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ON THE DIFFERENCE $f^3(x)-g^2(x)$

By

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In 1965 H. Davenport [2] proved that if $f(x)$, $g(x)$ are polynomials in x with arbitrary real or complex coefficients, then we have either

$$f^3(x)-g^2(x)=0 \qquad \text{identically},$$

or

$$\deg(f^3(x)-g^2(x)) \geqq \frac{1}{2}\deg f(x)+1\,. \tag{1}$$

It is known that for some pairs of polynomials $f(x)$, $g(x)$ the equality holds in (1). Clearly, we have for such pairs of polynomials

$$\deg f(x)=2k\,, \qquad \deg g(x)=3k$$

with some integral $k\geqq 1$. Indeed, if

$$f(x)=x^2+2\,, \qquad g(x)=x^3+3x\,,$$

then

$$f^3(x)-g^2(x)=3x^2+8\,,$$

and if

$$f(x)=x^4+2x\,, \qquad g(x)=x^6+3x^3+\frac{3}{2}\,,$$

then

$$f^3(x)-g^2(x)=-x^3-\frac{9}{4}\,.$$

Some other examples of pairs of polynomials $f(x)$, $g(x)$ of higher degrees satisfying the condition

$$\deg(f^3(x)-g^2(x))=\frac{1}{2}\deg f(x)+1 \tag{2}$$

are given by B. J. Birch, S. Chowla, Marshall Hall, Jr. and A. Schinzel [1]. They have shown in fact that

$$f(x)=x^6+4x^4+10x^2+6\,,$$

$$g(x)=x^9+6x^7+21x^5+35x^3+\frac{63}{2}x\,,$$

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and

$$f(x)=x^{10}+12x^{7}+60x^{4}+96x\ ,$$

$$g(x)=x^{15}+18x^{12}+144x^{9}+576x^{6}+1080x^{3}+432$$

are such pairs. It is interesting to note that, in all of these examples, $f(x)$ and $g(x)$ are polynomials of $\boldsymbol{Q}[x]$, where $\boldsymbol{Q}$ denotes as usual the field of rational numbers. It can be shown that there is also a pair of such polynomials $f(x)$ and $g(x)$ of degrees 8 and 12, respectively, with coefficients in $\boldsymbol{Q}$. Davenport [2] has found that there exist polynomials $f(x)$ and $g(x)$ satisfying the condition (2), being of degrees 16 and 24, respectively, and having coefficients in the field $\boldsymbol{C}$ of complex numbers. Actually, the polynomials $f(x)$, $g(x)$ in Davenport's example have coefficients in $\boldsymbol{Q}(\sqrt{-3})$. However, the question of whether there exist pairs of polynomials $f(x)$, $g(x)$ with coefficients in $\boldsymbol{C}$ and with $\deg f(x)$, and so $\deg g(x)$ also, arbitrarily large which satisfy (2) remains still open (cf. [1] and [2]).

Our principal aim in this note is to indicate that for $k=7$ and 11 there exist pairs of polynomials $f(x)$, $g(x)$ with real algebraic coefficients such that

$$\deg f(x)=2k\ , \qquad \deg g(x)=3k\ ,$$

and the condition (2) is therewith fulfilled. It may be of some interest to note that, in order to produce pairs of polynomials $f(x)$, $g(x)$ of that kind, we have taken full advantage of making machine computations whatever possible, with the aid of programmes in REDUCE-2, a language designed and used for algebraic manipulation of formulas.

1. General considerations.

We begin with describing some general methods, or algorithms, of finding particular pairs of polynomials $f(x)$, $g(x)$ that satisfy the condition (2). The first one is a slight modification of the method proposed in [1].

