TSUKUBA J. MATH.
Vol. 16 No. 2 (1992), 295–313

EXISTENCE RESULTS FOR QUASILINEAR DIRICHLET PROBLEM

By

J. CHABROWSKI

1. Introduction.

This paper deals with the Dirichlet problem

$$-\sum_{i,j=n}^{n} D_j(a_{ij}(x, u)D_i u)+c(x)u=b(x, u, Du) \quad \text{in } Q, \tag{1}$$

$$u(x)=\phi(x) \quad \text{on } \partial Q, \tag{2}$$

in a bounded domain $Q\subset \boldsymbol{R}_n$ with the boundary ∂Q of class C^2 and a function ϕ which, in general, is not a trace of an element from the space $W^{1,2}(Q)$. We consider two cases: $\phi\in L^\infty(\partial Q)$ (Section 3 and 4) and $\phi\in L^2(\partial Q)$ (Section 5).

In case where $\phi\in L^\infty(\partial Q)$ we establish some existence theorems for the problem (1), (2) under the assumption that the nonlinearity $b(x, u, p)$ grows quadratically in p. In recent years the problem (1), (2), with the nonlinearity b growing quadratically in p, has attracted some interest (see [1], [2], [7] and the references given there). In paper [1] the existence result was established in the space $\mathring{W}^{1,2}(Q)\cap L^\infty(Q)$ (that is, $\phi\equiv 0$ on ∂Q). The results of [2] show that under suitable assumptions on $b(x, u, p)$ one can also obtain unbounded solutions in $\mathring{W}^{1,2}(Q)$. The use of a weighted Sobolev space in [7] allowed one to obtain an existence theorem for the problem (1), (2) with $\phi\in L^\infty(\partial Q)$. In the case where $\phi\in L^2(\partial Q)$, we assume that the nonlinearity has a linear growth in p. The present paper is a generalization of [7].

The paper is organized as follows. In Section 2 we assemble definitions, assumptions and some terminology adopted in this work. Lemma 1, proved in this section, justifies our approach to the problem (1), (2) with the nonlinearity growing quadratically in p. Section 3 contains the main existence result of this paper which is closely related to Theorem 2.1 in [1] and Theorem 2 in [7]. The existence result in [1] was proved for more general quasilinear elliptic equations under the assumption of the existence of bounded sub and supersolutions but it can be applied only to the boundary data from $H^{1/2}(\partial Q)$.

Received August 12, 1991, Revised November 12, 1991.

The paper [7] contains some generalizations of this result for the problem (1), (2) with ϕ in $L^{\infty}(\partial Q)$. The method used in this paper requires the existence of bounded sequences of sub and supersolutions. The aim of this section is to relax this hypothesis by requiring the local boundedness of sequences of sub and supersolutions. In Section 4 we briefly discuss the existence of positive solutions of the problem (1), (2) with a_{ij} depending also on a gradient of u. In the final Section 5, we solve the Dirichlet problem with the L^2-boundary data. Finally, we point out that the methods used in this paper are not new and have appeared in [1], [2], [7], [8] and [12].

2. Preliminaries.

Throughout this paper we make the following assumptions:

(A) There exists a constant $\gamma>0$ such that

$$\gamma^{-1}|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \leqq \gamma|\xi|^2$$

for all $\xi\in \boldsymbol{R}_n$ and $(x, t)\in Q\times\boldsymbol{R}$. We also assume that $a_{ij}\in C(\bar{Q}\times\boldsymbol{R})$, $a_{ij}=a_{ji}$ $(i, j=1, \cdots, n)$ and that $a_{ij}(\cdot, t)\in C^1(\bar{Q})$ for each $t\in\boldsymbol{R}$ with bounded partial derivatives $D_i a_{ij}(x, t)$ on $\bar{Q}\times\boldsymbol{R}$ $(i, j=1, \cdots, n)$. Moreover, we assume that $c\in L^{\infty}(Q)$.

(B) The nonlinearity $b(x, t, p)$ satisfies the Carathéodory conditions, i.e.

(i) for each $(t, p)\in\boldsymbol{R}\times\boldsymbol{R}_n$, the function $x\to b(x, t, p)$ is measurable on Q.

(ii) for a.e. $x\in Q$, the function $(t, p)\to b(x, t, p)$ is continuous on $\boldsymbol{R}\times\boldsymbol{R}_n$.

We also assume there exist a constant $B>0$ and a non-negative function $f\in L^{\infty}(Q)$ such that

$$|b(x, t, p)|\leqq f(x)+B(|t|^r+|p|^2) \tag{3}$$

for all $(x, t, p)\in Q\times\boldsymbol{R}\times\boldsymbol{R}_n$ and some $0\leqq r<1$.

We briefly recall that a function $u\in W^{1,2}_{loc}(Q)$ is said to be a weak solution of (1) if u satisfies

$$\int_Q\Big(\sum_{i,j=1}^{n} a_{ij}(x, u)D_iuD_jv+c(x)uv\Big)dx=\int_Q b(x, u, Du)v\,dx \tag{4}$$

for every $v\in C^1(Q)$ with compact support in Q.

In Sections 3 and 4 we consider the Dirichlet problem (1), (2) with $\phi\in L^{\infty}(\partial Q)$. In general, functions from $L^{\infty}(\partial Q)$ are not traces of elements from $W^{1,2}(Q)$. Therefore we cannot expect a solution of (1), (2) to belong to $W^{1,2}(Q)$ The results of papers [3], [4], [5], [6], [7], [18] and [19] show that the suitable Sobolev space in our situation is

$$\tilde{W}^{1,2}(Q)=\left\{u\,;\,u\in W^{1,2}_{\mathrm{loc}}(Q)\ \text{and}\ \int_Q |Du(x)|^2 r(x)dx+\int_Q u(x)^2 dx<\infty\right\},$$

where $r(x)=\mathrm{dist}\,(x, \partial Q)$, equipped with the norm

$$\|u\|^2_{\tilde{W}^{1,2}(Q)}=\int_Q |Du(x)|^2 r(x)dx+\int_Q u(x)^2 dx\ .$$

The explain in what sense the solution recovers the boundary function ϕ, we need some definitions and terminology.

It follows from the regularity of the boundary ∂Q that there is a number $\delta_0>0$ such that for $\delta\in(0, \delta_0]$ the domain

$$Q_\delta=Q\cap\{x\,;\,\min_{y\in\partial Q}|x-y|>\delta\}$$

with the boundary ∂Q_δ possesses the following property: to each $x_0\in\partial Q$ there is a unique $x_\delta(x_0)\in\partial Q_\delta$ such that $x_\delta(x_0)=x_0-\delta\nu(x_0)$, where $\nu(x_0)$ is the outward normal to ∂Q at x_0, The above relation gives a one-to-one mapping of class C^1, of ∂Q onto ∂Q_δ.

According to Lemma 14.16 in [10], the distance $r(x)$ belongs to $C^2(\bar{Q}-Q_{\delta_0})$ if δ_0 is sufficiently small. We denote by $\rho(x)$ the extension of the function $r(x)$ into $\bar{Q}$ satisfying the following properties: $\rho(x)=r(x)$ for $x\in\bar{Q}-Q_{\delta_0}$, $\rho\in C^2(\bar{Q})$, $\rho(x)\geqq 3\delta_0/4$ in Q_{δ_0}, $\gamma_1^{-1}r(x)\leqq\rho(x)\leqq\gamma_1 r(x)$ in Q for some constant $\gamma_1>0$, $\partial Q_\delta=\{x\,;\,\rho(x)=\delta\}$ for $\delta\in(0, \delta_0]$ and finally $\partial Q=\{x\,;\,\rho(x)=0\}$.

