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THE JACOBSON RADICAL OF MONOID-GRADED ALGEBRAS

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It is always a pleasant surprise to find that certain well-known results seemingly of a different nature can be obtained as a consequence of a general approach which absorbs and unifies all the existing methods. This "right" viewpoint is often the main difficulty in any subject. It certainly applies to the topic under consideration here, which is the study of the Jacobson radical of monoid-graded algebras. These algebras include such classical objects as group-graded algebras, crossed products, twisted monoid rings, skew monoid rings, polynomial rings, skew polynomial rings, etc. The "correct" approach which we shall adopt is to consider the graded radical of a module and its important special case, namely, the graded Jacobson radical of a graded algebra. A detailed account of all relevant background for group-graded algebras can be found in Năstăsescu and Van Oystaeyen (1982a). As examples of successful applications of graded radicals we mention the works of Năstăsescu (1984), Năstăsescu and Van Oystaeyen (1982b) and Jespers and Puczylowski (1990).

The purdose of this paper is to prove a number of general results concerning the Jacobson radical of monoid-graded and group-graded algebras. One of the main theorems provides a large class of groups G for which any G-graded algebra has the property that its Jacobson radical is a graded ideal. We also demonstrate that most of what is known concerning the Jacobson radical of polynomial rings and skew polynomial rings is an easy consequence of our results.

1. Notation and terminology.

Throughout, A denotes an algebra over a commutative ring R, $J(A)$ the Jacobson radical of A and $J(V)$ the radical of an A-module V. Given R-submodules X and Y of A, we write XY for the R-submodule of A consisting of all finite sums

$$\sum x_i y_i \qquad \text{with} \quad x_i \in X,\ y_i \in Y$$

Let M be a multiplicative monoid, i.e. M is a multiplicative semigroup with

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identity element 1. We say that A is an *M-graded algebra* if there is a family

$$\{A_x \mid x \in M\}$$

of R-submodules of A indexed by the elements of M such that the following conditions hold:

$$A = \bigoplus_{x \in M} A_x \qquad \text{(direct sum of } R\text{-modules)} \tag{1}$$

$$A_x A_y \subseteq A_{xy} \qquad \text{for all} \quad x, y \in M \tag{2}$$

We shall refer to (1) as an M-grading of A and to A_x as the *x-component* of A. An element $a \in A$ is said to be *homogeneous of degree* x, if $a \in A_x$ for some $x \in M$.

Any algebra A may be considered M-graded, for any monoid M, by putting

$$A_1 = A, \quad A_x = 0 \qquad \text{for} \quad 1 \neq x \in M$$

Such an algebra A is said to be *trivially M-graded.*

Again assume that M is a monoid and A an M-graded algebra. Owing to (1), each element a in A can be written uniquely in the form

$$a = \sum_{x \in M} a_x \qquad (a_x \in A_x)$$

with finitely many $a_x \neq 0$. The *support* of a, written *Supp* a, is defined by

$$\textit{Supp } a = \{x \in M \mid a_x \neq 0\}$$

Thus *Supp* a is a finite set which is empty if and only if $a = 0$. The number of elements in *Supp* a is called the *length* of a. In case M is a group, the *supporting subgroup* of a is defined to be the subgroup generated by *Supp* a (by convention, the subgroup generated by an empty set is the identity subgroup).

When (2) is replaced by the stronger condition

$$A_x A_y = A_{xy} \qquad \text{for all} \quad x, y \in M$$

we say that A is a *strongly M-graded algebra*. Of course, if $R = \boldsymbol{Z}$, then we say that A is an *M-graded ring* (respectively, *strongly M-graded ring*) instead of A being an M-graded algebra (respectively, strongly M-graded algebra).

Let A be an M-graded algebra. An A-module V is said to be *M-graded* (or simply *graded*) if there exists a family

$$\{V_x \mid x \in M\}$$

of R-submodules of V indexed by M such that the following two conditions hold:

$$V = \bigoplus_{x \in M} V_x \qquad \text{(direct sum of } R\text{-modules)}$$

$$A_x V_y \subseteq V_{xy} \qquad \text{for all} \quad x,\ y \in M$$

The above definition certainly implies that the regular module ${}_AA$ is graded (with $V_x = A_x$ for all $x \in M$). A submodule W of a graded module V is said to be a *graded submodule if*

$$W = \oplus_{x \in M}(W \cap V_x)$$

A graded left A-submodule of A is called a *gradea left ideal* of A. The notions of a graded-simple and a graded-semisimple module are defined in an obvious manner. A graded submodule W of a graded A-module V is said to be *graded-maximal* if $W \neq V$ and W is not strictly contained in any proper graded submodule of V.

Let V be a graded A-module. Then the *graded radical* $J^g(V)$ on V is defined to be the intersection of all graded-maximal submodules of V. By convention, $J^g(V) = V$ if V has no graded-maximal submodules. The reader may easily verifythat if $V \neq 0$ is finitely generated, then $J^g(V) \neq V$.

The *graded Jacobson radical* $J^g(A)$ of A is defined by

$$J^g(A) = J^g({}_AA)$$

where ${}_AA$ is the regular left A-module. It is immediate that $J^g(A)$ is a graded ideal of A which contains all graded nil left ideals of A.

2. Monoid crossed products.

The most important example of a monoid-graded algebra is a monoid crossed product. It includes such well known constructions as ordinary crossed products, skew monoid rings, polynomial rings, skew polynomial rings, etc. The notion of a monoid crossed product is due to Bovdi (1963) and seems not to be well known to many people. In this short section we shall include all the relevant background required for our purposes.