(I) Let k be a given integer $\geqq 1$ and take $\nu=1, 2$, or 3. Define

$$h(x)=x^{5k-\nu}+t_1x^{5k-\nu-1}+\cdots+t_{5k-\nu}\ ,$$

where the t_j's are parameters whose values are to be determined later. We write

$$h^2(x)=\sum_{i=0}^{10k-2\nu}a_ix^{10k-2\nu-i}=x^{8k-2\nu}f(x)+A(x)$$

with

$$f(x)=\sum_{i=0}^{2k} a_i x^{2k-i}, \qquad \deg A(x)\leqq 8k-2\nu-1,$$

and

$$h^3(x)=\sum_{j=0}^{15k-3\nu} b_j x^{15k-3\nu-j}=x^{12k-3\nu}g(x)+B(x)$$

with

$$g(x)=\sum_{j=0}^{3k} b_j x^{3k-j}, \qquad \deg B(x)\leqq 12k-3\nu-1.$$

We have then

$$f^3(x)-g^2(x)=x^{-24k+6\nu}H(x),$$

where

$$H(x)=(h^2(x)-A(x))^3-(h^3(x)-B(x))^2,$$

so that $\deg(f^3(x)-g^2(x))=k+1$ if and only if $\deg H(x)=25k-6\nu+1$. Therefore, a sufficient condition for $f(x)$, $g(x)$ to satisfy (2) is that

(3) $$a_{2k+1}=a_{2k+2}=\cdots=a_{5k-2}=0,$$

(4) $$b_{3k+1}=b_{3k+2}=\cdots=b_{5k-2}=0,$$

and

(5) $$3a_{5k-1}-2b_{5k-1}\neq 0.$$

There are $5k-4$ equations in (3) and (4) with $5k-\nu$ unknowns $t_1, t_2, \cdots, t_{5k-\nu}$ ($\nu=1, 2,$ or 3), and we may find in general the values of the t_j satisfying (3), (4) and (5). With $\nu=1$, for instance, the conditions (3), (4) and (5) can be replaced by

$$a_{2k+1}=a_{2k+2}=\cdots=a_{5k-1}=0$$

and

$$b_{3k+1}=b_{3k+2}=\cdots=b_{5k-2}=0, \qquad b_{5k-1}\neq 0,$$

or by

$$a_{2k+1}=a_{2k+2}=\cdots=a_{5k-2}=0, \qquad a_{5k-1}\neq 0$$

and

$$b_{3k+1}=b_{3k+2}=\cdots=b_{5k-1}=0.$$

Note that we always have

$$a_m=\sum_{\substack{i+j=m\\ i\leqq j}}(2-\delta_{ij})t_i t_j, \qquad b_n=\sum_{i+j=n} a_i t_j,$$

where $a_0=b_0=t_0=1$, and $\delta_{ij}=1$ if $i=j$, and $=0$ if $i\neq j$.

The number of parameters t_j, which is the same as the degree of the basic polynomial $h(x)$, can be reduced as low as to $3k-1$, at the cost of imposing on the coefficients of $h^2(x)$ and of $h^3(x)$ somewhat more complicated conditions to satisfy than (3), (4) and (5).

A variant of the above method is the following. First, take $k=2l+1$, $l\geqq 0$, and put

$$h(x)=x^{5l+1}+t_1x^{5l}+\cdots+t_{5l+1}.$$

We write as before

$$h^2(x)=\sum_{i=0}^{10l+2} a_i x^{10l+2-i}=x^{8l+1}f_0(x)+A(x)$$

with

$$f_0(x)=\sum_{i=0}^{2l+1} a_i x^{2l+1-i}, \qquad \deg A(x)\leqq 8l,$$

and

$$h^3(x)=\sum_{j=0}^{15l+3} b_j x^{15l+3-j}=x^{12l+2}g_0(x)+B(x)$$

with

$$g_0(x)=\sum_{j=0}^{3l+1} b_j x^{3l+1-j}, \qquad \deg B(x)\leqq 12l+1.$$

Then, with $f(x)=f_0(x^2)$, $g(x)=xg_0(x^2)$ we have $\deg f(x)=2k$, $\deg g(x)=3k$, and

$$f^3(x)-g^2(x)=x^{-48l-6}H(x),$$

where

$$H(x)=(h^2(x^2)-A(x^2))^3-(h^3(x^2)-B(x^2))^2.$$

We have, therefore, $\deg(f^3(x)-g^2(x))=k+1$ if and only if $\deg H(x)=50l+8$, and sufficient condition for $f(x)$, $g(x)$ to satisfy (2) is that