We need the following result which justifies our approach to the Dirichlet problem (1), (2).

LEMMA 1. *Let u be a weak solution in $W^{1,2}_{\mathrm{loc}}(Q)$ of* (1) *such that*

$$\int_Q |Du(x)|^2(u(x)^2+1)r(x)dx+\int_Q u(x)^2 dx<\infty\,, \tag{5}$$

then there exists $\zeta\in L^2(\partial Q)$ such that

$$\lim_{\delta\to 0}\int_{\partial Q}(u(x_\delta(x))-\zeta(x))^2 dS_x=0\ .$$

PROOF. First we observe that by the hypothesis (5), u belongs to $\tilde{W}^{1,2}(Q)$. The same result was proved in [5] (see Theorem 2) under the assumption that b grows linearly in p. This proof can be adapted without any difficulty to the present situation. We only sketch the main steps of the proof. Let us define

$$v(x)=\begin{cases} u(x)(\rho(x)-\delta) & \text{on } Q_\delta\,,\\ 0 & \text{on } Q-Q_\delta\,.\end{cases}$$

It follows from (5) and the assumption (3) on $b(x, t, p)$ that v is a legitimate test function in (4). Integrating by parts we obtain

$$\int_{\partial Q_\delta} \sum_{i,j=1}^{n} \int_0^{u(x)} a_{ij}(x, s) s ds D_i \rho(x) dS_x$$

$$= -\int_{Q_\delta} \sum_{i,j=1}^{n} \int_0^{u(x)} a_{ij}(x, s) s ds D_{ij} \rho(x) dx$$

$$- \int_{Q_\delta} \int_0^{u(x)} \sum_{i,j=1}^{n} D_i a_{ij}(x, s) s ds D_j \rho(x) dx$$

$$+ \int_{Q_\delta} \sum_{i,j=1}^{n} a_{ij}(x, u) D_i u D_j u(\rho-\delta) dx$$

$$+ \int_{Q_\delta} c(x) u^2(\rho-\delta) dx - \int_{Q_\delta} b(x, u, Du) u(\rho-\delta) dx .$$

Here the values of u on ∂Q_δ are understood in the sense of traces (see [13], chap. 6). Using the assumptions on a_{ij} and b we derive the following estimate

$$\sup_{0<\delta\leqq\delta_1} \int_{\partial Q_\delta} \sum_{i,j=1}^{n} \int_0^{u(x)} a_{ij}(x, s) s ds D_i \rho D_j \rho dS_x \leqq C_1 \Big[\int_Q u(x)^2 dx$$

$$+ \int_Q |Du(x)|^2 (u(x)^2+1) \rho(x) dx + \int_Q u(x)^2 \rho(x) dx$$

$$+ \|f\|_\infty \int_Q |u(x)| \rho(x) dx + \int_Q |u(x)|^{r+1} \rho(x) dx \Big]$$

for some constants $C_1>0$ and $0<\delta_1\leqq\delta_0$. Hence, by the ellipticity assumption $\sup_{0<\delta\leqq\delta_1}\int_{\partial Q_\delta} u(x)^2 dS_x<\infty$. Consequently the set of functions

$$\left\{\int_0^{u(x_\delta)} \sum_{i,j=1}^{n} a_{ij}(x_\delta, s) ds D_i \rho(x_\delta) D_j \rho(x_\delta) dS_x \,;\, 0<\delta\leqq\delta_1\right\}$$

is bounded in $L^2(\partial Q)$. As in Lemma 2 from [5] we show that there exists a function $\beta\in L^2(\partial Q)$ such that

$$\int_0^{u(x_\delta)} \sum_{i,j=1}^{n} a_{ij}(x_\delta, s) ds D_i \rho(x_\delta) D_j \rho(x_\delta)$$

converges weakly to β in $L^2(\partial Q)$. Repeating the argument of Lemma 3 from [5] we show that the weak convergence can be replaced by the strong convergence in $L^2(\partial Q)$. Finally, following the argument used in the proof of Theorem 1 in [5] we conclude the existence of $\zeta\in L^2(\partial Q)$ satisfying the assertion of our lemma. We point out here that the following relation holds between ζ and β

$$\int_0^{\zeta(x)} \sum_{i,j=1}^n a_{ij}(x, s)dsD_i\rho(x)D_j\rho(x)=\beta(x)$$

a.e. on ∂Q.

Lemma 1 suggests the following approach to the Dirichlet problem (1), (2).

Let $\phi \in L^\infty(\partial Q)$. A weak solution $u \in W^{1,2}_{loc}(Q)$ of (1) is a solution of the Dirichlet problem with the boundary condition (2) if

(6) $$\lim_{\delta\to 0} \int_{\partial Q} [u(x_\delta(x))-\phi(x)]^2 dS_x=0 .$$

3. Existence of solutions of the Dirichlet problem (1), (2).

In this section, using the method of sub and supersolutions we establish the existence theorem for the problem (1), (2).

We briefly recall the definitions of sub and supersolution.

Let $\phi \in H^{1/2}(\partial Q)$. A function $\Phi \in W^{1,2}(Q)$ is a subsolution of (1) if $\Phi(x) \leqq \phi(x)$ on ∂Q in the sense of trace in $H^{1/2}(\partial Q)$ and

$$\int_Q \sum_{i,j=1}^n a_{ij}(x, \Phi)D_i\Phi D_j v dx+\int_Q c(x)\Phi v\, dx \leqq \int_Q b(x, \Phi, D\Phi)v\, dx$$

for all nonnegative $v \in C^1(Q)$ with compact support in Q.

A supersolution is defined by reversing the inequality sign in the above definition.

THEOREM 1. *Suppose that $c(x) \geqq c_0$ in Q for some $c_0>0$, $\phi \in L^\infty(\partial Q)$, and that there exists a sequence of $C^1(\partial Q)$-functions $\{\phi_k\}$ such that $\lim_{k\to\infty}\int_{\partial Q}[\phi_k(x)-\phi(x)]^2 dS_x=0$ and such that for each k the Dirichlet problem* (1), (2), *with $\phi=\phi_k$, admits a subsolution $\Phi_k(x)$ and a supersolution $\Psi_k(x)$ in $W^{1,\infty}(Q)$ satisfying $\Phi_k(x) \leqq \Psi_k(x)$ on Q. Moreover, we suppose that both sequences $\{\Phi_k\}$ and $\{\Psi_k\}$ are locally uniformly bounded in $L^\infty(Q)$. Then the problem* (1), (2) *admits a solution $u \in \widetilde{W}^{1,2}(Q)$ satisfying the estimate*

(7) $$\int_Q |Du(x)|^2(u(x)^2+1)e^{Tu(x)^2}r(x)dx+\int_Q u(x)^2 e^{Tu(x)^2}r(x)dx$$
$$+\sup_{0<\delta\leqq\delta_1}\int_{\partial Q_\delta} e^{Tu(x)^2}dS_x \leqq M_1\int_{\partial Q} e^{Tu(x)^2}dS_x+M_2$$

for some constants $M_1>0$, $M_2>0$, $T>0$ and $0<\delta_1\leqq\delta_0$.