Let A be an algebra over a commutative ring R, let M be a monoid and let $Aut_R(A)$ be the group of all R-algebra automorphisms of A. Denote by $U(A)$ the unit group of A. Given maps

$$\sigma : M \longrightarrow Aut_R(A)$$

and

$$\alpha : M \times M \longrightarrow U(A)$$

we say that (M, A, σ, α) is a *crossed system* for M over A if, for all $x, y, z \in M$ and $a \in A$, the following properties hold:

$${}^x({}^y a) = \alpha(x, y)\, {}^{xy}a\, \alpha(x, y)^{-1}$$

$$\alpha(x, y)\alpha(xy, z) = {}^{x}\alpha(y, z)\alpha(x, yz)$$

$$\alpha(x, 1) = \alpha(1, x) = 1$$

where by definition

$${}^{x}a = \sigma(x)(a) \qquad \text{for all} \quad a \in A,\ x \in M$$

Let (M, A, σ, α) be a crossed system for M over A and let $A*M$ be the free left A-module freely generated by the elements $\bar{x}$, $x \in M$, with multiplication defined distributively by using

$$(a_1\bar{x})(a_2\bar{y}) = a_1{}^{x}a_2\alpha(x, y)\overline{xy}$$

for all $a_1, a_2 \in A$ and $x, y \in M$. Then $A*M$ becomes an R-algebra and is called a crossed product of M over A. It is clear that $A*M$ is a strongly M-graded R-algebra with identity element $1\cdot\bar{1}$, $(A*M)_1 = A\cdot\bar{1}$ and with

$$(A*M)_x = A\bar{x} = \bar{x}A$$

It is clear that A may be embedded in $A*M$ via $a \mapsto a\cdot\bar{1}$ and we identify A with its image $A\cdot\bar{1}$ in $A*M$, From now on, we write $\bar{1}$ instead of $1\cdot\bar{1}$. With this convention, $\bar{1}$ is the identity element of both A and $A*M$.

If $\alpha(x, y) = 1$ for all $x, y \in M$, and σ is a homomorphism then $A*M$ is called a *skew monoid ring* of M over A and is denoted by $A^{\sigma}M$. On the other hand, if $\sigma(x) = 1$ for all $x \in M$ and $\alpha(x, y) \in U(Z(A))$ for all $x, y \in M$, then $A*M$ is called the *twisted monoid ring* of M over A and is denoted by $A^{\alpha}M$. It is clear that $A^{\alpha}M$ is a strongly M-graded $Z(A)$-algebra.

Let M be a free commutative monoid freely generated by the indeterminates $\{x_i \mid i \in I\}$ and, for each $i \in I$, let σ_i be an R-automorphism of A. Then there is a unique homomorphism $\sigma: M \to Aut_R(A)$ for which $\sigma(x_i) = \sigma_i$, $i \in I$. The corresponding skew monoid ring $A^{\sigma}M$ is called the *skew polynomial ring* in the commuting indeterminates x_i with coefficients in A. Of course, if each $\sigma_i = 1$, then $A^{\sigma}M$ is the ordinary polynomial ring in the commuting indeterminates x_i with coefficients in A.

3. Some auxiliary results.

In this section, we shall record some useful observations required for subsequent investigations. Throughout, M denotes a multiplicative monoid. All rings are associative with $1 \neq 0$ and subrings of a ring R are assumed to have the same identity element as R.

LEMMA 3.1. *Let A be an M-graded algebra where M is a right (or left)* *cancellative monoid. Then A_1 is a subalgebra of A.*

PROOF. It obviously suffices to verify that A_1 contains the identity element of A. Assume that M is a right cancellative monoid. There is an expansion $1=\sum_{x\in M} a_x$ with $a_x\in A_x$ for all $x\in M$ and all but a finite number of a_x are zero. Fix some $z\in M$ and $a'_z\in A_z$. Then the product $a_x a'_z$ lies in A_{xz} for all $x\in M$. Since M is a right cancellative monoid, we also have $\sum_{x\in M} A_{xz}=\bigoplus_{x\in M} A_{xz}$. Thus

$$a'_z=1\cdot a'_z=\sum_{x\in M} a_x a'_z\in(\bigoplus_{x\in M} A_{xz})\cap A_z$$

Hence all the $a_x a'_z$ for $x\neq 1$ must be zero and $a_1 a'_z$ must be a'_z. It follows that a_1 acts as a left identity on A_z for all $z\in M$. This forces a_1 to be a left identity for the algebra A. Consequently, $a_1\in A_1$ is the identity element of A, as required. A similar argument proves the case where M is a left cancellative monoid. ■

Given monoids M and M', by a *monoid homomorphism*, we understand any map $f: M\to M'$ such that $f(1)=1$ and $f(xy)=f(x)f(y)$ for all $x, y\in M$. Any multiplicatively closed subset of M which contains the identity element of M is called a *submonoid* of M.

LEMMA 3.2. *Let M be a monoid and let A be an M-graded algebra.*

(i) *If $f: M\to M'$ is a surjective monoid homomorphism, then A can be viewed as an M'-graded algebra via*

$$A_x=\sum_{f(y)=x} A_y \qquad \text{for all} \quad x\in M'$$

(ii) *If S is a submonoid of M and A_1 is a subalgebra of A, then the subalgebra $A^{(S)}$ of A defined by $A^{(S)}=\bigoplus_{x\in S} A_x$ is an S-graded algebra.*

PROOF. (i) It is clear that $A=\bigoplus_{x\in M'} A_x$. Since, for any $x, z\in M'$,

$$\begin{aligned}A_xA_z&=\Big(\sum_{f(y)=x} A_y\Big)\Big(\sum_{f(t)=z} A_t\Big)\\&\subseteq\sum_{f(y)=x}\sum_{f(t)=z} A_yA_t\subseteq\sum_{f(y)=x}\sum_{f(t)=z} A_{yt}\\&\subseteq\sum_{f(u)=xz} A_u=A_{xz},\end{aligned}$$

the required assertion follows.

(ii) It clearly suffices to verify that $A^{(S)}$ contains the identity element of A_1. Since $A^{(S)}\supseteq A_1$, the result foollows. ■

LEMMA 3.3. *Let H be a submonoid of a monoid M such that for any $h\in H$ and $g\in M-H$, $hg\notin H$ (or for any $h\in H$, $g\in M-H$, $gh\notin H$). If A is an M-graded algebra such that A_1 is a subalgebra of A, then*

(i) $U(A)\cap A^{(H)}=U(A^{(H)})$.

(ii) $A^{(H)}\cap J(A)\subseteq J(A^{(H)})$.

