

REMARKS ON THETA SERIES

By

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In this note we prove two propositions about theta series. The field of real numbers will be denoted by \mathbf{R} , the field of complex numbers by \mathbf{C} , and Siegel upper half domain of degree n by H_n . And $\exp(2\pi\sqrt{-1}z)$ will be denoted by $e(z)$. A theta series $\mathcal{G}\left[\begin{smallmatrix} k' \\ k'' \end{smallmatrix}\right](z|x)$ of theta characteristic $\left(\begin{smallmatrix} k' \\ k'' \end{smallmatrix}\right)$ is defined by

$$\mathcal{G}\left[\begin{smallmatrix} k' \\ k'' \end{smallmatrix}\right](z|x) = \sum_{r \in \mathbf{Z}^n} e\left(\frac{1}{2}{}^t(r+k')z(r+k') + {}^t(r+k')(x+k'')\right)$$

where $\left(\begin{smallmatrix} k' \\ k'' \end{smallmatrix}\right) \in \mathbf{R}^{2n}$ and $(z, x) \in H_n \times \mathbf{C}^n$.

One of the two propositions is the follows;

PROPOSITION 1. Let $f_i(z, x) = \mathcal{G}\left[\begin{smallmatrix} k'_i \\ k''_i \end{smallmatrix}\right](z|x)$ be m theta series of theta characteristic $\left(\begin{smallmatrix} k'_i \\ k''_i \end{smallmatrix}\right)$ ($i=1, 2, \dots, m$). If $\left(\begin{smallmatrix} k'_i \\ k''_i \end{smallmatrix}\right) \not\equiv \left(\begin{smallmatrix} k'_j \\ k''_j \end{smallmatrix}\right) \pmod{\mathbf{Z}^{2n}}$ for any i, j ($1 \leq i < j \leq m$) then f_1, f_2, \dots, f_m are linearly independent over \mathbf{C} .

In a special case, when $\left(\begin{smallmatrix} k'_i \\ k''_i \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} k'_j \\ k''_j \end{smallmatrix}\right) \pmod{\mathbf{Q}^{2n}}$ for any i, j ($1 \leq i < j \leq m$), the result was given in S. Koizumi's lecture at University of Tsukuba.

And the other is the following:

PROPOSITION 2. If $\mathcal{G}\left[\begin{smallmatrix} k' \\ k'' \end{smallmatrix}\right](z|0)$ is identically zero as a function of $z \in H_n$, then $2k' \equiv 2k'' \equiv 0 \pmod{\mathbf{Z}^{2n}}$ and $2{}^t k' k'' \not\equiv 0 \pmod{\mathbf{Z}}$.

In case that $\left(\begin{smallmatrix} k' \\ k'' \end{smallmatrix}\right) \in \mathbf{Q}^{2n}$, the result has been already known (Igusa [1], p. 174 Theorem 1). And we can see easily that the converse of proposition 2 is true.

Proposition 2 was proved first by A. Seyama [3] and later by the author independently, but since the two proofs are quite different in principle, the author believes that his proof is worth publishing.

1. PROOF OF PROPOSITION 1. The proof will be by induction on m . The result is trivial for $m=1$, so assume then that the result is true for $m-1$, and consider the proposition as stated above for m functions. Suppose the result is not true, and there exist m complex numbers a_1, a_2, \dots, a_m which are not all zero, such that $a_1f_1+a_2f_2+\dots+a_mf_m=0$. From the induction hypothesis, $a_i \neq 0$ for any i ($1 \leq i \leq m$). Then, for any $s', s'' \in \mathbb{Z}^n$ we have

$$0 = \sum_{i=1}^m a_i f_i(z, x + zs' + s'') = \sum_{i=1}^m a_i f_i(z, x) e\left(-{}^t s' x - \frac{1}{2} {}^t s' z s' - {}^t k'_i s' + {}^t k'_i s''\right).$$

Hence

$$\sum_{i=1}^m a_i f_i(z, x) e(-{}^t k'_i s' + {}^t k'_i s'') = 0.$$

On the other hand

$$\sum_{i=1}^m a_i f_i(z, x) e(-{}^t k''_i s' + {}^t k''_i s'') = 0.$$

Then we have

$$\sum_{i=1}^{m-1} a_i \{e(-{}^t k'_i s' + {}^t k'_i s'') - e(-{}^t k''_i s' + {}^t k''_i s'')\} f_i(z, x) = 0.$$