$$a_{2l+2}=a_{2l+3}=\cdots=a_{5l+1}=0$$

and

$$b_{3l+1}\neq 0, \qquad b_{3l+2}=b_{3l+3}=\cdots=b_{5l+1}=0.$$

Next, taking $k=3l+2$, $l\geqq 0$, we define, with

$$h(x)=x^{5l+2}+t_1x^{5l+1}+\cdots+t_{5l+2},$$

$$h^2(x)=\sum_{i=0}^{10l+4} a_i x^{10l+4-i}=x^{8l+3}f_0(x)+A(x),$$

$$f(x)=xf_0(x^3), \qquad f_0(x)=\sum_{i=0}^{2l+1} a_i x^{2l+1-i}, \qquad \deg A(x)\leqq 8l+2,$$

and

$$h^3(x)=\sum_{j=0}^{15l+6} b_j x^{15l+6-j}=x^{12l+4}g_0(x)+B(x),$$

$$g(x)=g_0(x^3), \qquad g_0(x)=\sum_{j=0}^{3l+2} b_j x^{3l+2-j}, \qquad \deg B(x)\leqq 12l+3,$$

so that $\deg f(x)=2k$, $\deg g(x)=3k$, and

$$f^3(x)-g^2(x)=x^{-72l-24}H(x)\,,$$

where

$$H(x)=(h^2(x^3)-A(x^3))^3-(h^3(x^3)-B(x^3))^2\,.$$

It follows from this that $\deg(f^3(x)-g^2(x))=k+1$ if and only if $\deg H(x)=75l+27$, and a sufficient condition for $f(x)$, $g(x)$ to satisfy (2) is that

$$a_{2l+1}\neq 0\,, \qquad a_{2l+2}=a_{2l+3}=\cdots=a_{6l+2}=0$$

and

$$b_{3l+3}=b_{3l+4}=\cdots=b_{6l+2}=0\,.$$

(II) Let k be again a given positive integer and consider the polynomials with coefficients in $\boldsymbol{C}$

$$f(x)=\sum_{i=0}^{2k}a_ix^{2k-i}\,, \qquad g(x)=\sum_{j=0}^{3k}b_jx^{3k-j}\,,$$

where it is assumed that $a_0=b_0=1$. We wish to show that the pair of polynomials $f(x)$, $g(x)$ satisfies the condition (2), if and only if

$$3f'(x)g(x)-2f(x)g'(x)=c\,, \tag{6}$$

where c is a non-zero constant.

For any monic polynomial $P=P(x)$ of degree $n\geqq 1$ and with coefficients in $\boldsymbol{C}$ we have

$$P(x)=\prod_{i=1}^{n}(x-\xi_i)$$

and therewith define

$$s_0(P)=n \quad\text{and}\quad s_\nu(P)=\sum_{i=1}^{n}\xi_i^\nu \qquad (\nu=1, 2, 3, \cdots)\,.$$

Now, in order to have

$$\deg(f^3(x)-g^2(x))=k+1$$

it is necessary and sufficient that the coefficients of x^m $(k+2\leqq m\leqq 6k-1)$ in $f^3(x)-g^2(x)$ do vanish and that of x^{k+1} does not, which is obviously equivalent to

$$s_\nu(f^3)-s_\nu(g^2)\begin{cases}=0 & (1\leqq\nu\leqq 5k-2)\,,\\ \neq 0 & (\nu=5k-1)\,.\end{cases} \tag{7}$$

Since we have $s_\nu(f^3)-s_\nu(g^2)=3s_\nu(f)-2s_\nu(g)$ for all values of ν, it follows from the formal power series expansion

$$\frac{3f'(x)}{f(x)}-\frac{2g'(x)}{g(x)}=\sum_{\nu=0}^{\infty}\frac{s_\nu(f^3)-s_\nu(g^2)}{x^{\nu+1}}$$