PROOF. Our assumptions guarantee that $A^{(H)}$ is a subalgebra of A such that

$$A=A^{(H)}\oplus(\oplus_{g\notin H}A_g)$$

(direct sum of left or right $A^{(H)}$-modules). This clearly implies (i). Since (ii) is a consequence of (i), the result follows. ■

COROLLARY 3.4. *Let $\{H_i|i\in I\}$ be the family of all finitely generated subgroups of a group G and let A be a G-graded algebra. Then*

$$J(A)\subseteq\bigcup_{i\in I}J(A^{(H_i)})$$

In particular,

(i) *If each $J(A^{(H_i)})=0$, then $J(A)=0$.*

(ii) *If each $J(A^{(H_i)})$ is nil, then $J(A)$ is nil.*

PROOF. If $a\in A$ and H is the supporting subgroup of a, then $a\in A^{(H)}$ and so

$$A=\bigcup_{i\in I}A^{(H_i)}$$

Hence any given $a\in J(A)$ lies in some $A^{(H_i)}$ and so, by Lemma 3.3

$$a\in J(A)\cap A^{(H_i)}\subseteq J(A^{(H_i)}),$$

as desired. ■

LEMMA 3.5. *Let A be a G-graded algebra, where G is an arbitrary group, let H be a subgroup of G and let $\{H_i|i\in I\}$ be a family of subgroups of G containing H and such that each finite subset of G is contained in some H_i. Then*

$$A^{(H)}\cap J(A)=\bigcap_{i\in I}(A^{(H)}\cap J(A^{(H_i)}))$$

PROOF. It is clear that every finitely generated subgroup of G is contained in some H_i. Hence $A=\bigcup_{i\in I}A^{(H_i)}$ which obviously implies that if $a\in J(A^{(H_i)})\cap A^{(H)}$ for all $i\in I$, then $a\in J(A)\cap A^{(H)}$. Conversely, if $a\in J(A)\cap A^{(H)}$, then

$$a\in J(A)\cap A^{(H_i)}\subseteq J(A^{(H_i)}) \qquad \text{for all } i\in I$$

by virtue of Lemma 3.3. Hence $a\in J(A^{(H_i)})\cap A^{(H)}$ for all $i\in I$, as we wished to show. ■

LEMMA 3.6. *Let G be a group and let A be a G-graded algebra. Assume that for any finitely generated subgroup H of G, $J(A^{(H)})$ is a graded ideal of $A^{(H)}$. Then $J(A)$ is a graded ideal of A.*

PROOF. Given $a\in J(A)$, we may write $a=\sum_{h\in H}a_h$ where each $a_h\in A_h$ and H is the supporting subgroup of a. Let $\{H_i|i\in I\}$ be the family of all finitely generated subgroups of G containing H. Then, by Lemma 3.5,

$$a\in J(A^{(H_i)}) \quad \text{for all} \quad i\in I$$

Hence, by hypothesis, each $a_h\in J(A^{(H_i)})$ for all $i\in I$. But then, by Lemma 3.5 each $a_h\in J(A)$, as required. ■

LEMMA 3.7. *Let H be a submonoid of a monoid M and let A be an H-graded algebra. Then A can be regarded as an M-graded algebra via $\tilde{A}_g=A_g$ if $g\in H$ and $\tilde{A}_g=0$ if $g\notin H$. Furthermore, any left ideal I of A is H-graded if and only if I is M-graded.*

PROOF. It is clear that $A=\bigoplus_{g\in M}\tilde{A}_g$. Furthermore, given x, $y\in M$, if $x\notin H$ or $y\notin H$, then

$$\tilde{A}_x\tilde{A}_y=0\subseteq\tilde{A}_{xy}$$

On the other hand, if x, $y\in H$, then

$$\tilde{A}_x\tilde{A}_y=A_xA_y\subseteq A_{xy}=\tilde{A}_{xy}$$

proving that A is an M-grded algebra. The remaining assertion being a consequence of the definition of $\tilde{A}_g$, the result follows. ■

LEMMA 3.8. *Let M be a monoid and let A be an M-graded algebra. Then*

(i) *$J^g(A)$ is the largest proper graded ideal I of A such that $1+ab$ is a unit of A for all $a\in I\cap A_1$, $b\in A_1$.*

(ii) *$J^g(A)$ is the largest proper graded ideal I of A such that $I\cap A_1\subseteq J(A_1)$.*

(iii) *If $J(A)$ is graded, then $J(A)\subseteq J^g(A)$.*

PROOF. (i) If $a\in J^g(A)\cap A_1$ and $b\in A_1$, then $u=1+ab\in A_1$ and $u+J^g(A)=1+J^g(A)$. The latter easily implies that u is a unit of A. Conversely, let I be any proper graded ideal of A such that $1+ab$ is a unit of A for all $a\in I\cap A_1$ $b\in A_1$. If $I\nsubseteq J^g(A)$, then $A=I+L$ for some graded-maximal left ideal L of A. Hence $1=a+b$ for some $a\in I\cap A_1$, $b\in L\cap A_1$. Therefore $b=1+(-a)$ is a unit of A, a contradiction.

(ii) If $a\in J^g(A)\cap A_1$, then by (i), $1+ab\in U(A)$ for all $b\in A_1$. Hence $1+ab$

$\in U(A_1)$ for all $b\in A_1$. Therefore $a\in J(A_1)$ and so $J^g(A)\cap A_1\subseteq J(A_1)$. Conversely, let I be a proper graded ideal of A with $I\cap A_1\subseteq J(A_1)$. Then $1+ab$ is a unit of A for all $a\in I\cap A_1$, $b\in A_1$. Hence, by (i), $I\subseteq J^g(A)$ as required.

(iii) By Lemma 3.3 (ii), $A_1\cap J(A)\subseteq J(A_1)$. Hence, by (ii), $J(A)\subseteq J^g(A)$ and the result follows. ■

LEMMA 3.9. *Let M be a free monoid (or a free commutative monoid) freely generated by a set S, let H be the submonoid generated by a subset T of S and let $a\in R*H$.*

(i) *$U(R*M)\cap R*H=U(R*H)$.*

(ii) *$R*H\cap J(R*M)\subseteq J(R*H)$.*

(iii) *If T is finite and $\{S_i\mid i\in I\}$ is the set of all finite subsets of S containing T, then $a\in J(R*M)$ if and only if $a\in J(R*M_i)$ for all $i\in I$, where M_i is the submonoid of M generated by S_i.*

PROOF. (i) and (ii). Our choice of H and M guarantees that for any $h\in H$ and $g\in M-H$, $hg\notin H$. Hence the required assertions follow by virtue of Lemma 3.3.