PROOF. Let $\{\phi_k\}$ be sequence of $C^1(\partial Q)$ functions satisfying the hypotheses of our theorem. Then it follows from [1] that for each k the problem (1), (2) with $u(x)=\phi_k(x)$ on ∂Q has a solution $u_k\in W^{1,2}(Q)\cap L^\infty(Q)$ and such that

$$\Phi_k(x)\leqq u_k(x)\leqq \Psi_k(x) \quad Q \tag{8}$$

for each k. Let us define

$$v(x)=u_k(x)e^{tu_k(x)^2}\rho(x)$$

for some $t>0$. It is clear that v is a legitimate test function in (4) and on substitution we obtain

$$\begin{aligned}(9)\qquad &\int_Q \sum_{i,j=1}^n a_{ij}(x, u_k)D_iu_kD_ju_ke^{tu_k^2}\rho dx\\ &+2t\int_Q \sum_{i,j=1}^n a_{ij}(x, u_k)D_iu_kD_ju_ku_k^2e^{tu_k^2}\rho dx\\ &+\int_Q \sum_{i,j=1}^n a_{ij}(x, u_k)D_iu_ku_ke^{tu_k^2}D_j\rho dx+\int_Q c(x)u_k^2e^{tu_k^2}\rho dx\\ &=\int_Q b(x, u_k, Du_k)u_ke^{tu_k^2}\rho dx\,.\end{aligned}$$

Let us denote the first three integrals of the left side of (9) by J_1, J_2 and J_3, respectively. If follows from (A) that

$$J_1+J_2\geqq\gamma^{-1}\Big[\int_Q|Du_k|^2e^{tu_k^2}\rho dx+2t\int_Q|Du_k|^2u_k^2e^{tu_k^2}\rho dx\Big]. \tag{10}$$

Integrating by parts we get

$$\begin{aligned}(11)\qquad J_3&=\int_Q \sum_{i,j=1}^n a_{ij}(x, u_k)D_iu_ku_ke^{tu_k^2}D_j\rho dx\\ &=\int_Q \sum_{i,j=1}^n D_i\Big(\int_0^{u_k} a_{ij}(x, s)se^{ts^2}ds\Big)D_j\rho dx\\ &\quad-\int_Q \sum_{i,j=1}^n \int_0^{u_k} D_ia_{ij}(x, s)se^{ts^2}dsD_j\rho dx\\ &=-\int_{\partial Q} \sum_{i,j=1}^n \int_0^{\phi_k} a_{ij}(x, s)se^{ts^2}dsD_i\rho D_j\rho dS_n\\ &\quad-\int_Q \sum_{i,j=1}^n \int_0^{u_k} a_{ij}(x, s)se^{ts^2}dsD_{ij}\rho dx\\ &\quad-\int_Q \sum_{i,j=1}^n \int_0^{u_k} D_ia_{ij}(x, s)se^{ts^2}dsD_j\rho dx\,.\end{aligned}$$

Combining (10), (11) and the assumptions (A) and (B) we derive from (9) that

(12) $$\gamma^{-1}\int_Q |Du_k|^2 e^{tu_k^2}\rho dx+2t\gamma^{-1}\int_Q |Du_k|^2 u_k^2 e^{tu_k^2}\rho dx+c_0\int_Q u_k^2 e^{tu_k^2}\rho dx$$
$$\geqq C_1\left[t^{-1}\int_{\partial Q} e^{t\phi_k^2}dS_x+t^{-1}\int_Q e^{tu_k^2}dx+\int_Q |u_k|e^{tu_k^2}\rho dx\right]$$
$$+B\left[\int_Q |Du_k|^2|u_k|e^{tu_k^2}\rho dx+\int_Q |u_k|^{r+1}e^{tu_k^2}\rho dx\right],$$

where a constant $C_1>0$ depends only on n, γ, $\sup_Q|\rho|$, $\sup_Q|D^2\rho|$, $\sup_{Q\times R}|D_i a^i_j(x, u)|$ $(i, j=1, \cdots, n)$, $\|f\|_\infty$ and n. Using the Young inequality we deduce from (12) that

(13) $$\frac{\gamma^{-1}}{2}\int_Q |Du_k|^2 e^{tu_k^2}\rho dx+\left(2t\gamma^{-1}-\frac{B^2\gamma}{2}\right)\int_Q |Du_k|^2 u_k^2 e^{tu_k^2}\rho dx$$
$$+\frac{c_0}{2}\int_Q u_k^2 e^{tu_k^2}\rho dx\leqq C_2\left[t^{-1}\int_{\partial Q} e^{t\phi_k^2}dS_x+t^{-1}\int_Q e^{tu_k^2}dx+\int_Q e^{tu_k^2}\rho dx\right],$$

where $C_2>0$ depends on C_1, c_0, r and B. Now taking as a test function in (4)

$$v(x)=\begin{cases} u_k(x)e^{tu_k(x)^2}(\rho(x)-\delta) & \text{on } Q_\delta, \\ 0 & \text{on } Q-Q_\delta, \end{cases}$$

and integrating by parts and letting $\delta\to 0$ we get the following estimate

(14) $$t^{-1}\gamma^{-1}\sup_{0<\delta\leqq\delta_1}\int_{\partial Q_\delta} e^{tu_k^2}dS_x\leqq\gamma\int_Q |Du_k|^2 e^{tu_k^2}\rho dx$$
$$+\|c\|_\infty\int_Q u_k^2 e^{tu_k^2}\rho dx+2t\gamma\int_Q |Du_k|^2 u_k^2 e^{tu_k^2}\rho dx$$
$$+C_3\left[t^{-1}\int_Q e^{tu_k^2}dx+\int_Q |u_k|e^{tu_k^2}\rho dx\right]$$
$$+B\left[\int_Q |Du_k|^2|u_k|e^{tu_k^2}\rho dx+\int_Q |u_k|^{1+r}e^{tu_k^2}\rho dx\right],$$

where $C_3>0$ is a constant of the same nature as C_1.