that (7) implies (6), and *vice versa.*

It is easily seen that if we write

$$3f'(x)g(x)-2f(x)g'(x)=\sum_{m=0}^{5k-1}c_m x^{5k-1-m},$$

then

$$c_m=\sum_{i=u}^{v}(2m-5i)a_i b_{m-i} \qquad (0\leqq m\leqq 5k-1),$$

where

$$u=\max(0, m-3k), \qquad v=\min(m, 2k).$$

We have

$$c_0=0 \qquad \text{automatically,}$$

and

$$c_{5k-1}=3a_{2k-1}b_{3k}-2a_{2k}b_{3k-1}.$$

Thus, the condition (6) is equivant to

$$c_1=c_2=\cdots=c_{5k-2}=0, \qquad c_{5k-1}=c \quad (\neq 0).$$

Our equation (6), which in some cases is slightly more convenient to deal with than the original condition (2), may be regarded as an *indefinite* differential equation in polynomials $f(x)$ and $g(x)$. The equation (6) admits a polynomial solution $g(x)$ when the polynomial $f(x)$ is given in such a way that the integral

$$\int (f(x))^{-5/2}dx$$

is a so-called pseudo-hyperelliptic integral; however, though this way of approaching the problem seems to be effective, the situation is in reality not so simple as expected.

Now, suppose that $k=2l+1$, $l\geqq 0$, and put

$$f_0(x)=\sum_{i=0}^{2l+1}a_i x^{2l+1-i}, \qquad g_0(x)=\sum_{j=0}^{3l+1}b_j x^{3l+1-j}.$$

Then, the polynomials $f(x)$, $g(x)$, defined by

$$f(x)=f_0(x^2), \qquad g(x)=xg_0(x^2),$$

so that $\deg f(x)=2k$, $\deg g(x)=3k$, satisfy the equation (6), if and only if

$$c_1=c_2=\cdots=c_{5l+1}=0, \qquad c_{5l+2}=\frac{1}{2}c \quad (\neq 0),$$

where

(8) $$c_m=\sum_{i=u}^{v}(2m-5i)a_ib_{m-i} \qquad (0\leqq m\leqq 5l+2)$$

with

$$u=\max(0,\ m-3l-1)\,, \qquad v=\min(m,\ 2l+1)\,.$$

If $k=3l+2$, $l\geqq 0$, and if

$$f_0(x)=\sum_{i=0}^{2l+1}a_ix^{2l+1-i}\,, \qquad g_0(x)=\sum_{j=0}^{3l+2}b_jx^{3l+2-j}\,,$$

then the polynomials $f(x)$, $g(x)$ defined by

$$f(x)=xf_0(x^3)\,, \qquad g(x)=g_0(x^3)\,,$$

so that $\deg f(x)=2k$, $\deg g(x)=3k$, satisfy the condition (6), if and only if

$$c_1=c_2=\cdots=c_{5l+2}=0\,, \qquad c_{5l+3}=\frac{1}{3}c \quad (\neq 0)\,,$$

where

(9) $$c_m=\sum_{i=u}^{v}(2m-5i)a_ib_{m-i} \qquad (0\leqq m\leqq 5l+3)$$

with

$$u=\max(0,\ m-3l-2)\,, \qquad v=\min(m,\ 2l+1)\,.$$

2. Some specific examples.

First we shall give examples of pairs of polynomials $f(x)$, $g(x)$, satisfying the condition (2) with $\deg f(x)=2k$, $\deg g(x)=3k$, for some small values of k. Our method of determining such pairs of polynomials $f(x)$, $g(x)$ will chiefly be the method (II) which we have just described above.

(i) $k=1$. Here, we have $k=2l+1$, $l=0$, and put

$$f_0(x)=a_0x+a_1\,, \qquad g_0(x)=b_0x+b_1\,,$$

where $a_0=b_0=1$. The coefficients c_m given by (8) with $l=0$ are:

$$c_0=0 \qquad \text{automatically,}$$

$$c_1=2a_0b_1-3a_1b_0=2b_1-3a_1\,,$$

$$c_2=-a_1b_1\,.$$

Accordingly, if we take

$$a_1=2z\,, \qquad b_1=3z \qquad \text{with } z\in \boldsymbol{Z},\ z\neq 0\,,$$

where $\boldsymbol{Z}$ denotes the set of rationnal integers, then we have

$$c_1=0, \qquad c_2=-6z^2\neq 0.$$

Thus we have

$$f_0(x)=x+2z, \qquad g_0(x)=x+3z,$$

$$f(x)=f_0(x^2)=x^2+2z, \qquad g(x)=xg_0(x^2)=x^3+3zx,$$

and

$$f^3(x)-g^2(x)=3z^2x^2+8z^3,$$

$$3f'(x)g(x)-2f(x)g'(x)=2c_2=-12z^2.$$

We note that the method (I) will furnish as a general solution of (6), i.e. of (2),

$$f(x)=x^2+2t_1x+t_1^2+2t_2,$$

$$g(x)=x^3+3t_1x^2+3(t_1^2+t_2)x+t_1(t_1^2+3t_2),$$

where t_1, t_2 are free parameters with $t_2\neq 0$.