It follows from the induction hypothesis that

$$e(-{}^t k'_i s' + {}^t k'_i s'') = e(-{}^t k''_i s' + {}^t k''_i s'') \quad \text{for any } s', s'' \in \mathbb{Z}^n \quad (1 \leq i \leq m-1).$$

Therefore

$$e(-{}^t(k'_i - k''_i)s' + {}^t(k'_i - k''_i)s'') = 1 \quad \text{for any } s', s'' \in \mathbb{Z}^n \quad (1 \leq i \leq m-1).$$

This implies $k''_i \equiv k'_i \pmod{\mathbb{Z}^n}$, $k'_i \equiv k''_i \pmod{\mathbb{Z}^n}$ ($1 \leq i \leq m-1$). But this contradicts the hypothesis, so the proof is completed.

REMARK. Let f_1, f_2, \dots, f_m be automorphic functions which respectively belong to factors of automorphy $\rho_1, \rho_2, \dots, \rho_m$ which are different from one another. Similarly to the above proof we can prove the following proposition: if ρ_i/ρ_j is a constant factor of automorphy for any i, j then f_1, f_2, \dots, f_m are linearly independent over \mathbb{C} .

But in general f_1, f_2, \dots, f_m are not always linearly independent. For example, define a factor of automorphy ρ on \mathbb{C} with respect to an analytic transformation group $G = \{1, -1\}$ by $\rho(1, z) = 1$ and $\rho(-1, z) = e^z$, and put $\rho_1 = 1$, $\rho_2 = \rho$, $\rho_3 = \rho^2$. Then $f_1 = -1$, $f_2 = 1 + e^{-z}$, $f_3 = -e^{-z}$ are automorphic functions which respectively belong to ρ_1, ρ_2, ρ_3 and are apparently linearly dependent.

2. Now in order to prove proposition 2 let us prepare two lemmas. Let $k' = \begin{pmatrix} k'_1 \\ k'_2 \\ \vdots \\ k'_n \end{pmatrix}$, $k'' = \begin{pmatrix} k''_1 \\ k''_2 \\ \vdots \\ k''_n \end{pmatrix} \in \mathbf{R}^n$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbf{C}^n$ and put $k_m^{*'} = \begin{pmatrix} k'_1 \\ \vdots \\ k'_m \end{pmatrix}$, $k_m^{*''} = \begin{pmatrix} k''_1 \\ \vdots \\ k''_m \end{pmatrix} \in \mathbf{R}^m$, $x_m^* = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \mathbf{C}^m$, and $k_{*'}^m = \begin{pmatrix} k'_{m+1} \\ \vdots \\ k'_n \end{pmatrix}$, $k_{*''}^m = \begin{pmatrix} k''_{m+1} \\ \vdots \\ k''_n \end{pmatrix} \in \mathbf{R}^{n-m}$, $x_{*'}^m = \begin{pmatrix} x_{m+1} \\ \vdots \\ x_n \end{pmatrix} \in \mathbf{C}^{n-m}$, where m is some integer such that $1 \leq m < n$.

LEMMA 1. Let m be any integer such that $1 \leq m < n$. For any $z_m^* \in H_m$, $z_{*'}^m \in H_{n-m}$,

$$\mathcal{G} \begin{bmatrix} k' \\ k'' \end{bmatrix} \left(\begin{pmatrix} z_m^* & 0 \\ 0 & z_{*'}^m \end{pmatrix} \middle| x \right) = \mathcal{G} \begin{bmatrix} k_m^{*'} \\ k_m^{*''} \end{bmatrix} (z_m^* | x_m^*) \mathcal{G} \begin{bmatrix} k_{*'}^m \\ k_{*''}^m \end{bmatrix} (z_{*'}^m | x_{*'}^m).$$

LEMMA 2. Let u be any element of $GL(n, \mathbf{Z})$. Then

$$\mathcal{G} \begin{bmatrix} k' \\ k'' \end{bmatrix} (z | x) = \mathcal{G} \begin{bmatrix} u^{-1}k' \\ {}^t_{uk''} \end{bmatrix} ({}^tuz | {}^tux).$$

Hence if $\mathcal{G} \begin{bmatrix} k' \\ k'' \end{bmatrix} (z | 0)$ is identically zero as a function of $z \in H_n$, so is $\mathcal{G} \begin{bmatrix} u^{-1}k' \\ {}^t_{uk''} \end{bmatrix} (z | 0)$.

These two lemmas are easily proved.