(iii) It is clear that $R*M=\bigcup_{i\in I}R*M_i$. Hence, if $a\in J(R*M_i)$ for all $i\in I$, then $a\in J(R*M)$. Conversely, if $a\in J(R*M)$ then

$$a\in J(R*M)\cap R*M_i\subseteq J(R*M_i) \qquad \text{for all} \quad i\in I$$

by virtue of (ii). ■

LEMMA 3.10. *Let M be a free monoid (or a free commutative monoid) freely generated by a set X with $|X|\geqq 2$. Let X_1, X_2 be nonempty subsets of X with $X=X_1\cup X_2$ (disjoint union) and let M_i be generated by X_i, $i=1, 2$. Denote by RM the monoid ring of M over an arbitrary ring R and put $S=RM_1$. Then there exists a surjective homomorphism $f: RM\to SM_2$ such that the restriction of f to S is an isomorphism from S onto S.*

PROOF. The map $\varphi(x)=x$ for all $x\in X$ determines a map $\lambda: M\to SM_2$ such that $\lambda(1)=1$ and $\lambda(g_1g_2)=\lambda(g_1)\lambda(g_2)$ for all $g_1, g_2\in M$. It follows that the map $f: RM\to SM_2$ given by $f(\sum_{g\in M}r_g g)=\sum_{g\in M}r_g\lambda(g)$, $r_g\in R$, is a surjective ring homomorphism such that

$$f(\sum_{g\in M_1}r_g g)=\sum_{g\in M_1}r_g g$$

as required. ■

4. Assumed results.

Our aim here is to quote a number of results of a miscellaneous nature which will be required for the rest of the paper.

Let G be a group and let A be a strongly G-graded algebra. An ideal X of A_1 is said to be *G-invariant* if

$$A_g X A_{g^{-1}} = X \qquad \text{for all} \quad g \in G$$

The following important result provides a complete description of graded ideals of strongly graded algebras.

THEOREM 4.1. (Dade (1970)). *Let G be a group and let A be a strongly G-graded algebra.*

(i) *If I is a graded ideal of A and $X = I \cap A_1$, then X is a G-invariant ideal of A_1 such that*

$$I = AX = XA \quad \text{and} \quad I \cap A_g = A_g X = X A_g \qquad \text{for all} \quad g \in G$$

(ii) *For any G-invariant ideal X of A_1, $I = XA = AX$ is a graded ideal of A such that*

$$I \cap A_g = A_g X = X A_g \qquad \text{for all} \quad g \in G$$

(iii) *$J(A_1)$ is a G-invariant ideal of A_1 and hence $A \cdot J(A_1) = J(A_1) \cdot A$ is a graded ideal of A.*

PROOF. See Dade (1970). ■

Next we quote a number of useful properties of a graded radical of a module.

THEOREM 4.2. (Năstăsescu 1984)). *Let G be a finite group of order n, let A be a G-graded algebra and let V be a graded A-module. Then*

(i) $J^g(V) = \bigoplus_{g \in G}(J(V) \cap V_g)$.

(ii) *$nJ(V) \subseteq J^g(V)$ and, in particular, if n is a unit of A, then $J^g(V) = J(V)$.*

(iii) *If $v = \sum_{g \in G} v_g \in J(V)$, $v_g \in V_g$, then $nv_g \in J(V)$ for all $g \in G$.*

(iv) $J(A)^n V \subseteq J^g(V)$.

PROOF. See Năstăsescu (1984). ■

LEMMA 4.3. *Let G be a finite group and let A be a strongly G-graded algebra. Then $J(A_1) = J(A) \cap A_1 = J^g(A) \cap A_1$.*

PROOF. The second equality follows from Theorem 4.2 (i). By Lemma 3.3, $J(A)\cap A_1 \subseteq J(A_1)$. To prove the opposite containment, it suffices to show that any simple A-module V is semisimple as an A_1-module. For the proof of this fact we refer to Karpilovsky (1987, p. 188). ■

COROLLARY 4.4. *Let G be a finite group of order n and let A be a strongly G-graded algebra. If n is a unit of A, then*

$$J(A)=A\cdot J(A_1)$$

PROOF. Apply Theorem 4.2 (ii), Theorem 4.1 (i) and Lemma 4.3. ■

Let C be a class of groups. A group G is said to be a *residually C-group* if G is a subdirect product of groups belonging to C. Thus G is a residually C-group if and only if given $1\neq g\in G$, there exists a normal subgroup N_g of G such that $g\notin N_g$ and $G/N_g\in C$. A group G is called a *locally C-group* if each finitely generated subgroup of G is a member of C.

THEOREM 4.5. (Iwasawa (1943)). *If p is any prime and G any free group, then G is a residually p-group.*

PROOF. See Iwasawa (1943). ■

THOREEM 4.6. (Gruenberg (1957)). *Any finitely generated torsion-free nilpotent group G is a residually finite p-group for every prime p.*

PROOF. See Gruenberg (1957). ■

Given a group G, we write $G^{(n)}$ for the n-th derived subgroup of G. A group G is said to be *free solvable* if G is of the form $F/F^{(n)}$ for some free group F and some $n\geqq 1$.

THEOREM 4.7. *Any free solvable group is a residually finite p-group for every prime p.*

PROOF. See Robinson (1972). ■

A monoid M is said to be a u.p.-*monoid* (unique product monoid) if, given any two nonempty finite subsets A and B of M, there exists at least one element $x\in M$ that has a unique representation in the form $x=ab$ with $a\in A$ and $b\in B$.

A monoid M is said to be a t.u.p.-*monoid* (two unique product monoid) if, given any two nonempty finite subsets A and B of M with $|A|+|B|>2$, there exist at least two distinct elements x and y of M that have unique representations in the form $x=ab$, $y=cd$ with $a, c\in A$ and $b, d\in B$.

A monoid M is said to be a *right-ordered monoid* if the elements of M can be linearly ordered with respect to the relation $<$ and if, for all $x, y, z\in M$, $x<y$ implies $xz<yz$.