(ii) $k=3$. In this case we have $k=2l+1$, $l=1$, and take

$$f_0(x)=a_0x^3+a_1x^2+a_2x+a_3,$$

$$g_0(x)=b_0x^4+b_1x^3+b_2x^2+b_3x+b_4,$$

where $a_0=b_0=1$. We have, by (8),

$$c_m=\sum_{i=u}^{v}(2m-5i)a_ib_{m-i} \qquad (0\leqq m\leqq 7)$$

with $u=\max(0, m-4)$, $v=\min(m, 3)$, where $c_0=0$ and the c_m $(1\leqq m\leqq 7)$ should satisfy

$$c_1=2b_1-3a_1=0,$$

$$c_2=4b_2-a_1b_1-6a_2=0,$$

$$c_3=6b_3+a_1b_2-4a_2b_1-9a_3=0,$$

$$c_4=8b_4+3a_1b_3-2a_2b_2-7a_3b_1=0,$$

$$c_5=5a_1b_4-5a_3b_2=0,$$

$$c_6=2a_2b_4-3a_3b_3=0,$$

$$c_7=-a_3b_4\neq 0.$$

We introduce a new set of parameters t_j $(1\leqq j\leqq 3)$ and write

$$a_1=2t_1,$$

$$a_2=t_1^2+2t_2,$$

$$a_3=2t_1t_2+2t_3.$$

Then, the equations $c_m=0$ $(1\leqq m\leqq 4)$ will give

$$b_1=3t_1\,,$$

$$b_2=3t_1^2+3t_2\,,$$

$$b_3=t_1^3+6t_1t_2+3t_3\,,$$

$$b_4=3t_1^2t_2+3t_1t_3+\frac{3}{2}t_2^2\,,$$

and the equation $c_5=0$ becomes, when reduced

$$t_2(t_1t_2+2t_3)=0\,.$$

Now, if $t_2=0$ then $c_6=0$ implies $t_3=0$, giving $b_4=0$. So, we must have $t_2\neq 0$, and

$$t_1t_2+2t_3=0\,.$$

On the other hand, it follows from $c_5=c_6=0$ that

$$3t_1^2t_2+3t_1t_3-2t_2^2=0\,.$$

Substituting in here $t_3=(-1/2)t_1t_2$, we get $t_2(3t_1^2-4t_2)=0$, or

$$3t_1^2-4t_2=0\,;$$

a general integer solution of this last equation is

$$t_1=2z\,,\qquad t_2=3z^2\qquad \text{with } z\in\boldsymbol{Z},\ z\neq 0\,,$$

and so

$$t_3=-3z^3\,.$$

Hence

$$a_1=4z\,,\qquad a_2=10z^2\,,\qquad a^3=6z^3:$$

$$b_1=6z\,,\qquad b_2=21z^2\,,\qquad b_3=35z^3\,,\qquad b_4=\frac{63}{2}z^4\,.$$

We thus have

$$f(x)=f_0(x^2)=x^6+4zx^4+10z^2x^2+6z^3\,,$$

$$g(x)=xg_0(x^2)=x^9+6zx^7+21z^2x^5+35z^3x^3+\frac{63}{2}z^4x\,,$$

where

$$3f'(x)g(x)-2f(x)g'(x)=-378z^7\,.$$

This with $z=1$ is one of the examples given in [1].

