introduce in a tubular neighborhood of $\partial\Omega$ a local coordinate system $(y_1, y_2, \cdots, y_N)$ such that:

$$\begin{cases} \Omega=\{y_N>0\}, \\ \partial\Omega=\{y_N=0\}. \end{cases}$$

Assume that, in terms of this coordinate system, the operator A^* is of the form

$$A^*=\sum_{i,j=1}^N \alpha^{ij}\frac{\partial^2}{\partial y_i\partial y_j}+\sum_{i=1}^N \beta^i\frac{\partial}{\partial y_i}+c^*.$$

If $\delta>0$ is sufficiently small, we choose a function $\phi_\delta\in C^\infty(\bar{\Omega})$ such that $0\leqq\phi_\delta\leqq 1$ on $\bar{\Omega}$ and that:

$$\phi_\delta=\begin{cases} 0 & \text{in the } \delta\text{-neighborhood } G_\delta \text{ of } \Sigma_0\cup\Sigma_1 \text{ and in } \Omega\backslash D, \\ 1 & \text{in } D \text{ outside the } 2\delta\text{-neighborhood } G_{2\delta} \text{ of } \Sigma_0\cup\Sigma_1. \end{cases}$$

One may assume that the function ϕ_δ depends only on the variable y_N and that we have as $\delta\downarrow 0$

$$\frac{\partial\phi_\delta}{\partial y_N}=O(\delta^{-1}),$$

$$\frac{\partial^2\phi_\delta}{\partial y_N^2}=O(\delta^{-2}).$$

Let v_1 be an arbitrary function in $C^2(\bar{D})$ satisfying $v_1=0$ on $\Sigma_1\cup\Sigma_3$. Then it follows that the function $v_1\phi_\delta$ belongs to $C^2(\bar{D})$ and satisfies $v_1\phi_\delta=0$ on $\partial\Omega\backslash\Sigma_2$. Thus, applying formula (3.14) to the function $v_1\phi_\delta$, we obtain that

$$\iint_D f\cdot v_1\phi_\delta dx=\iint_D u\cdot(A^*-\lambda)(v_1\phi_\delta)dx, \tag{3.15}$$

since $av_1\phi_\delta=0$ in Ω.

II-2c) We shall show that formula (3.15) tends to formula (3.12) as $\delta\downarrow 0$.

i) First, by the Lebesgue convergence theorem, it follows that the left-hand side of formula (3.15) tends to the left-hand side of formula (3.12) as $\delta\downarrow 0$:

$$\lim_{\delta\downarrow 0}\iint_D fv_1\phi_\delta dx=\iint_D fv_1 dx. \tag{3.16}$$

ii) We rewrite the right-hand side of formula (3.15) in the following form:

$$\begin{aligned}\iint_D u\cdot(A^*-\lambda)(v_1\phi_\delta)dx&=\iint_D u((A^*-\lambda)v_1)\phi_\delta dx \\ &\quad+\iint_D uv_1(A^*\phi_\delta-c^*\phi_\delta)dx \\ &\quad+2\iint_D\left(\sum_{i,j=1}^N a^{ij}\frac{\partial v_1}{\partial x_i}\frac{\partial\phi_\delta}{\partial x_j}\right)dx \\ &\equiv I_1^\delta+I_2^\delta+I_3^\delta.\end{aligned} \tag{3.17}$$

We calculate the limit of the terms I_1^δ, I_2^δ and I_3^δ as $\delta\downarrow 0$.

ii-a) For the term I_1^δ, we have by the Lebesgue convergence theorem

$$\lim_{\delta\downarrow 0}I_1^\delta=\iint_D u\cdot(A^*-\lambda)v_1 dx. \tag{3.18}$$

ii-b) For the terms I_2^δ and I_3^δ, we remark that the integrals I_2^δ and I_3^δ are taken over the 2δ-neighborhood $G_{2\delta}$ of the set $\Sigma_0\cup\Sigma_1$ where the functions

$\partial\phi_\delta/\partial x_i$ and $\partial^2\phi_\delta/\partial x_i\partial x_j$ may be different from zero. Thus, passing to the local coordinate system $(y_1, y_2, \cdots, y_N)$, we obtain that

$$I_2^\delta=\iint_{G_{2\delta}}\left(\alpha^{NN}\frac{\partial^2\phi_\delta}{\partial y_N^2}+\beta^N\frac{\partial\phi_\delta}{\partial y_N}\right)v_1u\kappa dy,$$

$$I_3^\delta=2\iint_{G_{2\delta}}\alpha^{NN}\frac{\partial v_1}{\partial y_N}\frac{\partial\phi_\delta}{\partial y_N}u\kappa dy+2\sum_{i=1}^{N-1}\iint_{G_{2\delta}}\alpha^{iN}\frac{\partial v_1}{\partial y_i}\frac{\partial\phi_\delta}{\partial y_N}u\kappa dy,$$

since the function ϕ_δ depends only on the variable y_N. Here κ is some C^∞ function.

First we consider the limit of the term I_2^δ as $\delta\downarrow 0$: Since we have $\alpha^{NN}=O(\delta^2)$, $\partial^2\phi_\delta/\partial y_N^2=O(\delta^{-2})$ near the set $\Sigma_0\cup\Sigma_1$ and since the measure $|G_{2\delta}|$ of $G_{2\delta}$ is of order δ, it follows that

$$\lim_{\delta\downarrow 0}\iint_{G_{2\delta}}\alpha_{NN}\frac{\partial^2\phi_\delta}{\partial y_N^2}v_1u\kappa dy=0. \tag{3.19}$$

On the other hand, we remark that $v_1=0$ on Σ_1 and that the function β^N coincides with the Fichera function b^* for the operator A^* on Σ_0. This implies that

$$v_1=O(\delta) \qquad \text{near } \Sigma_1,$$

$$\beta^N=O(\delta) \qquad \text{near } \Sigma_0.$$

Hence we have

$$\lim_{\delta\downarrow 0}\iint_{G_{2\delta}}\beta^N\frac{\partial\phi_\delta}{\partial y_N}v_1u\kappa dy=0, \tag{3.20}$$

since $\partial\phi_\delta/\partial y_N=O(\delta^{-1})$ and $|G_{2\delta}|=O(\delta)$.

Therefore, we obtain from formulas (3.19) and (3.20) that

$$\lim_{\delta\downarrow 0}I_2^\delta=0. \tag{3.21}$$

Next we consider the limit of the term I_3^δ as $\delta\downarrow 0$: Since we have $\alpha^{NN}=O(\delta^2)$ and $\partial\phi_\delta/\partial y_N=O(\delta^{-1})$ near the set $\Sigma_0\cup\Sigma_1$, it follows that

$$\lim_{\delta\downarrow 0}\iint_{G_{2\delta}}\alpha^{NN}\frac{\partial v_1}{\partial y_N}\frac{\partial\phi_\delta}{\partial y_N}u\kappa dy=0. \tag{3.22}$$

Furthermore, since the matrix (α^{ij}) is non-negative definite, we find that

$$\alpha^{iN}=0 \qquad \text{on } \Sigma_0\cup\Sigma_1,\ 1\leqq i\leqq N-1,$$

and so

$$\alpha^{iN}=O(\delta) \qquad \text{near } \Sigma_0\cup\Sigma_1,\ 1\leqq i\leqq N-1.$$

Thus we have

$$\lim_{\delta\downarrow 0}\sum_{i=1}^{N-1}\iint_{G_{2\delta}}\alpha^{iN}\frac{\partial v_1}{\partial y_i}\frac{\partial\phi_\delta}{\partial y_N}u\kappa dy=0, \tag{3.23}$$

since $\partial\phi_\delta/\partial y_N=O(\delta^{-1})$ and $|G_{2\delta}|=O(\delta)$.