THEOREM 4.8. (i) *Any right-ordered monoid is a t.u.p.-monoid.*

(ii) *A group G is a t.u.p-group if and only if G is a u.p.-group.*

(iii) *Every submonoid of a u.p.-group is a t.u.p.-monoid.*

PROOF. (i) The proof is straightforward and therefore will be omitted.

(ii) This was proved by Strojnowski (1980).

(iii) This is a direct consequence of (ii). ■

THEOREM 4.9. (Jespers, Krempa and Puczylowski (1982)). *Let M be a t.u.p.-monoid, let A be an M-graded algebra and let $I\subseteq J(A)$ be a nonzero ideal of A. If $a=a_1+\cdots+a_n$, $a_1\in A_1$, $a_i\in Ag_i$, $g_i\neq 1$, is an element of I of minimal positive length, then*

(i) *There exists $m\geqq 1$ such that $x_1\cdots x_m=0$ for all $x_i\in\{a_2, \cdots, a_n\}$.*

(ii) *$a_1\in J(A_1)$ and $aa_1=a_1a$.*

PROOF. See Jespers, Krempa and Puczylowski (1982). ■

THEOREM 4.10. *Let G be an arbitrary group and let A be a strongly G-graded algebra. Suppose that A_1 is semiprime and that the additive group of A has no n-torsion where n is the order of any finite subgroup of G. Then A is semiprime.*

PROOF. This is a special case of a result of Passman (1984). A detailed proof can also be found in Karpilovsky (1987, p. 309). ■

5. Graded radicals.

In this section, we shall provide a number of results concerning the Jacobson radical of graded algebras. We shall also exhibit numerous interrelationships among $J(A)$, $J^g(A)$ and $J(A_1)$, where A is a G-graded algebra, thereby obtaining deeper understanding of each of them.

Let π be an arbitrary set of prime numbers. A natural number n is called a *π-number* if each prime divisor of n belongs to π. A finite group G is said to be a π-group if $|G|$ is a π-number.

Let G be a group, let A be a G-graded algebra and let V be a graded A-module. Assume that $f: G \to H$ is a surjective homomorphism of groups. Then both A and V can be regarded as H-graded via

$$\hat{A}_x = \bigoplus_{f(g)=x} A_g \qquad (g \in G,\ x \in H)$$

$$\tilde{V}_y = \bigoplus_{f(g)=y} V_g \qquad (g \in G,\ y \in H)$$

Of course, a submodule of V can be H-graded without being G-graded.

THEOREM 5.1. *Let $\{G_i \mid i \in I\}$ be a collection of arbitrary groups, let G be a subdirect product of the G_i and let A be a G-graded algebra. Denote by V any graded A-module.*

(i) *If for each $i \in I$, $J(V)$ is a G_i-graded submodule of V (via the projection $f_i: G \to G_i$), then $J(V)$ is a graded submodule of V.*

(ii) *If each G_i is finite, then for any $v = \sum_{g \in G} v_g \in J(V)$, $v_g \in V_g$, there exists a positive integer n_v such that $n_v v_g \in J(V)$ for all $g \in G$. Furthermore, n_v divides $|H_1||H_2| \cdots |H_k|$ for some $k = k(v)$ and some $H_t \in \{G_i\}$, $1 \leqq t \leqq k$.*

(iii) *If there exist two disjoint sets π_1 and π_2 of prime numbers such that G is a residually π-group, $i = 1, 2$, then $J(V)$ is a graded submodule of V.*

PROOF. (i) and (ii). Given $v = \sum_{g \in G} v_g \in J(V)$, $v_g \in V_g$, write $Supp\, v = \{g \in G \mid v_g \neq 0\}$ and $l(v) = |Supp\, v|$. We argue by induction on $l(v)$. If $l(v) \leqq 1$, then $v = v_g$ for some $g \in G$ and there is nothing to prove. Assume that $l(v) = n \geqq 2$ and that the result is true for all $b \in J(V)$ with $l(b) < n$.

Since $l(v) \geqq 2$, we may choose two distinct elements $x, y \in Supp\, v$. By hypothesis, $f_\lambda(x) \neq f_\lambda(y)$ for some $\lambda \in I$. Define $m = 1$ if the hypothesis of (i) holds and $m = |G_\lambda|$ if the hypothesis of (ii) holds. Then, by Theorem 4.2 (iii), we may write $v = b + c$ with $b, c \in V$ such that $mb, mc \in J(V)$, $l(b) < n$ and $l(c) < n$. Since $v = b + c$, we have

$$v_g = b_g + c_g \qquad \text{for all} \quad g \in G \tag{1}$$

If the hypothesis of (i) holds, then by induction hypothesis, $b_g, c_g \in J(V)$ and hence each $v_g \in J(V)$, proving that $J(V)$ is a graded submodule. If the hypothesis of (ii) holds, then by induction hypothesis, applied to mb and mc, there exist positive integers l and s such that

$$lmb_g \in J(V) \ \text{ and } \ smc_g \in J(V) \qquad \text{for all} \quad g \in G \tag{2}$$

where $n_v = lsm$ divides $|H_1||H_2| \cdots |H_r|$ for some $r \geqq 1$ and some $H_j \in \{G_i\}$, $1 \leqq j \leqq r$. It follows from (2) and (3) that

$$n_v v_g = n_v b_g + n_v c_g \in J(V) \qquad \text{for all} \quad g \in G$$

proving (i) and (ii).

(iii) Let $v = \sum_{g\in G} v_g \in J(V)$ where all $v_g \in V_g$. Owing to (ii), there exists π_i-number n_i such that $n_i v_g \in J(V)$ for all $g \in G$, $i=1, 2$. Since the sets π_1 and π_2 are disjoint, $(n_1, n_2)=1$. Thus each $v_g \in J(V)$ and the result follows. ■

COROLLARY 5.2. *Let G be a group such that there exist two disjoint sets π_1 and π_2 of prime numbers for which G is a residually π_i-group, $i=1, 2$. Then, for any G-graded algebra A, $J(A)$ is a graded ideal of A.*

PROOF. This is a special case of Theorem 5.1 (iii) in which $V = {}_A A$. ■

COROLLARY 5.3. (Jespers and Puczylowski (1990)). *Assume that G is a residually finite p-group for two distinct primes p. Then, for any G-graded algebra A, $J(A))$ is a graded ideal of A.*

PROOF. This is a special case of Corollary 5.2. ■

The next corollary for the case where G is infiite cyclic is due to Năstăsescu and Van Oystayen (1982b).