Let us set

$$K\equiv\frac{1}{t}\int_Q e^{tu_k^2}dx+\frac{1}{t}\int_{\partial Q} e^{t\phi^2}dS_x+\int_Q e^{tu_k^2}\rho dx\,.$$

Applying the Young inequality we derive from (13) and (14) that

$$\frac{\gamma^{-1}}{2}\int_Q |Du_k|^2 e^{tu_k^2}\rho dx+\left(2t\gamma^{-1}-\frac{B^2\gamma}{2}\right)\int_Q |Du_k|^2 u_k^2 e^{tu_k^2}\rho dx$$
$$+\frac{c_0}{2}\int_Q u_k^2 e^{tu_k^2}\rho dx+\gamma^{-1}t^{-1}\sup_{0<\delta\leqq\delta_1}\int_{\partial Q_\delta} e^{tu_k^2}dS_x$$

$$\leq C_4K+\|c\|_\infty\int_Q u_k^2\rho dx$$

$$+(\gamma+B)\int_Q |Du_k|^2 e^{tu_k^2}\rho dx+(2t\gamma+B)\int_Q |Du_k|^2 u_k^2 e^{tu_k^2}\rho dx\,,$$

for some $C_4>0$ independent of $t>0$. Letting $t=T>B^2\gamma^2/2$ and combining the last estimate with (13) we arrive at the inequality

$$\int_Q |Du_k|^2 e^{Tu_k^2}\rho dx+\int_Q |Du_k^2|^2 u_k^2 e^{Tu_k^2}\rho dx$$
$$+\int_Q u_k^2 e^{Tu_k^2}\rho dx+\sup_{0<\delta\leq\delta_1}\int_{\partial Q_\delta} e^{Tu_k^2}dS_x\leq C_5K \tag{15}$$

for some constant $C_5>0$. To estimate the integrals $\int_Q e^{Tu_k^2}dx$ and $\int_Q e^{Tu_k^2}\rho dx$ we observe that by the local boundedness of $\{\Phi_k\}$ and $\{\Psi_k\}$ we obtain

$$\int_Q e^{Tu_k^1}dx=\int_{Q-Q_{\delta_1}} e^{Tu_k^2}dx+\int_{Q_{\delta_1}} e^{Tu_k^2}dx\leq\delta_1\sup_{0<\delta\leq\delta_1}\int_{\partial Q_\delta} e^{Tu_k^2}dS_x+M(\delta_1)\,,$$

for some constant $M(\delta_1)>0$. In a similar way we estimate $\int_Q e^{Tu_k^2}\rho dx$. Choosing δ_1 sufficiently small we obtain

$$\int_Q |Du_k|^2 e^{Tu_k^2}\rho dx+\int_Q |Du_k|^2 u_k^2 e^{Tu_k^2}\rho dx$$
$$+\int_Q u_k^2 e^{Tu_k^2}\rho dx+\sup_{0<\delta\leq\delta_1}\int_{\partial Q_\delta} e^{Tu_k^2}dS_x\leq L_1\int_{\partial Q} e^{T\phi^2}dS_x+L_2 \tag{16}$$

for some constants $L_1>0$ and $L_2>0$.

In the second step of the proof we show that for each open set Q_1, with $\bar{Q}_1\subset Q$, there exists $\varepsilon_1>0$ such that the sequence $\{u_k\}$ is bounded in $W^{1,2+\varepsilon}(Q_1)$. Since the argument of this claim was used in the proof of Theorem 2.1 in [1] and Theorem 2 in [6], we only sketch the proof of this fact. Let θ be a C^∞-function with properties $\theta(x)=1$ on $B(0, 1/2)$, $\theta(x)=0$ on $\boldsymbol{R}_n-B(0, 1/2)$ and $0\leq\theta(x)\leq 1$ on $\boldsymbol{R}_n$, where $B(x_0, r)$ denotes an open ball of radius r and centered at x_0. Let Q_2 be an open set such that $\bar{Q}_1\subset Q_2\subset\bar{Q}_2\subset Q$. We assign to each $x_0\in Q_1$ a number $R(x_0)$ defined by

$$R(x_0)=\sup\{R\,;\, R\in[0, \infty),\ \bar{B}(x_0, R)\subset Q_2\}\,.$$

Since Q is bounded, $R(x_0)$ is bounded independently of x_0. If $R<R(x_0)$ we define $\theta_R(x)=\theta(x-x_0/R)$ and set

$$v_k(x)=\theta_R(x)^2(u_k-K)e^{t(u_k-K)^2}$$

with

$$K=\frac{1}{|B(x_0, R)|}\int_{B(x_0,R)} u_k(x)dx .$$

Since the sequences $\}\Phi_k\}$ and $\{\Psi_k\}$ are bounded in $L^\infty(Q_2)$, the estimate (8) implies that the sequence $\{u_k\}$ is bounded in $L^\infty(Q_2)$. Using v_k as a test function in (4) and choosing t sufficiently large we arrive at the inequality

$$\text{(17)} \qquad \int_{B(x_0,R/2)}|Du_k|^2dz \leqq \frac{M_1}{R^2}\int_{B(x_0,R)}|u_k-K|^2dx+\int_{B(x_0,R)} g(x)dx ,$$

where $M_1>0$ is a constant independent of R and k and $g(x)$ is a bounded function on Q_2. Let $1/s=1/n+1/2$ if $1/n+1/2<1$ and $s=1$ if $1/n+1/2\geqq 1$. By the Sobolev embedding theorem we derive from (17) that

$$\frac{1}{|B(x_0, R/2)|}\int_{B(x_0,R/2)}|Du_k|^2dx \leqq M_2\Big(\frac{1}{|B(x_0, R)|}\int_{B(x_0,R)}|Du_k|^2dx\Big)^{2/s} + \frac{M_3}{|B(x_0, R)|}\int_{B(x_0,R)} gdx ,$$

where $M_2>0$ and $M_2>0$ are constants independent of R and k. Now by a standard argument with the aid of Gehring's Lemma [10] (see also Proposition 5.1 in [9]) we can show that there exists $\varepsilon>0$ such that $\{Du_k\}$ is bounded in $L^{2+\varepsilon}(\omega)$ for each open set ω with $\bar{\omega}\subset Q_2$. We note here that ε depends on Q_2. We now observe that by Fatou's lemma we may assume that the sequence $\int_{\partial Q} e^{T\phi_k^2}dS_x$ is bounded. Consequently, by (16) the sequence $\{u_k\}$ is bounded in $\widetilde{W}^{1,2}(Q)$. Therefore we may assume that there exists $u\in\widetilde{W}^{1,1}(Q)$ such that u_k converges to u weakly in $\widetilde{W}^{1,2}(Q)$. Moreover, by virture of Theorem 14.11 in [16] we may also assume that u_k converges to u in $L^2(Q)$ and a.e. on Q. Using the boundedness of $\{Du_k\}$ in $L^{2+\varepsilon}(\omega)$ for each $\omega\subset Q$, with $\bar{\omega}\subset Q$ and $\varepsilon=\varepsilon(\omega)$, one can show that for each open set ω, with $\bar{\omega}\subset Q$, there exists a subsequence $\{u_{k_m}\}$ such that Du_{k_m} converges to Du in $L^2(\omega)$ (for details see [7]). It is now obvious that u is a weak solution of (1) and that the estimates (7) asserted by our theorem holds for u. It remains to show that u satisfies the boundary condition (2) in the sense of L^2-convergence. According to Lemma 1, there exists $\zeta\in L^2(\partial Q)$ such that $\lim_{\delta\to 0}\int_{\partial Q}[u(x_\delta)-\zeta(x)]^2dS_x=0$. Therefore it suffices to show that $\zeta=\phi$ a.e. on ∂Q. The proof of this fact is similar to the corresponding part of Theorem 2 in [7] and therefore is omitted.

REMARK. Inspection of the proof of Theorem 1 shows that the assertion of this theorem remains true for a boundary data satisfying

$$\int_{\partial Q} e^{T\phi(x)^2} dS_x < \infty ,$$

where T is a constant satisfying $T > B^2\gamma^2/4$. Obviously this condition holds for bounded functions and one can give examples of unbounded functions satisfying this condition.