Hence, we obtain from formulas (3.22) and (3.23) that

$$\lim_{\delta\downarrow 0} I_3^\delta=0. \tag{3.24}$$

ii-c) Summing up, we obtain from formulas (3.17), (3.18), (3.21) and (3.24) that the right-hand side of formula (3.15) tends to the right-hand side of formula (3.12) as $\delta\downarrow 0$:

$$\lim_{\delta\downarrow 0}\iint_D u\cdot(A^*-\lambda)(v_1\phi_\delta)dx=\iint_D u\cdot(A^*-\lambda)v_1 dx. \tag{3.25}$$

iii) Therefore, formula (3.12) follows from formula (3.15) by combining formulas (3.16) and (3.25).

III) Now let f be an arbitrary function in the space $W^{1,\infty}(D)$. Then one can find a sequence $\{f_n\}_{n=1}^\infty$ in $C^{1+\theta}(\bar{D})$ such that

$$\|f_n\|_{C^1(\bar{D})}\leqq\|f\|_{1,\infty}, \tag{3.26a}$$

$$f_n\longrightarrow f \quad \text{in } C(\bar{D}) \text{ as } n\to\infty. \tag{3.26b}$$

By step II), it follows that there exists a weak solution $u_n\in W^{1,\infty}(D)$ of the Dirichlet problem:

$$\begin{cases}(A-\lambda)u_n=f_n & \text{in } D,\\ u_n=0 & \text{on } \Sigma_2\cup\Sigma_3,\end{cases}$$

and the solution u_n satisfies the estimate

$$\|u_n\|_{1,\infty}\leqq C_1(\lambda)\|f_n\|_{C^1(\bar{D})}\leqq C_1(\lambda)\|f\|_{1,\infty}. \tag{3.27}$$

But, by a Sobolev imbedding theorem (cf. [A, Lemma 5.17]), this implies that the sequence $\{u_n\}_{n=1}^\infty$ is uniformly bounded and Lipschitz continuous on $\bar{D}$ (and hence it is equicontinuous on $\bar{D}$). Thus, by virtue of the Ascoli-Arzelà theorem, one can choose a subsequence $\{u_{n'}\}$ which converges uniformly to a function u in $C(\bar{D})$ as $n'\to\infty$. Therefore, it follows from assertion (3.26b) that for all $v\in C^2(\bar{D})$ satisfying $v=0$ on $\Sigma_1\cup\Sigma_3$ we have

$$\begin{aligned}\iint_D u\cdot(A^*-\lambda)v\,dx&=\lim_{n'\to\infty}\iint_D u_{n'}\cdot(A^*-\lambda)v\,dx\\ &=\lim_{n'\to\infty}\iint_D f_{n'}v\,dx\\ &=\iint_D fv\,dx.\end{aligned}$$

On the other hand, just as in the proof of step II-1c) (cf. the proof of assertion (3.11)), we obtain from estimate (3.27) that

$$\begin{cases} u \in W^{1,\infty}(D), \\ \|u\|_{1,\infty} \leqq C_1(\lambda)\|f\|_{1,\infty}. \end{cases}$$

Summing up, we have proved that, for any $f \in W^{1,\infty}(D)$, there exists a weak solution u in the space $W^{1,\infty}(D)$ of problem (*) which satisfies inequality (3.1).

The proof of Theorem 3.1 is now complete.

3.2. Hölder Continuity for Weak Solutions

In this subsection, we study problem (D) in the framework of Hölder spaces. First we prove an existence theorem for problem (D) in the spaces $W^{m,\infty}(D)$ where $m \geqq 2$, generalizing Theorem 3.1 (cf. [OR, Theorem 1.8.2]; [C, Théorème 4.4]):

THEOREM 3.5. *Assume that hypothesis (H) is satisfied and that conditions (2.3) and (2.19) are satisfied. Then, for each integer $m \geqq 2$, one can find a constant $\lambda = \lambda(m) > 0$ such that, for any function f in the space $W^{2m+2,\infty}(D)$, there exists a weak solution $u \in W^{m,\infty}(D)$ of the Dirichlet problem:*

$$(*) \qquad \begin{cases} (A-\lambda)u = f & \text{in } D, \\ u = 0 & \text{on } \Sigma_2 \cup \Sigma_3. \end{cases}$$

Moreover, the solution u satisfies the inequality

$$(3.28) \qquad \|u\|_{m,\infty} \leqq C_m(\lambda)\|f\|_{2m+2,\infty},$$

where $C_m(\lambda) > 0$ is a constant independent of f.

PROOF. I) We modify the domain D and the operator A so that the set $\Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ is of type Σ_3, as in the proof of Theorem 3.1.

By hypothesis (H), one can choose a bounded domain Ω with C^∞ boundary $\partial\Omega$ such that (cf. Figure 4)

$$\begin{cases} D \cup \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \subset \Omega, \\ \Sigma_3 \subset \partial\Omega. \end{cases}$$

One may assume that

$$(2.3') \qquad c < 0 \quad \text{on } \bar{\Omega},$$

$$(2.19') \qquad c^* < 0 \quad \text{on } \bar{\Omega}.$$

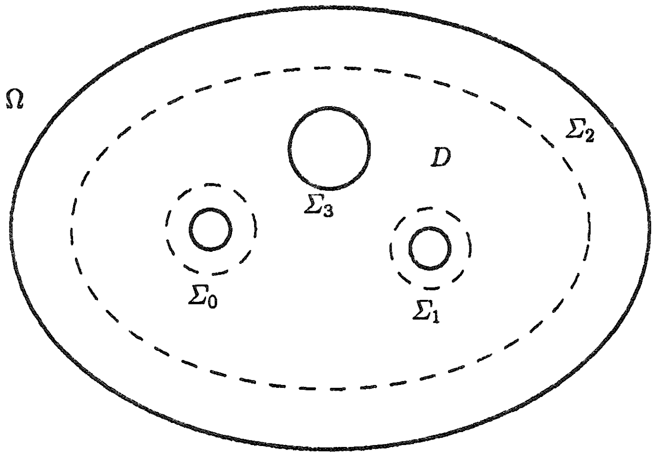

Figure 4.