COROLLARY 5.4. *Assume that a group G is of one of the following types:*

(a) *G is a free group.*

(b) *G is a finitely generated torsion-free nilpotent group.*

(c) *G is a free solvable group.*

Then for any G-graded algebra A and for any A-module V, $J(V)$ is a graded submodule of V.

PROOF. By Theorem 5.1 (iii), it suffices to show that G is a residually finite p-group for two distinct primes p. Since the latter is a consequence of Theorems 4.5, 4.6 and 4.7, the result follows. ■

As a main application of Theorem 5.1 we now record the following general result.

THEOREM 5.5. *Let A be a G-graded algebra, where G is a group of one of the following types:*

(a) *G is abelian and the orders of finite subgroups of G are units in A.*

(b) *G is locally finite and the orders of finite subgroups of G are units in A.*

(c) *G is locally free, or residually-free, or free solvable, or torsion-free nilpotent.*

(d) *G is a subdirect product of the groups G_i, $i \in I$, where each G_i is of one of the types* (a), (b) *or* (c).
Then the following properties hold:

(i) *$J(A)$ is a graded ideal.*

(ii) *$J(A) \subseteqq J^g(A)$.*

(iii) *$J(A)=(J(A) \cap A_1)A=A(J(A) \cap A_1)$, provided A is strongly G-graded.*

PROOF. (i) If G is of type (c), then (i) holds by virtue of Theorem 5.1 (i), Lemma 3.6 and Corollary 5.4. The case (d) follows from (a), (b) and (c) by applying Theorem 5.1 (i). To treat the cases (a) and (b), we may assume that G is finitely generated (Lemma 3.6). The case (b) now follows by applying Theorem 4.2 (ii). Finally, the case (a) follows from Theorems 5.1 (i) and 4.2 (ii) and case (c).

(ii) This follows from (i) and Lemma 3.8 (iii).

(iii) Apply (i) and Theorem 4.1 (i). ∎

COROLLARY 5.6. *Let N be a normal subgroup of a group G and let A be a strongly G-graded algebra. Assume that the factor group G/N is of one of the types* (a), (b), (c) *or* (d) *in Theorem* 5.5. *Then*

$$J(A)=(J(A) \cap A^{(N)})A \subseteqq J(A^{(N)})A$$

PROOF. Owing to Lemma 3.2 (i), we may view A as a strongly G/N-graded algebra with $A^{(N)}$ as the identity component. The desired conclusion is therefore a consequence of Theorem 5.5 and Lemma 3.3 (ii). ∎

Our next application of Theorem 5.5 deals with the Jacobson radical of monoid-graded algebras.

COROLLARY 5.7. *Let M be a submonoid of a group G, where G is of one of the types* (a), (b), (c) *or* (d) *in Theorem* 5.5 (*e.g. M is a free monoid or M is a free commutative monoid*). *If A is any M-graded algebra, then*

(i) *$J(A)$ is a graded ideal of A.*

(ii) *$J(A) \subseteqq J^g(A)$.*

PROOF. (i) Owing to Lemma 3.7, we may harmlessly assume that $M=G$. Now apply Theorem 5.5.

(ii) This follows from (i) and Lemma 3.8 (iii). ∎

THEOREM 5.8. *Let G be a group, let A be a strongly G-graded algebra and let G have a finite chain of subgroups:*

$$1=G_0 \subseteqq G_1 \subseteqq G_2 \subseteqq \cdots \subseteqq G_n=G \tag{3}$$

such that $G_{i-1} \lhd G_i$ *and each* G_i/G_{i-1} *is of one of the types* (a), (b), (c) *or* (d) *in Theorem* 5.5. *Then*

$$J(A) \subseteqq J(A_1)A$$

PROOF. We argue by induction on $i=0, 1, \cdots, n$ that

$$J(A^{(G_i)}) \subseteqq J(A_1)A^{(G_i)}$$

Since $G_n=G$, this will obviously complete the proof. Because $G_0=1$, the case $i=0$ is clear. Suppose now that $i \leqq n$ and that

$$J(A^{(G_{i-1})}) \subseteqq J(A_1)A^{(G_{i-1})} \tag{4}$$

Since G_i/G_{i-1} is of one of the types (a), (b), (c) or (d) in Theorem 5.5, it follows from Corollary 5.6 (applied to $N=G_{i-1}$ and $G=G_i$) that

$$J(A^{(G_i)}) \subseteqq J(A^{(G_{i-1})})A^{(G_i)} \tag{5}$$

Hence, by (4) and (5),

$$\begin{aligned} J(A^{(G_i)}) &\subseteqq J(A_1)A^{(G_{i-1})}A^{(G_i)} \\ &= J(A_1)A^{(G_i)} \end{aligned}$$

and the result follows. ■

COROLLARY 5.9. *Let* N *be a normal subgroup of a group* G *such that* G/N *is solvable and let* A *be a strongly* G*-graded algebra over a field* F *of characteristic* 0. *Then* $J(A) \subseteqq J(A^{(N)})A$ *and, in particular,* $J(A^{(N)})=0$ *implies* $J(A)=0$.

PROOF. We may view A as a strongly G/N-graded algebra with $A^{(N)}$ as the identity component. Since G/N is solvable, it has a series (3) in which each G_i/G_{i-1} is abelian. Furthermore, since *char* $F=0$, the orders of finite subgroups of G_i/G_{i-1} are units in A. This shows that each G_i/G_{i-1} is of type (a) in Theorem 5.5. Hence, by Theorem 5.8, $J(A) \subseteqq J(A^{(N)})A$ and the result follows. ■

The special case of the above result in which $N=1$ (hence G is solvable) and $A=FG$ is the group algebra of G over F (hence $A^{(N)}=A_1=F$) is due to Villamayor (1959).

Our next result is also a generalization of a theorem established by Villamayor (1958) in the context of group algebras.

THEOREM 5.10. *Let* N *be a normal subgroup of a group* G *such that* G/N

is locally finite and let A be a G-graded algebra. Then

(i) $J(A^{(N)})=J(A)\cap A^{(N)}$.