To illustrate Theorem 1 let us consider the problem (1), (2) with $b(x, u, Du) = f(x) - |Du|^2 g(x, u)$, where $f \in L^\infty(Q)$, $g \in L^\infty(Q \times \boldsymbol{R})$ and $g(x, u)u \geqq 0$ for all $(x, u) \in Q \times R$. Moreover, we assume that there exists functions $A_{ij} \in C^1(\bar{Q})$ such that

(i) $$\lim_{|u| \to \infty} a_{ij}(x, u) = A_{ij}(x)$$

and

(ii) $$\lim_{|u| \to \infty} D_x a_{ij}(x, u) = D_x A_{ij}(x)$$

$(i, j = 1, \cdots, n)$ uniformly on $\bar{Q}$. Let $\phi \in L^\infty(\partial Q)$ and let $\{\phi_k\}$ be a sequence of $C^1(\partial Q)$-functions such that

$$\lim_{k \to \infty} \int_{\partial Q} [\phi(x) - \phi_k(x)]^2 dS_x = 0.$$

For each k the Dirichlet problem

$$-\sum_{ij=1}^{n} D_j(a_{ij}(x, u) D_i u) + c(x)u = |f(x)| \quad \text{in } Q ,$$

$$u(x) = |\phi_k(x) \quad \text{on } \partial Q ,$$

has a solution $\Phi_k \in W^{1,1}(Q) \cap L^\infty(Q)$, which by the maximum principle in non negative on Q. Since $g(x, \Phi_k) \geqq 0$ on Q, Φ_k is a supersolution of the problem (1), (2). A subsolution Ψ_k is determined as a solution of the problem

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x, u) D_i u) + c(x)u = -|f(x)| \quad \text{in } Q$$

$$u(x) = -|\phi_k(x)| \quad \text{on } \partial Q .$$

As in [6] one can show that the sequences $\{\Phi_k\}$ and $\{\Psi_k\}$ are bounded in $\tilde{W}^{1,1}(Q)$. We sketch the proof of this fact here for $\{\Phi_k\}$. Using as a test function

$$v(x) = \begin{cases} \Phi_k(x)(\rho(x) - \delta) & x \in Q_\delta , \\ 0 & \text{on } Q - Q_\delta , \end{cases}$$

integrating by parts and letting $\delta - 0$, we get

$$\int_Q \sum_{i,j=1}^n a_{ij}(x, \Phi_k)D_i\Phi_k D_j\Phi_k \rho dx$$
$$=\frac{1}{2}\int_Q \sum_{i,j=1}^n \int_0^{\Phi_k^2} a_{ij}(x, s)dsD_i\rho D_j\rho dS_x+\frac{1}{2}\int_Q \sum_{i,j=1}^n \int_0^{\Phi_k^2} a_{ij}(x, s)dsD_{ij}\rho dx$$
$$+\int_Q \sum_{i,j=1}^n \int_0^{\Phi_k^2} D_i a_{ij}(x, s)dsD_j\rho dx-\int_Q c(x)\Phi_k^2\rho dx+\int_Q |f(x)|\Phi_k\rho dx .$$

It is now clear that to show the boundedness of $\{\Phi_k\}$ in $\tilde{W}^{1,2}(Q)$ it is sufficient to show that this sequence is bounded in $L^2(Q)$. In the contrary case we may assume that $\lim_{k\to\infty}\|\Phi_k\|_{L^2(Q)}=\infty$. Letting $v_k=\Phi_k\|\Phi_k\|_{L^2(Q)}^{-1}$, the above identity shows that $\{v_k\}$ is bounded in $\tilde{W}^{1,2}(Q)$. Since $\tilde{W}^{1,2}(Q)$ is compactly embedded in $L^2(Q)$ (see [16]), we may assume that $v_k\to v$ weakly in $\tilde{W}^{1,2}(Q)$, strongly in $L^2(Q)$ and a.e. on Q. As in [6] we can show that v satisfies the equation

$$-\sum_{i,j=1}^n D_j(A_{ij}(x)D_i v)+c(x)v=0 \quad \text{in } Q .$$

According to [3] or [4], v must have trace $\zeta\in L^2(\partial Q)$, in the sense that $v(x_\delta)\to\zeta$ in $L^2(\partial Q)$ as $\delta\to 0$. It is now a routine to show that $\zeta\equiv 0$ on ∂Q, that is, $v\in \mathring{W}^{1,2}(Q)$. Since $c\geqq 0$ on Q we get $v\equiv 0$, and this contradicts the fact that $\|v\|_{L^2(Q)}=1$. If we additionally assume that $D_u a_{ij}\in L^\infty(Q\times \boldsymbol{R})$ $(i, j=1, \cdots, n)$ then $\Phi_k\in W^{1,\infty}(Q)$ for each k.

It is worth mentioning that Theorem 1 is closely related to Theorem 2 from [7]. However, applying the latter to our example, we can only conclude the existence of a solution for $\phi\in L^\infty(\partial Q)$ with small norm and some additional restriction on the coefficient c.

4. Nonnegative solutions.

The objective of this section is to establish the existence of nonnegative solutions. To achieve this we assume the existence of nonnegative subsolutions and supersolutions. This assumption allows to consider the quasilinear equations with the coefficients a_{ij} depending also on Du.

In this section we assume that the coefficients $a_{ij}(x, u, p)$ $(i, j=1, \cdots, n)$ are defined and continuous on $\bar{Q}\times R\times R_n$ and satisfy the ellipticity condition from Section 2 (see asumption (A)). The functions $a_{ij}(x, u, 0)$ have bounded partial derivatives $D_i a_{ij}(x, u, 0)$ on $\bar{Q}\times R$ $(i, j=1, \cdots, n)$ and moreover

(18) $$|a_{ij}(x, u, p)-a_{ij}(x, u, 0)|\leqq \frac{A}{|p|+1} \quad (i, j=1, \cdots, n)$$

for all $(x, u, p)\in Q\times R\times R_n$ and for some constant $A>0$. The nonlinearity satisfies the assumption (B) from Section (2).

We derive here the existence result for the Dirichlet problem

(19) $$-\sum_{i,j=1}(D_j a_{ij}(x, u, Du)D_j u)+c(x)u=b(x, u, Du) \quad \text{in } Q,$$

(20) $$u(x)=\phi(x) \quad \text{on } \partial Q.$$

To proceed further we observe first that Lemma 1 continues to hold for weak solutions $u\in W_{loc}^{1,2}(Q)$ of (19) satisfying the condition (5) of Lemma 1. Indeed, using the same test function as in the proof of Lemma 1 we arrive at the identity

$$\begin{aligned}
&\int_{\partial Q_\delta}\sum_{i,j=1}^{n}\int_0^{u(x)} a_{ij}(x, s, 0)sdsD_i\rho D_j\rho dS_x \\
&=-\int_{Q_\delta}\sum_{i,j=1}^{n}\int_0^{u(x)} a_{ij}(x, s, 0)sdsD_{ij}\rho dx \\
&-\int_{Q_\delta}\sum_{i,j=1}^{n}\int_0^{u(x)} D_i a_{ij}(x, s, 0)sdsD_j\rho dx \\
&+\int_{Q_\delta}\sum_{i,j=1}^{n} a_{ij}(x, u, Du)D_iuD_ju(\rho-\delta)dx \\
&+\int_{Q_\delta}\sum_{i,j=1}^{n}[a_{ij}(x, u, Du)-a_{ij}(x, u, 0)]D_iuuD_j\rho dx \\
&+\int_{Q_\delta}c(x)u^2(\rho-\delta)dx-\int_{Q_\delta}b(x, u, Du)u(\rho-\delta)dx .
\end{aligned}$$

By virtue of the assumption (18) the fourth integral on the right side can be estimated by

$$\sup_Q|D\rho(x)|n^2A\int_Q|u|dx .$$

It is now a routine to show that

$$\sup_{0<\delta\leq\delta_1}\int_{\partial Q_\delta}|u(x)|^2dS_x<\infty .$$

Repeating the argument of the proof of Lemma 1 (see also Theorem 1 in [5]) one can show that there exists $\zeta\in L^2(\partial Q)$ such that $\lim_{\delta\to 0}\int_{\partial Q}[u(x_\delta)-\zeta(x)]^2dS_x =0$.