Now we take a function $a \in C^{\infty}(\bar{\Omega})$ such that

(3.29) $$\begin{cases} a=0 & \text{in } D, \\ a>0 & \text{in } \bar{\Omega}\backslash\bar{D}, \end{cases}$$

and consider the Dirichlet problem for the *elliptic* operators $\varepsilon\Delta+A+a\Delta-\lambda(\varepsilon>0)$:

$(\tilde{D}_\varepsilon)$ $$\begin{cases} (\varepsilon\Delta+A+a\Delta-\lambda)u_\varepsilon=f & \text{in } \Omega, \\ u_\varepsilon=0 & \text{on } \partial\Omega. \end{cases}$$

We remark that by condition (3.29)

$$\sum_{i,j=1}^{N} a^{ij}(x')n_i n_j+a(x')\sum_{i=1}^{N} n_i^2>0 \qquad \text{on } \partial\Omega.$$

In other words, the boundary $\partial\Omega$ is of type Σ_3 for the operators $A+a\Delta-\lambda$, $\lambda>0$.

II) First let f be an arbitrary function in the space $C^{2m+2+\theta}(\bar{D})$, $0<\theta<1$. We show that there exists a weak solution $u \in W^{m,\infty}(D)$ of problem (*) which satisfies inequality (3.28).

One may assume that

$$f \in C^{2m+2+\theta}(\bar{\Omega}),$$

and that

$$\|f\|_{C^{2m+2}(\bar{\Omega})} \leqq \|f\|_{C^{2m+2}(\bar{D})}.$$

II-1) We construct a function $w \in C^{m+2+\theta}(\bar{\Omega})$ such that the function $(A-\lambda)w-f$ vanishes on Σ_2, together with its derivatives of order $\leqq m$, and that

(3.30) $$\|w\|_{C^{m+2}(\bar{\Omega})} \leqq C(\lambda)\|f\|_{C^{2m+2}(\bar{\Omega})} \leqq C(\lambda)\|f\|_{C^{2m+2}(\bar{D})}.$$

Let x_0' be an arbitrary point of the set Σ_2. We construct the function w locally in a neighborhood of x_0'. To do so, we introduce a local coordinate system $(y_1, y_2, \cdots, y_N)$ in a neighborhood of x_0' such that

$$\begin{cases} x_0'=0, \\ D=\{y_N>0\}, \\ \Sigma_2=\{y_N=0\}, \end{cases}$$

and assume that the equation $(A-\lambda)v=f$ takes the form:

(3.31) $$\sum_{i,j=1}^{N} \alpha^{ij} \frac{\partial^2 v}{\partial y_i \partial y_j} + \sum_{i=1}^{N} \beta^i \frac{\partial v}{\partial y_i} + (c-\lambda)v = f.$$

Since the matrix (α^{ij}) is non-negative definite and $\alpha^{NN}=0$ on Σ_2, it follows that

$$\frac{\partial \alpha^{NN}}{\partial y_N}=0 \qquad \text{on } \Sigma_2,$$

and that

$$\alpha^{Nj}=0 \qquad \text{on } \Sigma_2, \quad 1 \leqq j \leqq N-1,$$

$$\frac{\partial \alpha^{Nj}}{\partial y_k}=0 \qquad \text{on } \Sigma_2, \quad 1 \leqq j,\ k \leqq N-1.$$

Thus we have

(3.32) $$\beta^N = \beta^N - \sum_{j=1}^{N} \frac{\partial \alpha^{Nj}}{\partial y_j} = b < 0 \qquad \text{on } \Sigma_2,$$

and also

(3.31′) $$(A-\lambda)v = \beta^N \frac{\partial v}{\partial y_N} + \left(\sum_{i,j=1}^{N-1} \alpha^{ij} \frac{\partial^2 v}{\partial y_i \partial y_j} + \sum_{i=1}^{N-1} \beta^i \frac{\partial v}{\partial y_i} + (c-\lambda)v \right)$$
$$= f \qquad \text{on } \Sigma_2.$$

Now assume that

$$v=0 \qquad \text{on } \Sigma_2.$$

Then we obtain from formulas (3.31′) and (3.32) that

$$\frac{\partial v}{\partial y_N}(y', 0) = \frac{f(y', 0)}{\beta^N(y', 0)}.$$

Furthermore, differentiating equation (3.31′) with respect to the variable y_N, we obtain that

$$\frac{\partial^2 v}{\partial y_N^2}(y', 0) = -\frac{1}{\beta^N(y', 0)}\Big[\frac{\partial \beta^N}{\partial y_N}(y', 0)\frac{f(y', 0)}{\beta^N(y', 0)}$$
$$+\sum_{i,j=1}^{N-1}\alpha^{ij}(y', 0)\frac{\partial^2}{\partial y_i \partial y_j}\Big(\frac{f(y', 0)}{\beta^N(y', 0)}\Big)$$
$$+\sum_{i=1}^{N-1}\beta^i(y', 0)\frac{\partial}{\partial y_i}\Big(\frac{f(y', 0)}{\beta^N(y', 0)}\Big)$$
$$+(c(y', 0)-\lambda)\frac{f(y', 0)}{\beta^N(y', 0)} - \frac{\partial f}{\partial y_N}(y', 0)\Big].$$

Similarly, continuing this process, we may find all the derivatives $(\partial^l v/\partial y_N^l)(y', 0)$ for $1 \leqq l \leqq m$.

We define, in a neighborhood of x_0',

$$w(y', y_N) = \sum_{l=1}^{m}\frac{1}{l!}\frac{\partial^l v}{\partial y_N^l}(y', 0) y_N^l.$$

Then it is easy to verify that the function w satisfies inequality (3.30) and that the function

$$(A-\lambda)w - f$$

vanishes in a neighborhood of $x_0' \in \Sigma_2$, together with its derivatives of order $\leqq m$, and is of class $C^{m+\theta}$.

In order to construct the function w in the entire domain Ω, we cover the set Σ_2 by a finite number of coordinates patches $\{\omega_j\}_{j=1}^d$ such that, in each ω_j, one may pass to a local coordinate system $y=(y_1, y_2, \cdots, y_N)$ and construct a function w_j as above. Let $\{\psi_j\}_{j=1}^d$ be a partition of unity subordinate to the covering $\{U_j\}_{j=1}^d$. Then it is easy to verify that the function

$$w = \sum_{j=1}^{d}\psi_j w_j$$

satisfies the desired conditions. Furthermore, by hypothesis (H), one can (re)-construct the function w so that

$$w=0 \quad \text{on } \Sigma_3.$$

II-2) We let

$$\tilde{f} = f - (A-\lambda)w.$$

Then it follows that the function $\tilde{f}$ vanishes on Σ_2, together with its derivatives of order $\leqq m$, and belongs to the space $C^{m+\theta}(\bar{\Omega})$. Thus, letting

$$f_1 = \begin{cases} 0 & \text{in the tubular neighborhood } \mathcal{U} \text{ of } \Sigma_2 \text{ in } \bar{\Omega}\backslash\bar{D}, \\ \tilde{f} & \text{in } \bar{\Omega}\backslash\mathcal{U}, \end{cases} \tag{3.33}$$

we obtain that $f_1 \in C^{m+\theta}(\bar{\Omega})$. Furthermore, it follows from inequality (3.30) that

(3.34) $$\|f_1\|_{C^m(\bar{\Omega})} \leq C(\lambda)\|f\|_{C^{2m+2}(\bar{D})}.$$

Now we know (cf. [GT]) that the Dirichlet problem

$(\tilde{D}_\varepsilon)$ $$\begin{cases} (\varepsilon\Delta + A + a\Delta - \lambda)u_\varepsilon = f_1 & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution u_ε in the space $C^{m+2+\theta}(\bar{\Omega})$, since $f_1 \in C^{m+\theta}(\bar{\Omega})$.