(iii) $k=2$. For this value of k we have $k=3l+2$, $l=0$, and we easily find, referring to (9), a pair

$$f(x)=x^4+2zx\,,$$
$$g(x)=x^6+3zx^3+\frac{3}{2}z^2$$

with $z\in\boldsymbol{Z}$, $z\neq 0$. Here

$$3f'(x)g(x)-2f(x)g'(x)=9z^3\,.$$

(iv) $k=5$. In this case $k=3l+2$, $l=1$, and one may proceed in a manner similar to that for the case of $k=3$, with (9) in place of (8), to obtain a pair

$$f(x)=x^{10}+6zx^7+15z^2x^4+12z^3x\,,$$
$$g(x)=x^{15}+9zx^{12}+36z^2x^9+72z^3x^6+\frac{135}{2}z^4x^3+\frac{27}{2}z^5$$

with $z\in\boldsymbol{Z}$, $z\neq 0$, where

$$3f'(x)g(x)-2f(x)g'(x)=486z^8\,.$$

For $z=2$ this reduces to another example found in [1].

So far everything is quite simple and no machine computations are needed.

3. Further examples.

We are now going to describe the examples newly found of pairs of polynomials $f(x)$, $g(x)$ satisfying the condition (2), or equivalently the equation (6), such that

$$\deg f(x)=2k\,,\qquad \deg g(x)=3k\,,$$

where $k=4$, 7, 8 and 11. Our computations were done on an electronic computer HITAC M-200H (VOS 3) in the Information Processing Center, Hiroshima University, Hiroshima.

(v) $k=4$. We have

$$f(x)=\sum_{i=0}^{8}a_ix^{8-i}\,,\qquad g(x)=\sum_{j=0}^{12}b_jx^{12-j}\,,$$

where $a_0=b_0=1$ and, with free parameters s, t, $t\neq 0$,

$$a_1=8s+2t\,,$$
$$a_2=28s^2+14st+7t^2\,,$$
$$a_3=56s^3+42s^2t+42st^2+6t^3\,,$$
$$a_4=70s^4+70s^3t+105s^2t^2+30st^3+11t^4\,,$$
$$a_5=56s^5+70s^4t+140s^3t^2+60s^2t^3+44st^4-4t^5\,,$$

$$a_6=28s^6+42s^5t+105s^4t^2+60s^3t^3+66s^2t^4-12st^5,$$

$$a_7=8s^7+14s^6t+42s^5t^2+30s^4t^3+44s^3t^4-12s^2t^5-12t^7,$$

$$a_8=s^8+2s^7t+7s^6t^2+6s^5t^3+11s^4t^4-4s^3t^5-12st^7+t^8;$$

$$b_1=12s+3t,$$

$$b_2=66s^2+33st+12t^2,$$

$$b_3=220s^3+165s^2t+120st^2+19t^3,$$

$$b_4=495s^4+495s^3t+540s^2t^2+171st^3+39t^4,$$

$$b_5=792s^5+990s^4t+1440s^3t^2+684s^2t^3+312st^4+24t^5,$$

$$b_6=924s^6+1386s^5t+2520s^4t^2+1596s^3t^3+1092s^2t^4+168st^5+30t^6,$$

$$b_7=792s^7+1386s^6t+3024s^5t^2+2394s^4t^3+2184s^3t^4+504s^2t^5+180st^6-36t^7,$$

$$b_8=495s^8+990s^7t+2520s^6t^2+2394s^5t^3+2730s^4t^4+840s^3t^5$$
$$+450s^2t^6-180st^7-15t^8,$$

$$b_9=220s^9+495s^8t+1440s^7t^2+1596s^6t^3+2184s^5t^4+840s^4t^5$$
$$+600s^3t^6-360s^2t^7-60st^8-60t^9,$$

$$b_{10}=66s^{10}+165s^9t+540s^8t^2+684s^7t^3+1092s^6t^4+504s^5t^5$$
$$+450s^4t^6-360s^3t^7-90s^2t^8-180st^9+\frac{27}{2}t^{10},$$

$$b_{11}=12s^{11}+33s^{10}t+120s^9t^2+171s^8t^3+312s^7t^4+168s^6t^5$$
$$+180s^5t^6-180s^4t^7-60s^3t^8-180s^2t^9+27st^{10}-\frac{9}{2}t^{11},$$

$$b_{12}=s^{12}+3s^{11}t+12s^{10}t^2+19s^9t^3+39s^8t^4+24s^7t^5+30s^6t^6$$
$$-36s^5t^7-15s^4t^8-60s^3t^9+\frac{27}{2}s^2t^{10}-\frac{9}{2}st^{11}+\frac{29}{2}t^{12}.$$