THEOREM 2. *Suppose that* $c(x)\geqq c_0$ *on* Q *for some* $c_0>0$. *Let* ϕ *be a nonnegative function in* $L^\infty(\partial Q)$ *and suppose that there exists a sequence of* $C^1(\partial Q)$-*functions* $\{\phi_k\}$ *with* $\lim_{k\to\infty}\int_{\partial Q}[\phi_k(x)-\phi(x)]^2dS_x=0$ *and such that for each* k *the*

Dirichlet problem (19), (20) *with* $\phi=\phi_k$ *admits a subsolution* Φ_k *and a supersolution* Ψ_k *in* $W^{1,\infty}(Q)$ *satisfying* $0\leqq\Phi_k(x)\leqq\Psi_k(x)$ *on* Q *for each* k. *Moreover we suppose that the sequence* $\{\Psi_k\}$ *is locally uniformly bounded in* $L^\infty(Q)$. *Then the problem* (19), (20) *admits a solution* $u\in W^{1,2}_{\mathrm{loc}}(\bar{Q})$ *satisfying the estimate*

$$(21)\qquad \int_Q |Du(x)|^2 e^{Tu(x)}r(x)dx+\int_Q u(x)e^{Tu(x)}r(x)dx + \sup_{0<\delta\leqq\delta_1}\int_{\partial Q_\delta} e^{Tu(x)}dS_x \leqq M_1\int_{\partial Q} e^{T\phi(x)}dS_x+M_2$$

for some constants $M_1>0$, $M_2>0$ *and* $0<\delta\leqq\delta_0$.

PROOF. The proof is similar to that of Theorem 1. We only give the proof of the analogue of the energy estimate (16).

Let $\{\phi_k\}$ be sequence of $C^1(\partial Q)$-functions satisfying the hypotheses of the theorem. According of Theorem 2.1 in [1] for each k the Dirichlet problem (19), (20), with $\phi=\phi_k$ admits a solution $u\in W^{1,2}(Q)\cap L^\infty(Q)$ such that

$$\Phi_k(x)\leqq u_k(x)\leqq\Psi_k(x)\quad \text{on } Q\,.$$

Taking as a test function

$$v(x)=e^{tu_k(x)}\rho(x)$$

for some $t>0$ we obtain

$$(22)\qquad t\int_Q \sum_{i,j=1}^n a_{ij}(x, u_k, Du_k)D_iu_kD_ju_ke^{tu_k}\rho dx + \int_Q \sum_{i,j=1}^n a_{ij}(x, u_k, 0)D_iu_ke^{tu_k}D_j\rho dx + \int_Q \sum_{i,j=1}^n [a_{ij}(x, u_k, Du_k)-a_{ij}(x, u_k, 0)]D_iu_ke^{tu_k}D_j\rho dx + \int_Q cu_ke^{tu_k}\rho dx=\int_Q b(x, u_k, Du_k)e^{tu_k}\rho dx\,.$$

Let us denote the first three integrals on the left side by J_1, J_2 and J_3, respec--tively. We then have

$$(23)\qquad J_1\geqq t\gamma^{-1}\int_Q |Du_k|^2e^{tu_k}\rho dx$$

and by the assumption (18)

$$(24)\qquad |J_3|\leqq An^2\sup_Q |D\rho(x)|\int_Q e^{tu_k}dx.$$

Integrating by parts we obtain

(25)
$$J_2=-\int_{\partial Q}\sum_{i,j=1}^{n}\int_0^{\phi_k}a_{ij}(x,\ s,\ 0)e^{ts}dsD_i\rho D_j\rho dx$$
$$-\int_Q\sum_{i,j=1}^{n}\int_0^{u_k}D_ia_{ij}(x,\ s,\ 0)e^{ts}dsD_j\rho dx$$
$$-\int_Q\sum_{i,j=1}^{n}\int_0^{u_k}a_{ij}(x,\ s,\ 0)e^{ts}dsD_{ij}\rho dx\ .$$

It follows from (22), (23). (24) and (25) that

$$(t\gamma^{-1}-B)\int_Q|Du_k|^2e^{tu_k}\rho dx+\frac{c_0}{2}\int_Q u_ke^{tu_k}\rho dx$$
$$\leqq\gamma t^{-1}\int_{\partial Q}e^{t\phi_k}dS_x+C_1\int_Q e^{tu_k}dx$$

for some $C_1>0$ independent of t. Similarly using as a test function

$$u(x)=\begin{cases} e^{tu_k(x)}(\rho(x)-\delta) & \text{on } Q_\delta\,, \\ 0 & \text{on } Q-Q_\delta\,, \end{cases}$$

we arrive at the estimate

$$\sup_{0<\delta\leqq\delta_1}\int_{\partial Q}e^{tu_k}dS_x\leqq C_2\left[\int_Q|Du_k|^2e^{tu_k}\rho dx+\int_Q u_ke^{tu_k}\rho dx+\int_Q e^{tu_k}\rho dx\right]$$

for some $C_2>0$ independent of t. Finally, using Lemma 2, we deduce from the last two estimates, as in the proof of Theorem 1, the estimate (21).

REMARK. If $b(x, 0, 0)\leqq 0$ on Q, then we can take $\Phi_k\equiv 0$ ($k=1, 2, \cdots$) as subsolutions and to guarantee the existence of a nontrivial solution we can assume that either $b(x, 0, 0)$ or ϕ is not identically equal to 0.

We conclude this section with the following comment. Theorem 1, unlike Theorem 2, has been proved for the equation (1) with the coefficients a_{ij} independent of Du. Comparing the proofs of these theorems we see that the dependence of a_{ij} on Du would lead, in the derivation of the energy estimate (16), to an extra term

$$\int_Q\sum_{i,j=1}^{n}[a_{ij}(x,\ u,\ Du)-a_{ij}(x,\ u,\ 0)]D_iuue^{tu^2}D_j\rho dx\ .$$

Assuming (18), this term can be estimated by $n^2A\int_Q|u|e^{tu^2}dx$ and we were unable to get the estimate (16) in this situation.