II-3) We show that there exists a subsequence $\{u_{\varepsilon_k}\}_{k=1}^{\infty}$ which, together with all its derivatives of order $\leq m$, converges weakly to some function $\tilde{u} \in W^{m,\infty}(\Omega)$, as $\varepsilon_k \downarrow \infty$.

We only show that $\tilde{u} \in W^{2,\infty}(\Omega)$. The proof that $\tilde{u} \in W^{m,\infty}(\Omega)$ for each positive integer $m \geq 3$ can be carried out in a similar way.

II-3a) The next result, analogous to Lemma 3.2, may be proved just as in the proof of Théorème 4.1 of Cancelier [C].

LEMMA 3.6. *If $\varphi \in C^\infty(\bar{D})$, we let*

$$p_2(x) = \sum_{i,j=1}^{N} \left| \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x) \right|^2, \; x \in \bar{D},$$

and

$$R_2(x) = A p_2(x) - \sum_{i,j=1}^{N} B_A\left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j}, \frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right)(x), \; x \in \bar{D}.$$

Then, for each $\eta > 0$, there exist constants $\beta_1 > 0$ and $\beta_2 > 0$ such that we have for all $\varphi \in C^\infty(\bar{D})$

$$|R_2(x)| \leq \eta \sum_{i,j=1}^{N} B_A\left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j}, \frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right)(x) + \beta_1 \|\varphi\|^2_{C^1(\bar{D})}$$
$$+ \beta_2 \|\varphi\|^2_{C^2(\bar{D})} + \frac{1}{2}\|A\varphi\|^2_{C^2(\bar{D})}, \; x \in \bar{D}.$$

We remark that the constants β_1 and β_2 are *uniform* for the operators $A + \varepsilon\Delta - \lambda I$, $0 \leq \varepsilon \leq 1$, $\lambda \geq 0$.

II-3b) The proof that $\tilde{u} \in W^{2,\infty}(\Omega)$ is based on the following lemma (cf. [OR, Lemma 1.8.1]):

LEMMA 3.7. *Assume that hypothesis (H) is satisfied with $\partial D = \Sigma_3$ and that condition (2.3) is satisfied. Then one can find a constant $\lambda > 0$ such that if f is a function in the space $C^{2+\theta}(\bar{D})$, then the unique solution $u_\varepsilon \in C^{4+\theta}(\bar{D})$ of the Dirichlet problem*

$$\begin{cases} (A+\varepsilon\Delta-\lambda)u_\varepsilon=f & \text{in } D, \\ u_\varepsilon=0 & \text{on } \partial D \end{cases}$$

satisfies the estimate

$$\|u_\varepsilon\|_{C^2(\bar{D})} \leqq C_2(\lambda)\|f\|_{C^2(\bar{D})}, \tag{3.35}$$

where $C_2(\lambda)>0$ is a constant independent of $\varepsilon>0$.

PROOF. We recall that

$$\|u_\varepsilon\|_{C^1(\bar{D})} \leqq C_1(\lambda)\|f\|_{C^1(\bar{D})}. \tag{3.3}$$

Thus, to prove estimate (3.35), it suffices to show that

$$\Big(\max_{\bar{D}} \sum_{i,j=1}^{N} \Big|\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}\Big|^2\Big)^{1/2} \leqq M_2(\lambda)(\|u_\varepsilon\|_{C^1(\bar{D})}+\|f\|_{C^2(\bar{D})}), \tag{3.36}$$

where $M_2(\lambda)>0$ is a constant independent of $\varepsilon>0$.

We let

$$p_2^\varepsilon(x)=\sum_{i,j=1}^{N}\Big|\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}(x)\Big|^2, \quad x\in\bar{D}.$$

i) First we assume that the function $p_2^\varepsilon(x)$ attains its positive maximum at a point x_0 of D. Then, since the matrix (a^{ij}) is non-negative definite, we obtain that

$$(A+\varepsilon\Delta)p_2^\varepsilon(x_0) \leqq c(x_0)p_2^\varepsilon(x_0). \tag{3.37}$$

But it follows from an application of Lemma 3.6 with $\eta=1/2$ that

$$(A+\varepsilon\Delta-\lambda)p_2^\varepsilon(x)=\sum_{i,j=1}^{N} B_{A+\varepsilon\Delta-\lambda I}\Big(\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}, \frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\Big)(x)+R_2(x),$$

with

$$|R_2(x)| \leqq \frac{1}{2}\sum_{i,j=1}^{N} B_{A+\varepsilon\Delta-\lambda I}\Big(\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}, \frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\Big)(x) + \beta_1\|u_\varepsilon\|_{C^1(\bar{D})}^2+\beta_2\|u_\varepsilon\|_{C^2(\bar{D})}^2+\frac{1}{2}\|f\|_{C^2(\bar{D})}^2. \tag{3.38}$$

Here the constants β_1 and β_2 are independent of $\varepsilon>0$ and $\lambda>0$.

Hence we obtain from inequalities (3.37), (3.38) and (3.3) that

$$\begin{aligned}
\lambda p_2^\varepsilon(x_0) &\leqq (\lambda-c(x_0))p_2^\varepsilon(x_0) \\
&\leqq (\lambda-A-\varepsilon\Delta)p_2^\varepsilon(x_0) \\
&= -\Big((A+\varepsilon\Delta-\lambda)p_2^\varepsilon(x_0)-\sum_{i,j=1}^{N} B_{A+\varepsilon\Delta-\lambda I}\Big(\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}, \frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\Big)(x_0)\Big) \\
&\quad -\sum_{i,j=1}^{N} B_{A+\varepsilon\Delta-\lambda I}\Big(\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}, \frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\Big)(x_0)
\end{aligned}$$

$$\leqq \beta_1 \|u_\varepsilon\|^2_{C^1(\bar{D})} + \beta_2(\|u_\varepsilon\|^2_{C^1(\bar{D})} + p_2^\varepsilon(x_0)) + \frac{1}{2}\|f\|^2_{C^2(\bar{D})}$$

$$\leqq (\beta_1+\beta_2)C_1(\lambda)^2\|f\|^2_{C^1(\bar{D})} + \beta_2 p_2^\varepsilon(x_0) + \frac{1}{2}\|f\|^2_{C^2(\bar{D})}.$$

Therefore, if $\lambda>0$ is so large that

$$\lambda>\beta_2,$$

then it follows that

$$p_2^\varepsilon(x_0) \leqq C(\lambda)\|f\|^2_{C^2(\bar{D})}, \tag{3.39}$$

where $C(\lambda)>0$ is a constant *independent* of $\varepsilon>0$.