(ii) $J(A)=J(A^{(N)})A=AJ(A^{(N)})$, *provided A is strongly G-graded and the orders of finite subgroups of G/N are units in A.*

(iii) *If $(G:N)=n<\infty$ and A is strongly G-graded, then*

$$J(A)^n \subseteq J(A^{(N)})A \subseteq J(A)$$

PROOF. We may view A as a G/N-graded algebra with $A^{(N)}$ as the identity component. Furthermore, if A is strongly G-graded, then A is also strongly G/N-graded. Hence we may assume that $N=1$, in which case $A^{(N)}=A_1$ and G is locally finite. Thus, if (i) holds, then (ii) holds by Theorem 5.5 (iii).

To prove (i) it suffices, by Lemma 3.3 (ii), to show that $J(A_1)\subseteq J(A)$. Since G is locally finite, there is a family $\{H_i \mid i\in I\}$ of finite subgroups of G with $A=\bigcup_{i\in I}A^{(H_i)}$. Hence $\bigcap_{i\in I}J(A^{(H_i)})\subseteq J(A)$. But, by Lemma 4.3,

$$J(A_1)\subseteq J(A^{(H_i)}) \qquad \text{for all } i\in I$$

Hence $J(A_1)\subseteq J(A)$, proving (i).

Finally, assume that G is of finite order n. Then, by Theorems 4.2 (iv) and 4.1 together with Lemma 4.3, we have

$$\begin{aligned} J(A)^n &\subseteq J^g(A)=(J^g(A)\cap A_1)A \\ &=J(A_1)A\subseteq J(A) \end{aligned}$$

where the last containment follows from (i). This proves (iii) and hence the result. ∎

The special case of the following result where A is a group algebra (or a twisted group algebra) over a field is due to Villamayor (1959) in characteristic 0 and to Passman (1970), Wallace (1970) and Zalesskii (1970) in characteristic $p>0$.

THEOREM 5.11. *Let A be a strongly G-graded algebra where G is a locally solvable group such that the orders of finite subgroups of G are units in A, If $J(A_1)=0$, then $J(A)=0$.*

PROOF. Owing to Corollary 3.4 (ii), we may harmlessly assume that G is solvable. Hence there is a finite chain of subgroups

$$1=G_0\subseteq G_1\subseteq G_2\subseteq \cdots \subseteq G_n=G$$

such that $G_{j-1}\lhd G_j$ and G_j/G_{j-1} is abelian, $j=1, \cdots, n$. We argue by induction on $i=0, 1, \cdots, n$ that $J(A^{(G_i)})=0$. Since $G_0=1$, the case $i=0$ follows from the assumption that $J(A_1)=0$. Suppose now that $i\leqq n$ and that $J(A^{(G_{i-1})})=0$. Given $a\in J(A^{(G_i)})$, let H be the subgroup of G_i generated by G_{i-1} and the support of a. Let $\{H_i|i\in I\}$ be the family of all groups L with $G_i\supseteqq L\supseteqq H$ and L/G_{i-1} finitely generated. Then, by Lemma 3.5, it suffices to show that $J(A^{(L)})=0$ for each such L.

Now, by hypothesis, L/G_{i-1} is a finitely generated abelian group. Therefore there exists an intermediate group K with $L\supseteqq K\supseteqq G_{i-1}$, L/K torsion-free abelian and K/G_i finite of order, say m. Then, by Theorem 5.10 (iii),

$$J(A^{(K)})^m\subseteqq J(A^{(G_{i-1})})A^{(K)}=0$$

and hence $J(A^{(K)})$ is nilpotent. Since, by Theorem 4.10, $A^{(K)}$ is semiprime, we have $J(A^{(K)})=0$. Furthermore, because L/K is torsion-free abelian, Corollary 5.6 yields

$$J(A^{(L)})\subseteqq J(A^{(K)})A=0$$

Therefore $J(A^{(L)})=0$, and the induction step is proved. Since $G=G_n$, the result follows. ■

6. The Jacobson radical of monoid crossed products.

Throughout this section, R denotes an arbitrary ring and M a multiplicative monoid. We write $R*M$ for the crossed product of M over R corresponding to a crossed system (M, R, σ, α).

Recall that each element of $R*M$ can be uniquely written in the form

$$a=\sum_{x\in M} a_x\bar{x} \qquad (a_x\in R)$$

with finitely many $a_x\neq 0$. By definition, $Supp\ a=\{x\in M|\ a_x\neq 0\}$ and the length $l(a)$ of a is defined by $l(a)=|Supp\ a|$. The multiplication in $R*M$ is determined by

$$\bar{x}r={}^x r\bar{x} \qquad (x\in M, r\in R) \tag{1}$$

$$\bar{x}\bar{y}=\alpha(x, y)\overline{xy} \qquad (x, y\in M) \tag{2}$$

where ${}^x r=\sigma(x)(r)$ for all $r\in R$, $x\in M$.

LEMMA 6.1. *Given an ideal L of $R*M$, an arbitrary subset S of M and $s\in S$, put*

$$i_L(S, s)=\{a_s\in R|\ \sum_{x\in S} a_x\bar{x}\in L \quad \text{for some } a_g\in R,\ g\in S-\{s\}\}$$

Then $i_L(S, s)$ is an ideal of R.

PROOF. It is clear that $i_L(S, s)$ is an additive subgroup of R. Assume that $a_s \in i_L(S, s)$ and choose $a_g \in R$, $g \in S-\{s\}$ such that

$$\sum_{x\in S} a_x \bar{x} \in L$$

Then, for any given $r \in R$, we have

$$r(\sum_{x\in S} a_x \bar{x}) = \sum_{x\in S} (ra_x)\bar{x} \in L$$

which shows that $ra_s \in i_L(S, s)$. On the other hand, by (1), we have

$$(\sum_{x\in S} a_x \bar{x})r = \sum_{x\in S} a_x {}^x r \bar{x} \in L$$

and so $a_s {}^s r \in i_L(S, s)$. But $r \mapsto {}^s r$ is an automorphism of R, hence $i_L(S, s)$ is also a right ideal of R, as required. ∎

The following terminology is extracted from C. Jordan (1975) and Bedi and Ram (1980). Let λ be an automorphism of R. An element $r \in R$ is said to be *λ-nilpotent* if, for any positive integer n, there exists a positive integer $m=m(n)$ such that

$$r\lambda^n(r)\lambda^{2n}(r) \cdots \lambda^{(m-1)n}(r)=0 \tag{3}$$

An ideal I of R is called a *λ-nil ideal* if every element of I is λ-nilpotent. An automorphism λ of R is said to be of *locally finite order* if for every $r \in R$ there exists an integer $n=n(r) \geqq 1$ such that $\lambda^n(r)=r$. For example, every automorphism of finite order is also of locally finite order. It is clear that if λ is of locally finite order and $r \in R$ is λ-nilpotent, then r is nilpotent.