The polynomials $f(x)$, $g(x)$ satisfy

$$3f'(x)g(x)-2f(x)g'(x)=-513t^{19}.$$

(vi) $k=7$. For this value of k we have $k=2l+1$, $l=3$, and put

$$f(x)=\sum_{i=0}^{7}a_ix^{14-2i}, \qquad g(x)=\sum_{j=0}^{10}b_jx^{21-2j}$$

with $a_0=b_0=1$. The coefficients a_i $(1\leqq i\leqq 7)$ and b_j $(1\leqq j\leqq 10)$ are given in the following way.

Let r be any one of the roots of the irreducible quintic equation

$$x^5-\frac{8}{3}x^3+\frac{15}{4}x^2-\frac{10}{3}x+\frac{4}{3}=0,$$

which admits a uniqe real root in the interval $(-3, -2)$, and set

$$s=\left(\frac{24r^2-24r}{3r^2+2r-4}\right)^{1/5}.$$

Then, with a free parameter $t\neq 0$,

$$a_1=\frac{-144r^4-108r^3+288r^2-348r+243}{16}s^4t,$$

$$a_2=\frac{46212r^4+36852r^3-88576r^2+108647r-78957}{1248}s^3t^2,$$

$$a_3=\frac{-24072r^4-20172r^3+44764r^2-54790r+40731}{312}s^2t^3,$$

$$a_4=\frac{103068r^4+88872r^3-187572r^2+230573r-170597}{1248}st^4,$$

$$a_5=\frac{-23520r^4-20244r^3+41216r^2-54880r+37653}{624}t^5,$$

$$a_6=\frac{42576r^4+35100r^3-78496r^2+98508r-77623}{9216}s^4t^6,$$

$$a_7=\frac{-60r^4-48r^3+112r^2-145r+100}{96}s^3t^7;$$

$$b_1=\frac{-432r^4-324r^3+864r^2-1044r+729}{32}s^4t,$$

$$b_2=\frac{9408r^4+7536r^3-17984r^2+22069r-16083}{104}s^3t^2,$$

$$b_3=\frac{-469512r^4-397068r^3+862412r^2-1051214r+788907}{1352}s^2t^3,$$

$$b_4=\frac{2210508r^4+1962390r^3-3987258r^2+4789471r-3641527}{2704}st^4,$$

$$b_5=\frac{-250644r^4-229116r^3+443516r^2-530080r+401607}{208}t^5,$$

$$b_6=\frac{158004r^4+162756r^3-271392r^2+233244r-314765}{2304}s^4t^6,$$

$$b_7=\frac{-97404r^4-91020r^3+174280r^2-187081r+168035}{768}s^3t^7,$$

$$b_8=\frac{294840r^4+271572r^3-522144r^2+601874r-488901}{3072}s^2t^8,$$

$$b_9=\frac{-257748r^4-236268r^3+437440r^2-557307r+420451}{9216}st^9,$$

$$b_{10}=\frac{60r^4+48r^3-40r^2+193r-100}{64}t^{10}.$$

Here we have

$$3f'(x)g(x)-2f(x)g'(x)=\frac{102r^4+102r^3-170r^2+187r-153}{48}s^3t^{17}.$$

(vii) $k=8$. Here, we have $k=3l+2$, $l=2$, and

$$f(x)=\sum_{i=0}^{5}a_ix^{16-3i}, \qquad g(x)=\sum_{j=0}^{8}b_jx^{24-3j},$$

where $a_0=b_0=1$ and, with $t\neq 0$,

$$a_1=\frac{-26+6\sqrt{-3}}{3}t,$$

$$a_2=\frac{241-105\sqrt{-3}}{9}t^2,$$

$$a_3=\frac{-140+72\sqrt{-3}}{3}t^3,$$

$$a_4=\frac{301-147\sqrt{-3}}{6}t^4,$$

$$a_5=\frac{-64+24\sqrt{-3}}{3}t^5;$$

$$b_1=(-13+3\sqrt{-3})t,$$

$$b_2=\frac{383-183\sqrt{-3}}{6}t^2,$$

$$b_3=\frac{-4600+3384\sqrt{-3}}{27}t^3,$$

$$b_4=\frac{1835-1725\sqrt{-3}}{6}t^4,$$

$$b_5=\frac{-1177+1221\sqrt{-3}}{3}t^5,$$

$$b_6=\frac{1043-1008\sqrt{-3}}{3}t^6,$$

$$b_7=(-178+138\sqrt{-3})t^7,$$