ii) Next we assume that the function $p_2^\varepsilon(x)$ attains its positive maximum at a point x_0' of ∂D, and let

$$q_2^\varepsilon = \sqrt{p_2^\varepsilon(x_0')} = \left(\max_{x'\in\partial D}\sum_{i,j=1}^{N}\left|\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}(x')\right|^2\right)^{1/2}.$$

ii-a) Since $\partial u_\varepsilon/\partial x_j=0$ on ∂D for $1\leqq j\leqq N-1$, applying estimate (2.10) to the functions $\partial u_\varepsilon/\partial x_j$, we obtain that:

For *every* $\eta>0$, there exists a constant $M_\eta>0$ independent of $\varepsilon>0$ such that

$$\left(\max_{x'\in\partial D}\sum_{j=1}^{N-1}\left|\frac{\partial^2 u_\varepsilon}{\partial x_j \partial x_N}(x')\right|^2\right)^{1/2} \leqq \eta\left\|(A+\varepsilon\Delta-\lambda)\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right\|_{C(\bar{D})} + M_\eta\left\|\frac{\partial u_\varepsilon}{\partial x_j}\right\|_{C(\bar{D})}. \tag{3.40}$$

But it follows that

$$\begin{aligned}
(A+\varepsilon\Delta-\lambda)\left(\frac{\partial u_\varepsilon}{\partial x_j}\right) &= \frac{\partial}{\partial x_j}((A+\varepsilon\Delta-\lambda)u_\varepsilon) + \left[A+\varepsilon\Delta-\lambda, \frac{\partial}{\partial x_j}\right]u_\varepsilon \\
&= \frac{\partial f}{\partial x_j} + \left[A+\varepsilon\Delta-\lambda, \frac{\partial}{\partial x_j}\right]u_\varepsilon \\
&= \frac{\partial f}{\partial x_j} - \left(\sum_{l,m=1}^{N}\frac{\partial a^{lm}}{\partial x_j}\frac{\partial^2 u_\varepsilon}{\partial x_l \partial x_m} + \sum_{l=1}^{N}\frac{\partial b^l}{\partial x_j}\frac{\partial u_\varepsilon}{\partial x_l} + \frac{\partial c}{\partial x_j}u_\varepsilon\right) \\
&\quad -\varepsilon\left(\sum_{l,m=1}^{N}\frac{\partial \mu^{lm}}{\partial x_j}\frac{\partial^2 u_\varepsilon}{\partial x_l \partial x_m} + \sum_{l=1}^{N}\frac{\partial \nu^l}{\partial x_j}\frac{\partial u_\varepsilon}{\partial x_l}\right).
\end{aligned}$$

Hence we have with a constant $C>0$ independent of $\varepsilon>0$

$$\left\|(A+\varepsilon\Delta-\lambda)\left(\frac{\partial u_\varepsilon}{\partial x_j}\right)\right\|_{C(\bar{D})} \leqq \|f\|_{C^1(\bar{D})} + C(q_2^\varepsilon + \|u_\varepsilon\|_{C^1(\bar{D})}).$$

Therefore, combining this inequality with estimate (3.40), we obtain that

$$\left(\max_{x'\in\partial D}\sum_{j=1}^{N-1}\left|\frac{\partial^2 u_\varepsilon}{\partial x_j \partial x_N}(x')\right|^2\right)^{1/2} \leqq \frac{1}{2}q_2^\varepsilon + C'(\|u_\varepsilon\|_{C^1(\bar{D})} + \|f\|_{C^1(\bar{D})}). \tag{3.41}$$

Here $C'>0$ is a constant independent of $\varepsilon>0$.

ii-b) In order to estimate the term

$$\frac{\partial^2 u_\varepsilon}{\partial y_N^2}(x_0'),$$

we choose a local coordinate system $(y_1, y_2, \cdots, y_N)$ in a neighborhood of x_0' such that

$$\begin{cases} x_0'=0, \\ D=\{y_N>0\}, \\ \partial D=\{y_N=0\}, \end{cases}$$

and assume that, in terms of this coordinate system, the equation

$$(A+\varepsilon\Delta-\lambda)u_\varepsilon=f$$

is of the form

$$\begin{aligned}(A+\varepsilon\Delta-\lambda)u_\varepsilon=&(\alpha^{NN}+\varepsilon\mu^{NN})\frac{\partial^2 u_\varepsilon}{\partial y_N^2}+\sum_{i,j=1}^{N-1}(\alpha^{ij}+\varepsilon\mu^{ij})\frac{\partial^2 u_\varepsilon}{\partial y_i\partial y_j}\\ &+(\beta^N+\varepsilon\nu^N)\frac{\partial u_\varepsilon}{\partial y_N}+\sum_{i=1}^{N-1}(\beta^i+\varepsilon\nu^i)\frac{\partial u_\varepsilon}{\partial y_i}+(c-\lambda)u_\varepsilon\\ =&f.\end{aligned}$$

Since $u_\varepsilon=0$ on ∂D and $x_0'\in\partial D=\Sigma_3$ (so $\alpha^{NN}(0)>0$), it follows that

$$\frac{\partial^2 u_\varepsilon}{\partial y_N^2}(0)=\frac{1}{\alpha^{NN}(0)+\varepsilon\mu^{NN}(0)}\Big(f(0)-(\beta^N(0)+\varepsilon\nu^N(0))\frac{\partial u_\varepsilon}{\partial y_N}(0)\Big).$$

Hence we have

(3.42) $$\left|\frac{\partial^2 u_\varepsilon}{\partial y_N^2}(0)\right|\leqq C''(\|u_\varepsilon\|_{C^1(\bar D)}+\|f\|_{C(\bar D)}),$$

with a constant $C''>0$ independent of $\varepsilon>0$.

ii-c) Finally we remark that

$$\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}=0 \quad \text{on } \partial D, \quad 1\leqq i,\ j\leqq N-1.$$

Therefore, combining estimates (3.41) and (3.42), we find that

$$q_2^\varepsilon\leqq\frac{1}{2}q_2^\varepsilon+C'''(\|u_\varepsilon\|_{C^1(\bar D)}+\|f\|_{C^1(\bar D)}),$$

so that

(3.43) $$q_2^\varepsilon\leqq 2C'''(\|u_\varepsilon\|_{C^1(\bar D)}+\|f\|_{C^1(\bar D)}).$$

Here $C'''>0$ is a constant independent of $\varepsilon>0$.

iii) The desired estimate (3.36) (and hence estimate (3.35)) follows by combining estimates (3.39) and (3.43).

The proof of Lemma 3.7 is complete.