5. The Dirichlet preblem with L^2-boundary data.

In this section we extend our method to solve the problem (1), (2) with $\phi\in L^2(\partial Q)$. However, we must introduce more restrictive assumptions on the nonlinearity b. We consider the equation (1) with $c(x)\equiv 0$ on Q, that is,

$$(1')\qquad -\sum_{i,j=1}^{n} D_j(a_{ij}(x, u)D_i u)+b(x, u, Du)=0 \quad \text{in } Q\,,$$

with the boundary condition (2), where $\phi\in L^2(\partial Q)$.

We assume that $b(x, u, p)$ satisfies the Carathéodory conditions and

$$(29)\qquad |b(x, u, p)|\leqq f(x)+B(|u|+|p|)$$

for all $(x, u, p)\in Q\times \boldsymbol{R}\times \boldsymbol{R}_n$, where $f\in L^2(Q)$ and $B>0$ is a constant.

We point out here that, according to Theorem 1 in [5], if u is a solution in $\tilde{W}^{1,2}(Q)$ of (1′) then there exists $\zeta\in L^2(\partial Q)$ such that (6) holds. Obviously this result justifies our approach to the problem (1′) (2) with the boundary condition (2) understood by the relation (6).

THEOREM 3. *Let $\phi\in L^2(\partial\bar{Q})$ and suppose that there exists a sequence of $C^1(\partial\bar{Q})$-functions $\{\phi_k\}$ such that $\lim_{k\to\infty}\int_{\partial Q}[\phi_k(x)-\phi(x)]^2 dS_x=0$ and such that for each k the Dirichlet problem (1′), (2), with $\phi=\phi_k$, admits a subsolution Φ_k and a supersolution Ψ_k in $W^{1,\infty}(\bar{Q})$ satisfying $\Phi_k(x)\leqq\phi_k(x)\leqq\Psi_k(x)$ on Q. Moreover, we assume that both sequences $\{\Phi_k\}$ and $\{\Psi_k\}$ are locally uniformly bounded in $L^\infty(\bar{Q})$. Then the problem (1′), (2) admits a solution $u\in\tilde{W}^{1,2}(Q)$ satisfying the estimate*

$$(27)\qquad \int_Q |Du(x)|^2 r(x)dx+\int_Q u(x)^2 dx+\sup_{0<\delta\leqq\delta_1}\int_{\partial Q_\delta} u(x)^2 dS_x \leqq M_1\int_{\partial Q}\phi(x)^2 dS_x+M_2$$

for some constants $M_1>0$, $M_2>0$ and $0<\delta_1\leqq\delta_0$.

PROOF. The proof is similar to the proofs of Theorem 1 and 2. We only change test functions. First, it follows from [1] that for each k the problem (1′), (2), with $u(x)=\phi_k(x)$ on ∂Q, has a solution $u_k\in W^{1,2}(Q)\cap L^\infty(Q)$ such that

$$\Phi_k(x)\leqq u_k(x)\leqq\Psi_k(x) \quad \text{on } Q\,.$$

Taking $v(x)=u_k(x)\rho(x)$ as a test function we obtain, integrating by parts, that

$$\int_Q \sum_{ij}^n a_{ij}(x, u_k)D_iu_kD_ju_k\rho dx=\frac{1}{2}\int_{\partial Q}\sum_{i,j=1}^n\int_0^{\phi_k^2} a_{ij}(x, s)dsD_i\rho D_j\rho dS_n$$

$$+\int_Q\sum_{i,j=1}^n\int_0^{u_k^2} a_{ij}(x, s)dsD_{ij}\rho dx$$

$$+\frac{1}{2}\int_Q\sum_{i,j=1}^n\int_0^{u_k^2} D_ia_{ij}(x, s)dsD_i\rho dx-\int_Q b(x, u_k, Du_k)u_k\rho dx$$

Using the ellipticity, (6) and the Young inequality we arrive at the estimate

$$\int_Q |Du_k|\rho dx\leqq C_1\left[\int_{\partial Q}\phi_k^2dS_x+\int_Q u_k^2dx+\int_Q f^2dx+\int_Q u_k^2\rho dx\right], \tag{28}$$

where $C_1>0$ is a constant. Similarly the use of the test function

$$v(x)=\begin{cases} u_k(x)(\rho(x)-\delta) & \text{on } Q_\delta, \\ 0 & \text{on } Q-Q_\delta, \end{cases}$$

yields the estimate

$$\sup_{0<\delta\leqq\delta_1}\int_{\partial Q}\leqq C_2\left[\int_Q |Du_k|^2\rho dx+\int_Q u_k^2dx+\int_Q u_k^2\rho dx+\int_Q f^2dx\right]. \tag{29}$$

The estimates (28) and (29) combined together give

$$\int_Q |Du_k|^2\rho dx+\sup_{0<\delta\leqq\delta_1}\int_{\partial Q_\delta} u_k^2dS_x\leqq C_3\left[\int_Q u_k^2dx+\int_Q u_k^2\rho dx+\int_Q f^2dx\right] \tag{30}$$

for some constant $C_2>0$. We now observe that for each $-1<\mu\leqq 1$ we have

$$\int_Q u_k^2\rho^\mu dx\leqq\frac{\delta_1^{1+\mu}}{1+\mu}\sup_{0<\delta\leqq\delta_1}\int_{\partial Q_\delta} u_k^2dS_x+\int_{Q_{\delta_1}} u_k^2\rho^\mu dx$$

$$\leqq\frac{\delta_1^{1+\mu}}{1+\mu}\sup_{0<\delta\leqq\delta_1}\int_{\partial Q_\delta} u_k^2dS_x+M\max_{Q_{\delta_1}}\rho^\mu,$$

where $M=\sup_{k\geqq 1}\sup_{Q_{\delta_1}}u_k(x)^2$. Hence taking δ_1 sufficiently small we get from (29) that

$$\int_Q |Du_k|^2\rho dx+\sup_{0<\delta\leqq\delta_1}\int_{\partial Q_\delta} u_k^2dS_x\leqq C_4\int_{\partial Q}\phi_k^2dS_x+C_5 \tag{31}$$

for some constants $C_4>0$ and $C_5>0$. On the other hand, according to Lemma 1 in [4], we have for $0<d\leqq\delta_0/2$

$$\int_{Q_\delta} u_k^2dS_x\leqq K\left[\int_{Q_\delta} u_k^2dx+d\int_{\partial Q_\delta} u_k^2dS_x+d\int_Q |Du_k|^2\rho dx\right],$$

for all $0<\delta\leqq d$, where $K>0$ is a constant independent of k, δ, d and δ_1. Combining this with the estimate (31) we obtain (27). The estimate (27) shows that the sequence $\{u_k\}$ is bounded in $\tilde{W}^{1,2}(Q)$. Consequently we may assume that u_k converges weakly in $\tilde{W}^{1,2}(Q)$ to a function u. By virtue of Theorem

14.11 in [16] we may also assume that u_k converges to u in $L^2(Q)$ and a.e. on Q. It is clear that u is a solution of (1′). By Theorem 2 in [5] u has a trace $\zeta\in L^2(\partial Q)$ in the sense of the L^2-convergence (see (6)). It is now a routine to show that $\zeta=\phi$ a.e. on ∂Q (for more details see the proof of Theorem 3 in [5]).