$$b_8=(20-12\sqrt{-3})t^8.$$

With these polynomials we find

$$3f'(x)g(x)-2f(x)g'(x)=(-416+1248\sqrt{-3})t^{13}.$$

For $t=1$ our polynomials $f(x)$, $g(x)$ will reduce to the ones found by Davenport [2].

(viii) $k=11$. In this case we have $k=3l+2$, $l=3$, and

$$f(x)=\sum_{i=0}^{7} a_i x^{22-3i}, \qquad g(x)=\sum_{j=0}^{11} b_j x^{33-3j},$$

where $a_0=b_0=1$ as before and the coefficients a_i $(1\leqq i\leqq 7)$ and b_j $(1\leqq j\leqq 11)$ are given as follows.

Let r be any one of the roots of the irreducible cubic equation

$$x^3+3x^2+7x+\frac{31}{3}=0,$$

which has a unique real root in the interval $(-3, -2)$, and set

$$s=\left(\frac{24r^2+36r+156}{7}\right)^{1/5}.$$

Then, with a free parameter $t\neq 0$,

$$a_1=\frac{15r^2+18r-49}{64}s^4t,$$

$$a_2=\frac{-267r^2+1878r+6661}{896}s^3t^2,$$

$$a_3=\frac{-369r^2-2566r-5097}{224}s^2t^3,$$

$$a_4=\frac{4797r^2+18798r+31093}{896}st^4,$$

$$a_5=\frac{-3063r^2-9162r-15391}{448}t^5,$$

$$a_6=\frac{693r^2+5502r+11501}{4096}s^4t^6,$$

$$a_7=\frac{-3r^2-18r-43}{64}s^3t^7;$$

$$b_1=\frac{45r^2+54r-147}{128}s^4t,$$

$$b_2=\frac{-153r^2+1188r+3993}{224}s^3t^2,$$

$$b_3=\frac{-12141r^2-79248r-143775}{1568}s^2t^3,$$

$$b_4=\frac{146799r^2+536490r+720999}{3136}st^4,$$

$$b_5=\frac{-7641r^2-19494r-21249}{64}t^5,$$

$$b_6=\frac{20163r^2+77202r+90699}{1024}s^4t^6,$$

$$b_7=\frac{-23949r^2-36423r-75717}{1024}s^3t^7,$$

$$b_8=\frac{75141r^2+191070r+199485}{4096}s^2t^8,$$

$$b_9=\frac{-38467r^2-85074r-106315}{4096}st^9,$$

$$b_{10}=\frac{2493r^2+4926r+8277}{896}t^{10},$$

$$b_{11}=\frac{-3r^2-12r-21}{256}s^4t^{11}.$$

Here, we could replace, of course, t by st (or by $s^{-4}t$), thus eliminating s in the above expressions for the a_i and the b_j; this replacement, however, would not seem to bring any particular improvement on these expressions.*)

We have

$$3f'(x)g(x)-2f(x)g'(x)=\frac{27r^2+18r+27}{128}s^2t^{18}.$$

References

[1] Birch, B. J., S. Chowla, Marshall Hall, Jr., and A. Schinzel, On the difference x^3-y^2. Norske Vid. Selsk. Forh. (Trondheim), **38** (1965), 65-69.

[2] Davenport, H., On $f^3(t)-g^2(t)$. Norske Vid. Selsk. Forh. (Trondheim), **38** (1965), 86-87.

*) The same remark applies also in expressing the coefficients of polynomials of our example for $k=7$ given in (vi) above.

Cf. also: The Collected Works of Harold Davenport, edited by B. J. Birch, H. Halberstam, C. A. Rogers, Vol. IV. Academic Press, London et al., 1977. Especially, pp. 1743-1744, 1757.

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