II-3c) Now, since $f_1 \in C^{m+\theta}(\bar{\Omega})$, it follows from an application of Lemm 3.7 with $A = A + a\Delta$ that the unique solution $u_\varepsilon \in C^{m+2+\theta}(\bar{\Omega})$ of the Dirichlet problem

$$(\tilde{D}_\varepsilon) \qquad \begin{cases} (\varepsilon\Delta + A + a\Delta - \lambda)u_\varepsilon = f_1 & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies the estimate

$$\|u_\varepsilon\|_{C^2(\bar{\Omega})} \leqq C_2(\lambda) \|f_1\|_{C^2(\bar{\Omega})}.$$

Hence, combining this estimate with inequality (3.34), we obtain that

$$\|u_\varepsilon\|_{C^2(\bar{\Omega})} \leqq C_m(\lambda) \|f\|_{C^{2m+2}(\bar{D})},$$

where $C_m(\lambda) > 0$ is a constant independent of $\varepsilon > 0$.

Therefore, arguing as in step II-1c) of the proof of Theorem 3.1, we can choose a subsequence $\{u_{\varepsilon_k}\}_{k=1}^{\infty}$ which, together with all its derivatives of order $\leqq 2$, converges weakly to some function $\tilde{u}$ in the Hilbert space $L^2(\Omega)$ as $\varepsilon_k \downarrow \infty$. Thus, passing to the limit in problem $(\tilde{D}_\varepsilon)$, we obtain that $\tilde{u}$ belongs to the space $W^{2,\infty}(\Omega)$ and satisfies

$$(\tilde{D}) \qquad \begin{cases} (A + a\Delta - \lambda)\tilde{u} = f_1 & \text{in } \Omega, \\ \tilde{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\|\tilde{u}\|_{2,\infty} \leqq C_m(\lambda) \|f\|_{C^{2m+2}(\bar{D})}.$$

Hence it follows from formulas (3.33) and (3.29) that

$$(A - \lambda)\tilde{u} = \tilde{f} = f - (A - \lambda)w \qquad \text{in } D,$$

and that

$$\tilde{u} = 0 \qquad \text{on } \Sigma_3,$$

since the boundary $\partial\Omega$ contains the set Σ_3.

II-4) Finally we show that $\tilde{u} = 0$ on Σ_2 and hence $\tilde{u} = 0$ on $\Sigma_2 \cup \Sigma_3$.

By formulas (3.33) and (3.29), we find that

$$(A + a\Delta - \lambda)\tilde{u} = 0 \qquad \text{in } \mathcal{U},$$

where $\mathcal{U}$ is the tubular neighborhood of Σ_2 in $\Omega \backslash \bar{D}$. But we remark that the set Σ_2 is of type Σ_1 for the operator $A + a\Delta - \lambda$ in the domain $\mathcal{U}$ and that condition (2.19′) is satisfied. Hence it follows from an application of the uniqueness theorem for the Dirichlet problem (Theorem 2.6) that

$$\tilde{u} = 0 \qquad \text{in } \mathcal{U},$$

so that

$$\tilde{u}=0 \quad \text{on } \Sigma_2 .$$

Therefore, since $w=0$ on $\Sigma_2 \cup \Sigma_3$, we obtain that the function

$$u=\tilde{u}+w \in W^{2,\infty}(D)$$

is a weak solution of problem $(*)$ which satisfies the inequality

$$\|u\|_{2,\infty} \leqq C_2(\lambda)\|f\|_{C^{2m+2}(\bar{\Omega})} \leqq C_2(\lambda)\|f\|_{C^{2m+2}(\bar{D})} .$$

III) Now let f be an arbitrary function in the space $W^{2m+2,\infty}(D)$. Then one can find a sequence $\{f_n\}_{n=1}^{\infty}$ in the space $C^{2m+2+\theta}(\bar{D})$ such that

$$\begin{cases} \|f_n\|_{C^{2m+2}(\bar{D})} \leqq \|f\|_{2m+2,\infty}, \\ f_n \longrightarrow f \quad \text{in } C(\bar{D}) \text{ as } n \to \infty . \end{cases}$$

By step II), it follows that there exists a weak solution $u_n \in W^{m,\infty}(D)$ of the Dirichlet problem:

$$\begin{cases} (A-\lambda)u_n=f_n & \text{in } D, \\ u_n=0 & \text{on } \Sigma_2 \cup \Sigma_3, \end{cases}$$

and the solution u_n satisfies the estimate

$$\|u_n\|_{m,\infty} \leqq C_m(\lambda)\|f_n\|_{C^{2m+2}(\bar{D})} \leqq C_m(\lambda)\|f\|_{2m+2,\infty} .$$

Therefore, just as in the proof of step III) of Theorem 3.1, we obtain that the limit function u of u_n when $n \to \infty$ is a weak solution in the space $W^{m,\infty}(D)$ of problem $(*)$ which satisfies inequality (3.28).

The proof of Theorem 3.5 is now complete.

3.3. Proof of Theorem 2

Theorem 2 follows from Theorem 3.5 by a well-known interpolation argument (cf. [Tr]), since the space $C^{k+\theta}(\bar{D})$ is a real interpolation space between the spaces $W^{k,\infty}(D)$ and $W^{k+1,\infty}(D)$:

$$C^{k+\theta}(\bar{D})=(W^{k,\infty}(D), W^{k+1,\infty}(D))_{\theta,\infty} .$$

Furthermore, we can prove the following existence and uniqueness theorem for problem (D) in the framework of Hölder spaces:

THEOREM 3.8. *Assume that hypothesis* (H) *is satisfied and that conditions* (2.3) *and* (2.19) *are satisfied. Then, for each integer* $m \geqq 2$, *one can find a constant* $\lambda=\lambda(m)>0$ *such that, for any* $f \in C^{2m+2+2\theta}(\bar{D})$ *and any* $g \in C^{2m+4+2\theta}(\Sigma_2 \cup \Sigma_3)$, *there exists a unique solution* $u \in C^{m+\theta}(\bar{D})$ *of the Dirichlet problem*:

$$(D)\qquad \begin{cases} (A-\lambda)u=f & \text{in } D, \\ u=g & \text{on } \Sigma_2\cup\Sigma_3. \end{cases}$$

4. Proof of Theorem 1

The proof of Theorem 1 is based on Theorem 1.4 which is a Feller semigroup version of the Hille-Yosida theorem in terms of the maximum principle. We shall verify conditions (α), (β) and (γ) of the same theorem.