PROOF OF PROPOSITION 2. When $n=1$, since

$$\mathcal{G} \begin{bmatrix} k' \\ k'' \end{bmatrix} (z | 0) = \mathcal{G} \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z | zk' + k'') e\left(\frac{1}{2}k'zk' + k'k''\right),$$

both $\mathcal{G} \begin{bmatrix} k' \\ k'' \end{bmatrix} (z | 0)$ and $\mathcal{G} \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z | zk' + k'')$ are identically zero at the same time.

But $\mathcal{G} \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z | x)$ is zero if and only if $x \in \left(\frac{1}{2} + \mathbf{Z}\right)z + \left(\frac{1}{2} + \mathbf{Z}\right) = \left\{az + b \mid a \equiv \frac{1}{2}, b \equiv \frac{1}{2} \pmod{\mathbf{Z}}\right\}$. Hence $\mathcal{G} \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z | zk' + k'')$ is identically zero if and only if $zk' + k'' \in \left(\frac{1}{2} + \mathbf{Z}\right)z + \left(\frac{1}{2} + \mathbf{Z}\right)$ for any $z \in H_1$. This implies that if $\mathcal{G} \begin{bmatrix} k' \\ k'' \end{bmatrix} (z | 0)$ is identically zero, $k' \equiv \frac{1}{2}$ and $k'' \equiv \frac{1}{2} \pmod{\mathbf{Z}}$ and the result is true in this case.

When $n=2$, by Lemma 1

$$\mathcal{G} \begin{bmatrix} k' \\ k'' \end{bmatrix} \left(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \middle| 0 \right) = \mathcal{G} \begin{bmatrix} k'_1 \\ k''_1 \end{bmatrix} (z_1 | 0) \mathcal{G} \begin{bmatrix} k'_2 \\ k''_2 \end{bmatrix} (z_2 | 0) \quad \text{for any } z_1, z_2 \in H_1.$$

Since the result is true when $n=1$ it follows that

$$k'_1 \equiv k''_1 \equiv \frac{1}{2} \quad \text{or} \quad k'_2 \equiv k''_2 \equiv \frac{1}{2} \pmod{\mathcal{Z}}.$$

We may assume $k'_1 \equiv k''_1 \equiv \frac{1}{2}$. Then putting $u = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, by Lemma 2 $\mathcal{G} \left[\begin{smallmatrix} u^{-1}k' \\ {}^t u k'' \end{smallmatrix} \right] (z|0)$ is identically zero, where $u^{-1}k' = \begin{pmatrix} k'_1 - k'_2 \\ -k'_1 + 2k'_2 \end{pmatrix}$ and ${}^t u k'' = \begin{pmatrix} 2k''_1 + k''_2 \\ k''_1 + k''_2 \end{pmatrix}$. Then similarly to above we gain

$$k'_1 - k'_2 \equiv 2k''_1 + k''_2 \equiv \frac{1}{2} \quad \text{or} \quad -k'_1 + 2k'_2 \equiv k''_1 + k''_2 \equiv \frac{1}{2} \pmod{\mathcal{Z}},$$

therefore

$$k'_2 \equiv 0, \quad k''_2 \equiv \frac{1}{2} \quad \text{or} \quad 2k'_2 \equiv 0, \quad k''_2 \equiv 0 \pmod{\mathcal{Z}},$$

and in either case the result is true.

Now assuming $n > 2$, we shall complete the proof by induction on n . Suppose $\mathcal{G} \left[\begin{smallmatrix} k' \\ k'' \end{smallmatrix} \right] (z|0)$ is identically zero.

Case 1) If there exists i ($1 \leq i \leq n$) such that $2k'_i \not\equiv 0$ or $2k''_i \not\equiv 0$ or $2^t k'_i k''_i \equiv 0 \pmod{\mathcal{Z}}$, we may assume $i=1$. Then by Lemma 1

$$\mathcal{G} \left[\begin{smallmatrix} k_1^{*'} \\ k_1^{*''} \end{smallmatrix} \right] (z_1^*|0) \mathcal{G} \left[\begin{smallmatrix} k_1^{\downarrow'} \\ k_1^{\downarrow''} \end{smallmatrix} \right] (z_1^\downarrow|0) = 0 \quad \text{for any } z_1^* \in H_1, z_1^\downarrow \in H_{n-1}.$$