Remark. Examination of the proof of Theorem 3 shows that the assumption (26) on the growth of the nonlinearity b can be replaced by

$$|b(x, u, p)|\leqq f(x)+B(r(x)^{-\alpha}|p|+r(x)^{-\beta}|u|)$$

for all $(x, u, p)\in Q\times \boldsymbol{R}\times \boldsymbol{R}_n$, where α and β are constants such that $0<\alpha<1$ and $0<\beta<2$.

We close up this paper with an example illustrating the use of Theorem 3. Let

$$b(x, u, Du)=f(x)-\sqrt{|Du|^2+1}\,g(u)F(x),$$

where $f\rho^{\theta/2}\in L^2(Q)$ for some $2\leqq\theta<3$, F is a measurable function such that $|F(x)|\leqq Br(x)^{-\alpha}$ on Q with $0<\alpha<1$ and $g\in L^\infty(\boldsymbol{R})$ with $g(u)u\geqq 0$ on $\boldsymbol{R}$. We assume that a_{ij} satisfy the conditions (i) and (ii) (see the example following Theorem 1) and moreover $D_u a_{ij}\in L^\infty(Q\times\boldsymbol{R})$. Let $\{\phi_k\}$ be a C^1-sequence converging to ϕ in $L^2(\partial Q)$. Since f, in general, is not in $L^2(Q)$, we take a sequence $\{f_l\}$ in $L^\infty(Q)$ such that $\int_Q f_l^2\rho^\theta dx\to\int_Q f^2\rho^\theta dx$ as $l\to\infty$. First, we consider for each $l\geqq 1$ the Dirichlet problem

$$-\sum_{i,j=1}^n D_j(a_{ij}(x, u)D_i u)+b_l(x, u, Du)=0 \quad \text{in} \tag{32}$$

$$u(x)=\phi(x) \quad \partial Q, \tag{33}$$

where $b_l(x, u, Du)=f_l(x)-\sqrt{|Du|^2+1}\,g(u)F(x)$. To solve the problem (32), (33) we construct a sequence of supersolutions $\{\Psi_k^l\}$ (subsolutions $\{\Phi_k^l\}$) obtained as solutions of the Dirichlet problem

$$-\sum_{i,j=1}^n D_j(a_{ij}(x, u)D_i u)=|f_l(x)| \quad \text{in } Q, \text{ (resp. } -|f_l(x)|)$$

$$u(x)=\phi_k(x) \quad \text{on } \partial Q\,. \quad \text{resp. } -|\phi_k(x)|$$

Using the assumptions (i) and (ii) we can show that $\{\Phi_k^l\}$ and $\{\Psi_k^l\}$ are bounded in $\widetilde{W}^{1,2}(Q)$ independently of k and l. Moreover, both sequences are locally uniformly bounded in $L^\infty(Q)$. By Theorem 3 for each l the problem (32), (33) has a solution $u_l\in\widetilde{W}^{1,2}(Q)$. It is clear that we may assume that $\lim_{k\to\infty}\Phi_k^l(x)=\Phi_l$ and $\lim_{k\to\infty}\Psi_k^l(x)=\Psi_l(x)$ a.e. on Q with $\{\Phi_l\}$ and $\{\Psi_l\}$ bounded in $L^2(Q)$. We also have for each l

$$\Phi_l(x)\leqq u_l(x)\leqq\Psi_l(x) \quad \text{a.e. on } Q\,.$$

Using this inequality and repeating the estimates from the proof Theorem 3 one can show that

$$\int_Q|Du|^2\rho dx+\int_Q u_l^2 dx+\sup_{0<\delta\leqq\delta_1}\int_{\partial Q}u_l^2 dS_x\leqq M_1\Big(\int_{\partial Q}\phi^2 dS_x+\int_Q f^2\rho^\theta dx\Big)+M_2$$

for some constants $M_1>0$, $M_2>0$ and $\delta_1>0$. Obviously this estimate implies the solvability of the problem (32), (33) with b replaced by b_l.

References

[1] L. Boccardo, F. Murat and J.P. Puel, Résultats d'existence pour certains problèmes elliptiques quasilinéaires, Ann. Sc. Norm. Pisa **11(2)** (1984), 213-235.

[2] L. Boccardo, F. Murat and J.P. Puel, Existence de solutions non bornées pour certaines équations quarilinéaires, Portugaliae Math. **4-1** (1982), 507-534.

[3] J. Chabrowski and B. Thompson, On traces of solutions of semilinear partial differential equation of elliptic type, Ann. Polon. Math. **42** (1983), 45-71.

[4] J. Chabrowski and H.B. Thompson, On the boundary values of the solutions of linear elliptic equations, Bull. Austral. Math. Soc. **27** (1983), 1-30.

[5] J. Chabrowski, On boundary values of solutions of a quasi-linear partial differential equation of elliptic type, Rocky Mountain J. Math. **16** (1986), 223-236.

[6] J. Chabrowski, On the Dirichlet problem for a quasi-linear elliptic equations, Rend. Circ. Mat. Palermo **35** (1986), 159-168.

]7] J. Chabrowski, On the existence of solutions of the Dirichlet problem for nonlinear elliptic equations, Rend. Circ. Mat. Palermo **37** (1988), 65-87.

[8] J. Deuel and P. Hess, A criterion for the existence of solutions on non-linear elliptic boundary value problems' Proc. Roy. Roc. Edinburgh Sect. A **83(3)** (1974/75), 45-54.

[9] M. Giaquinta and M. Modica, Regularity results for some classes of higher order nonlinear elliptic systems, J. Reine Angw. **311-312** (1973), 145-169.

[10] D. Gilbarg and N.S. Trrdinger, Elliptic partial differential equations of second order, Springer—Verlag, Berlin—Heidelber—New York—Tokyo, 1983 (second edition).

[11] F.W. Gehring, The L^p-integrability of the partial derivatives of a quasi-conformal mapping, Acta Math. **130** (1973), 265-277.

[12] P. Hess, On the solvability of nonlinear elliptic boundary value problems, Indiana Univ. Math. J. **25(5)** (1976), 461-466.

[13] A. Kufner, O. John and S. Fučik, Function spaces, Nordhof, Academia, Leyden-Prague, 1977.

[14] O.A. Ladyzhenskaya and N.N. Ural'ceva, Linear and quasilinear elliptic equations, Academia Press, 1968.

[15] J.L. Lions, Quelques méthodes de résolution de problèmes aux limites nonlinéaires, Dunod, Gauthier—Villars, Paris, 1968.

[16] R.D. Meyer, Some embedding theorems for generalized Sobolev spaces and applications to degenerate elliptic differential operrtors, J. Math. Mech. **16** (1967), 739-760.

[17] V.P. Mikhailov, On the boundary values of the solutions of elliptic equations, Appl. Math, Optim. **6** (1980), 193-199.

[18] V. P. Mikhailov, Boundary values of the solutions of elliptic equations in domains with smooth boundary, Mat. Sb. **101** (**43**) (1974), 163-188.

[19] Yu. A. Mikhailov, Solvability of the Dirichlet problem in the space H^1_{loc} for second order elliptic equations (Russian), Differential equations and operator theory, Mat. Issled. Akademija Nauk Moldavskoi SSSR **67** (1982), 109-123.

[20] J. Serrin, Local behavior of solutions of quasi-linear elliptic equatiins, Acta Math. **III** (1924), 247-302.

Department of Mathematics,
The University of Queensland
St. Lucia 4067, Queensland,
Australia