4.1. The Space $C_0(\bar{D}\backslash M)$

First we consider a one-point compactification $K_\partial=K\cup\{\partial\}$ of the space $K=\bar{D}\backslash M$, where

$$M=\Sigma_2\cup\Sigma_3.$$

We say that two points x and y of $\bar{D}$ are equivalent modulo M if either $x=y$ or $x, y\in M$. We denote by $\bar{D}/M$ the totality of equivalence classes modulo M. On the set $\bar{D}/M$, we define the quotient topology induced by the projection $q: \bar{D}\to\bar{D}/M$. Then it is easy to see that the topological space $\bar{D}/M$ is a *one-point compactification* of the space $\bar{D}\backslash M$ and that the *point at infinity* ∂ corresponds to the set M:

$$K_\partial=\bar{D}/M,$$

$$\partial=M.$$

Furthermore we have the following isomorphism:

$$(4.1)\qquad C(K_\partial)\cong\{u\in C(\bar{D});\ u \text{ is constant on } \Sigma_2\cup\Sigma_3\}.$$

Now we introduce a closed subspace of $C(K_\partial)$ as in Subsection 1.1:

$$C_0(K)=\{u\in C(K_\partial);\ u(\partial)=0\}.$$

Then we have by assertion (4.1)

$$(4.2)\qquad C_0(K)\cong C_0(\bar{D}\backslash M)=\{u\in C(\bar{D});\ u=0 \text{ on } \Sigma_2\cup\Sigma_3\}.$$

4.2. Proof of Theorem 1

The next theorem summarizes the basic results of Sections 2 and 3 about the Dirichlet problem in the framework of Hölder spaces:

THEOREM 4.1. *Assume that hypothesis* (H) *is satisfied. Then, for each integer* $m\geq 2$, *one can find a constant* $\alpha=\alpha(m)>0$ *such that, for any* $f\in C^{2m+2+2\theta}(\bar{D})$ *and any* $\varphi\in C^{2m+4+2\theta}(\Sigma_2\cup\Sigma_3)$, $0<\theta<1$, *there exists a unique solution* $u\in C^{m+\theta}(\bar{D})$ *of the Dirichlet problem*:

$$(D) \qquad \begin{cases} (\alpha-A)u=f & \text{in } D, \\ u=\varphi & \text{on } \Sigma_2\cup\Sigma_3. \end{cases}$$

Moreover, the solution u satisfies the inequality

$$\max_{\bar{D}}|u|\leqq\max\left(\frac{1}{\alpha}\max_{\bar{D}}|f|,\ \max_{\Sigma_2\cup\Sigma_3}|\varphi|\right). \tag{4.3}$$

Theorem 4.1 with $m=2$ tells us that problem (D) has a unique solution u in the space $C^{2+\theta}(\bar{D})$ for any $f\in C^{6+2\theta}(\bar{D})$ and any $\varphi\in C^{8+2\theta}(\Sigma_2\cup\Sigma_3)$, if $\alpha>0$ is sufficiently large. Therefore, we can introduce linear operators

$$G_\alpha^0:\ C^{6+2\theta}(\bar{D})\longrightarrow C^{2+\theta}(\bar{D})$$

and

$$H_\alpha:\ C^{8+2\theta}(\Sigma_2\cup\Sigma_3)\longrightarrow C^{2+\theta}(\bar{D})$$

as follows.

a) For any $f\in C^{6+2\theta}(\bar{D})$, the function $G_\alpha^0 f\in C^{2+\theta}(\bar{D})$ is the unique solution of the problem:

$$\begin{cases} (\alpha-A)G_\alpha^0 f=f & \text{in } D, \\ G_\alpha^0 f=0 & \text{on } \Sigma_2\cup\Sigma_3. \end{cases}$$

b) For any $\varphi\in C^{8+2\theta}(\Sigma_2\cup\Sigma_3)$, the function $H_\alpha\varphi\in C^{2+\theta}(\bar{D})$ is the unique solution of the problem:

$$\begin{cases} (\alpha-A)H_\alpha\varphi=0 & \text{in } D, \\ H_\alpha\varphi=\varphi & \text{on } \Sigma_2\cup\Sigma_3. \end{cases}$$

The operator C_α^0 is called the *Green operator* and the operator H_α is called the *harmonic operator*, respectively.

Then we have the following result:

LEMMA 4.2. *The operator $G_\alpha^0(\alpha>0)$, considered from $C(\bar{D})$ into itself, is non-negative and continuous with norm*

$$\|G_\alpha^0\|=\|G_\alpha^0 1\|=\max_{x\in\bar{D}}G_\alpha^0 1(x)\leqq\frac{1}{\alpha}. \tag{4.4}$$

PROOF. First, in order to prove the non-negativity of G_α^0, we assume that:

$$f\in C^{6+2\theta}(\bar{D}) \quad\text{and}\quad f\geqq 0 \text{ on } \bar{D}.$$

Then one can find a unique solution $u_\varepsilon\in C^{8+2\theta}(\bar{D})$ of the Dirichlet problem for the elliptic operators $A-\alpha+\varepsilon\Delta(\varepsilon>0)$:

$$\begin{cases} (\alpha-A-\varepsilon\Delta)u_\varepsilon=f & \text{in } D, \\ u_\varepsilon=0 & \text{on } \partial D. \end{cases}$$

Since we have

$$\begin{cases} (A+\varepsilon\Delta-\alpha)u_\varepsilon=-f\leqq 0 & \text{in } D, \\ u_\varepsilon=0 & \text{on } \partial D, \end{cases}$$

it follows from an application of the maximum principle (Theorem A.1) that

$$u_\varepsilon\geqq 0 \quad \text{on } \bar{D}.$$

But we know (cf. the proof of Theorem 3.5) that a subsequence $\{u_{\varepsilon_k}\}$ converges uniformaly to the function $G_\alpha^0 f\in C^{2+\theta}(\bar{D})$, as $\varepsilon_k\downarrow 0$. Hence we have

$$G_\alpha^0 f\geqq 0 \quad \text{on } \bar{D}.$$

This proves the non-negativity of G_α^0.

Therefore, inequality (4.4) follows from inequality (4.3) by taking $f=1$ and $\varphi=0$.

The proof of Lemma 4.2 is complete.

Similarly, we have the following:

LEMMA 4.3. *The operator $H_\alpha(\alpha>0)$, considered from $C(\Sigma_2\cup\Sigma_3)$ into $C(\bar{D})$, is non-negative and continuous with norm*

$$\|H_\alpha\|=\|H_\alpha 1\|=\max_{x\in\bar{D}} H_\alpha 1(x)=1.$$

PROOF OF THEOREM 1. We recall that $\mathcal{A}$ is a linear operator from the space $C_0(\bar{D}\backslash M)$ into itself defined by the following:

(1) The domain $D(\mathcal{A})$ of $\mathcal{A}$ is the space

$$D(\mathcal{A})=\{u\in C^2(\bar{D})\,;\ u=Au=0 \text{ on } \Sigma_2\cup\Sigma_3\}.$$

(2) $\mathcal{A}u=Au,\ u\in D(\mathcal{A})$.

I) First we verify condition (α), that is, the density of the domain $D(\mathcal{A})$ in the space $C_0(\bar{D}\backslash M)$.