Since $\mathcal{G} \left[\begin{smallmatrix} k_1^{*'} \\ k_1^{*''} \end{smallmatrix} \right] (z_1^*|0)$ is not identically zero, $\mathcal{G} \left[\begin{smallmatrix} k_1^{\downarrow'} \\ k_1^{\downarrow''} \end{smallmatrix} \right] (z_1^\downarrow|0)$ is identically zero, $\mathcal{G} \left[\begin{smallmatrix} k_1^{\downarrow'} \\ k_1^{\downarrow''} \end{smallmatrix} \right] (z_1^*|0)$ is identically zero hence we have $2k_1^{\downarrow'} \equiv 2k_1^{\downarrow''} \equiv 0 \pmod{\mathcal{Z}^{n-1}}$ and $2^t k_1^{\downarrow'} k_1^{\downarrow''} \not\equiv 0 \pmod{\mathcal{Z}}$ from the induction hypothesis. Then there exists an integer $j \geq 2$ such that $k_j \equiv k_j'' \equiv \frac{1}{2} \pmod{\mathcal{Z}}$ (if not, $2^t k_1^{\downarrow'} k_1^{\downarrow''} \equiv 0 \pmod{\mathcal{Z}}$). We may assume $j=2$. Now by Lemma 1,

$$\mathcal{G} \left[\begin{smallmatrix} k_2^{*'} \\ k_2^{*''} \end{smallmatrix} \right] (z_2^*|0) \mathcal{G} \left[\begin{smallmatrix} k_2^{\downarrow'} \\ k_2^{\downarrow''} \end{smallmatrix} \right] (z_2^\downarrow|0) = 0 \quad \text{for any } z_2^* \in H_2, z_2^\downarrow \in H_{n-2},$$

Since $2^t k_2^{\downarrow'} k_2^{\downarrow''} \equiv 2^t k_2^{\downarrow'} k_2^{\downarrow''} - \frac{1}{2} \equiv 0 \pmod{\mathcal{Z}}$, $\mathcal{G} \left[\begin{smallmatrix} k_2^{\downarrow'} \\ k_2^{\downarrow''} \end{smallmatrix} \right] (z_2^\downarrow|0)$ is not identically zero. This implies that $2k'_1 \equiv 2k''_1 \equiv 0$ and $2k'_1 k''_1 \equiv 0 \pmod{\mathcal{Z}}$, hence $2k' \equiv 2k'' \equiv 0 \pmod{\mathcal{Z}^n}$ and $2^t k' k'' = 2k'_1 k''_1 + 2^t k_1^{\downarrow'} k_1^{\downarrow''} \not\equiv 0 \pmod{\mathcal{Z}}$.

Case 2) Now we assume for any i ($1 \leq i \leq n$) $2k'_i \equiv 2k''_i \equiv 0$, $2k'_i k''_i \not\equiv 0 \pmod{\mathcal{Z}}$, that is, $k'_i \equiv k''_i \equiv \frac{1}{2} \pmod{\mathcal{Z}}$ for any i . In this case, if $\mathcal{G} \left[\begin{smallmatrix} k' \\ k'' \end{smallmatrix} \right] (z|0)$ is identically zero, n must be odd. In fact, if n is even by lemma 1

$$\mathcal{G}\left[\begin{array}{c} k_{\frac{1}{2}}^{*'} \\ k_{\frac{1}{2}}^{*''} \end{array}\right](z_{\frac{1}{2}}^*|0)\mathcal{G}\left[\begin{array}{c} k_{\frac{1}{2}}^{2'} \\ k_{\frac{1}{2}}^{2''} \end{array}\right](z_{\frac{1}{2}}^2|0)=0 \quad \text{for any } z_{\frac{1}{2}}^* \in H_2, z_{\frac{1}{2}}^2 \in H_{n-2}.$$

But from the induction hypothesis neither $\mathcal{G}\left[\begin{array}{c} k_{\frac{1}{2}}^{*'} \\ k_{\frac{1}{2}}^{*''} \end{array}\right](z_{\frac{1}{2}}^*|0)$ nor $\mathcal{G}\left[\begin{array}{c} k_{\frac{1}{2}}^{2'} \\ k_{\frac{1}{2}}^{2''} \end{array}\right](z_{\frac{1}{2}}^2|0)$ is identically zero. This contradicts the hypothesis. Now if n is odd, clearly $2k' \equiv 2k'' \equiv 0 \pmod{\mathbf{Z}^n}$ and $2^t k' k'' \not\equiv 0 \pmod{\mathbf{Z}}$. Hence the proof is completed.

References

- [1] Igusa, J., Theta functions. Die Grundlehren der Math. Wiss., bd. 194, Springer-Verlag, Berlin (1972).
- [2] Koizumi, S., Theta functions. (in Japanese) Lecture note, Sophia. Univ. Tokyo (1978).
- [3] Seyama, A., On the vanishing of theta constants. (to appear)