Now we assume that:

$$f\in C^\infty(\bar{D}) \quad \text{and} \quad f=0 \text{ on } \Sigma_2\cup\Sigma_3.$$

Then we obtain that

$$AG_\alpha^0 f=\alpha G_\alpha^0 f-f=0 \quad \text{on } \Sigma_2\cup\Sigma_3,$$

so that

$$G_\alpha^0 f\in D(\mathcal{A}).$$

But it follows from an application of the uniqueness theorem for the Dirichlet problem (Theorem 2.6) that

$$f-\alpha G_\alpha^0 f=G_\alpha^0((\beta-A)f)-\beta G_\alpha^0 f,\ \beta>0.$$

Indeed, the both sides have the same boundary value 0 on the set $\Sigma_2\cup\Sigma_3$ and satisfy the same equation: $(\alpha-A)u=-Af$ in D. In view of inequality (4.4), we have

$$\|f-\alpha G_\alpha^0 f\|\leqq\frac{1}{\alpha}\|(\beta-A)f\|+\frac{\beta}{\alpha}\|f\|,$$

and hence

$$\lim_{\alpha\to+\infty}\|f-\alpha G_\alpha^0 f\|=0.$$

This verifies condition (α), since the space

$$C^\infty(\bar{D})\cap C_0(\bar{D}\setminus M)=\{f\in C^\infty(\bar{D});\ f=0 \text{ on } \Sigma_2\cup\Sigma_3\}$$

is everywhere dense in the space $C_0(\bar{D}\setminus M)$.

II) Next, in order to verify condition (β), we assume that:

$$u\in D(\mathcal{A}) \quad\text{and}\quad \max_{\bar{D}\setminus(\Sigma_2\cup\Sigma_3)} u>0.$$

Then we have the following two cases:

(i) There exists a point x_0 of D such that

$$u(x_0)=\max_{\bar{D}\setminus(\Sigma_2\cup\Sigma_3)} u>0.$$

(ii) There exists a point x_0' of $\Sigma_0\cup\Sigma_1$ such that

$$u(x_0')=\max_{\bar{D}\setminus(\Sigma_2\cup\Sigma_3)} u>0.$$

Case (i): In this case, we have

$$\mathcal{A}u(x_0)=Au(x_0)=\sum_{i,j=1}^{N} a^{ij}(x_0)\frac{\partial^2 u}{\partial x_i\partial x_j}(x_0)+c(x_0)u(x_0)\leqq 0,$$

since the matrix (a^{ij}) is non-negative definite and $c\leqq 0$ in D.

Case (ii): We choose a local coordinate system $(y_1, y_2, \cdots, y_N)$ in a neighborhood of $x_0'\in\Sigma_0\cup\Sigma_1$ such that

$$\begin{cases} x_0'=0, \\ D=\{y_N>0\}, \\ \partial D=\{y_N=0\}, \end{cases}$$

and assume that, in terms of this coordinate system, the operator A is of the form

$$A=\alpha^{NN}\frac{\partial^2}{\partial y_N^2}+\beta^N\frac{\partial}{\partial y_N}+\sum_{i,j=1}^{N-1}\alpha^{ij}\frac{\partial^2}{\partial y_i\partial y_j}+\sum_{i=1}^{N-1}\beta^i\frac{\partial}{\partial y_i}+c. \tag{4.5}$$

We remark that:

(ii-a) $\alpha^{NN}(0)=0$ and $\beta^N(0)>0$ if $x_0'\in\Sigma_1$.

(ii-b) $\alpha^{NN}(0)=0$ and $\beta^N(0)=0$ if $x_0'\in\Sigma_0$.

But we have

$$\begin{cases} u(0)>0, \\ \dfrac{\partial u}{\partial y_i}(0)=0, \quad 1\leqq i\leqq N-1, \\ \dfrac{\partial u}{\partial y_N}(0)\leqq 0, \end{cases}$$

and also

$$\sum_{i,j=1}^{N-1}\alpha^{ij}(0)\frac{\partial^2 u}{\partial y_i\partial y_j}(0)\leqq 0.$$

Hence it follows from formula (4.5) that

$$\mathcal{A}u(x_0')=Au(x_0')\leqq\begin{cases} \beta^N(0)\dfrac{\partial u}{\partial y_N}(0)+c(0)u(0)\leqq 0 & \text{if } x_0'\in\Sigma_1, \\ c(0)u(0)\leqq 0 & \text{if } x_0'\in\Sigma_0. \end{cases}$$

Therefore, we have proved the following:

CLAIM. *If* $u\in D(\mathcal{A})$ *and* $\max_{\bar{D}\setminus M}u>0$, *then there exists a point* $x\in\bar{D}\setminus M$ *such that*

$$\begin{cases} u(x)=\max_{\bar{D}\setminus M}u, \\ \mathcal{A}u(x)\leqq 0. \end{cases}$$

This claim verifies condition (β).

III) It remains to verify condition (γ). By Theorem 4.1, we find that if $\alpha>0$ is sufficiently large, then the range $R(\alpha I-\mathcal{A})$ contains the space $C^\infty(\bar{D})\cap C_0(\bar{D}\setminus M)$. This implies that the range $R(\alpha I-\mathcal{A})$ is everywhere dense in the space $C_0(\bar{D}\setminus M)$, for $\alpha>0$ sufficiently large.

Summing up, we have proved that the operator $\mathcal{A}$ satisfies conditions (α) through (γ) in Theorem 1.4. Hence, in view of assertion (4.2), it follows from an application of the same theorem that the operator $\mathcal{A}$ is closable in the space $C_0(\bar{D}\setminus M)$, and its minimal closed extension $\bar{\mathcal{A}}$ is the infinitesimal generator of some Feller semigroup $\{T_t\}_{t\geqq 0}$ on $\bar{D}\setminus M$.

The proof of Theorem 1 is now complete.

Appendix The Maximum Principle

Let D be a bounded domain of Euclidean space $\boldsymbol{R}^N$, with boundary ∂D, and let A be a second-order, *degenerate* elliptic differential operator with real coefficients such that

$$Au(x)=\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x)+\sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x_i}(x)+c(x)u(x),$$

where:

1) $a^{ij}\in C(\boldsymbol{R}^N)$, $a^{ij}=a^{ji}$ and

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j\geqq 0, \quad x\in\boldsymbol{R}^N, \quad \xi=(\xi_1, \cdots, \xi_N)\in\boldsymbol{R}^N.$$

2) $b^i\in C(\boldsymbol{R}^N)$, $1\leqq i\leqq N$.

3) $c\in C(\boldsymbol{R}^N)$ and $c\leqq 0$ in D.

First we have the following result:

THEOREM A.1 (The weak maximum principle). *Assume that a function* $u\in C(\bar{D})\cap C^2(D)$ *satisfies either*

$$Au\geqq 0 \quad \textit{and} \quad c<0 \textit{ in } D$$

or

$$Au>0 \quad \textit{and} \quad c\leqq 0 \textit{ in } D.$$

Then the function u may take its positive *maximum only on the boundary* ∂D.

As an application of the weak maximum principle, we can obtain a pointwise estimate for solutions of the inhomogeneous equation $Au=f$:

THEOREM A.2. *Assume that*

$$c<0 \quad \textit{on } \bar{D}=D\cup\partial D.$$

Then we have for all $u\in C(\bar{D})\cap C^2(D)$

$$\max_{\bar{D}}|u|\leqq\max\left\{\frac{1}{c_0}\sup_{D}|Au|, \max_{\partial D}|u|\right\},$$

where

$$c_0=\max_{\bar{D}}(-c)>0.$$

For a proof of Theorems A.1 and A.2, the reader might refer to Bony-Courrège-Priouret [BCP], Oleĭnik-Radkevič [OR] and Taira [Ta].

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