LEMMA 6.2. *Let $g \in M$, $r \in R$ and let $\lambda=\sigma(g)$, Then*

(i) $\bar{g}^n r = \lambda^n(r)\bar{g}^n$ *for all* $n \geqq 1$.

(ii) $(r\bar{g}^n)^m = r\lambda^n(r)\lambda^{2n}(r)\cdots\lambda^{(m-1)n}(r)\bar{g}^{nm}$ *for all* $n \geqq 1$, $m \geqq 1$.

(iii) *r is λ-nilpotent if and only if $r\bar{g}^n$ is nilpotent for all* $n \geqq 1$.

PROOF. (i) The case $n=1$ being a consequence of (1), we argue by induction on n. Assume that $\bar{g}^n r = \lambda^n(r)\bar{g}^n$. Then, by (1), we have

$$\bar{g}^{n+1} r = \bar{g}(\bar{g}^n r) = (\bar{g}\lambda^n(r))\bar{g}^n = \lambda^{n+1}(r)\bar{g}^{n+1}$$

as required.

(ii) Again, the case $m=1$ is obvious. Assume that (ii) is true for m. Then, by (i),

$$\begin{aligned}(r\bar{g}^n)^{m+1} &= (r\bar{g}^n)(r\lambda^n(r)\cdots\lambda^{(m-1)n}(r)\bar{g}^{nm}) \\ &= r\lambda^n(r)\lambda^{2n}(r)\cdots\lambda^{n+(m-1)n}(r)\bar{g}^{n(m+1)}\end{aligned}$$

as desired.

(iii) Given $k\geqq 1$, we have $\bar{g}^k=u\overline{g^k}$ for some $u\in U(R)$ (see (2)). Hence, if $a\bar{g}^k=0$ for some $a\in R$, then $a=0$. Now apply (3) and (ii). ■

LEMMA 6.3. *Let $M\neq 1$ be a t.u.p.-monoid. If I is a nonzero ideal of $R*M$, then there exists an element x in I of minimal positive length and with $1\notin Supp\, x$.*

PROOF. Let $y=r_1\bar{g}_1+\cdots+r_t\bar{g}_t$, $0\neq r_i\in R$, $g_i\in M$ be any element of I of minimal positive length. Choose any nonidentity g in M. Since M is a t.u.p.-monoid, there exists $m\geqq 0$ such that

$$g_i g^m\neq 1 \qquad \text{for all} \quad i\in\{1, \cdots, t\}$$

Now put $x=y\bar{g}^m$. Then $x\in I$, $Supp\, x\subseteq\{g_1g^m, \cdots, g_tg^m\}$ and $1\notin Supp\, x$. Hence it suffices to verify that $x\neq 0$. By (2), we may write $\bar{g}_i\bar{g}^m=u_i\overline{g_ig^m}$ for some $u_i\in U(R)$, in which case

$$x=r_1u_1\overline{g_1g^m}+\cdots+r_tu_t\overline{g_tg^m}$$

Since M is cancellative, we deduce that $x\neq 0$, as desired. ■

We have now accumulated all the information necessary to prove the following result.

THEOREM 6.4. *Let $M\neq 1$ be a t.u.p.-monoid such that $J(R*M)\neq 0$.*

(i) *There exists an element x in $J(R*M)$ of minimal positive length and with $1\notin Supp\, x$.*

(ii) *Let $x\in J(R*M)$ be of minimal positive length, let $S=Supp\, x$ and let $L=J(R*M)$. Then, for any $s\in S-\{1\}$, $i_L(S, s)$ is a nonzero $\sigma(s)$-nil ideal of R.*

(iii) *If at least one $\sigma(g)$, $1\neq g\in M$, is of locally finite order and M is a u.p.-group, then R has a nonzero nil ideal.*

PROOF. (i) This is a special case of Lemma 6.3.

(ii) Write $x=r_1\bar{g}_1+\cdots+r_t\bar{g}_t$ with $0\neq r_i\in R$, $g_i\in M$ and $s=g_1\neq 1$. Since $r_1\neq 0$ and $x\in L$, we see that $r_1\in i_L(S, s)\neq 0$.

Now fix $a_1\in i_L(S, s)$. Then there exist $a_1, \cdots, a_t$ in R such that

$$y=a_1\bar{g}_1+a_2\bar{g}_2+\cdots+a_t\bar{g}_t\in L$$

If $y=0$, then $a_1=0$ and a_1 is obviously $\sigma(s)$-nilpotent. If $y\neq 0$, then y is an element of L of minimal positive length (in particular, $a_1\neq 0$). Hence, for any positive integer n,

$$y\bar{g}_1^{n-1}=a_1\bar{g}_1^n+a_2(\bar{g}_2\bar{g}_1^{n-1})+\cdots+a_t(\bar{g}_t\bar{g}_1^{n-1})$$

is also an element of L of minimal positive length. Since $g_1^n \neq 1$, it follows from Theorem 4.9 (i) that $a_1 \bar{g}_1^n$ is nilpotent. Thus, by Lemma 6.2 (iii), a_1 is $\sigma(s)$-nilpotent and so $i_L(S, s)$ is a $\sigma(s)$-nil ideal of R.

(iii) Let x be as in (ii). Multiplying x on the right by $\bar{g}_1^{-1}\bar{g}$, we may assume that $g_1 = s = g$. Hence, by (ii), R has a nonzero $\sigma(g)$-nil ideal. But $\sigma(g)$ is of locally finite order, hence R has a nonzero nil ideal. ∎

COROLLARY 6.5. *Let $M \neq 1$ be a t.u.p.-monoid and let $R*M$ be a crossed product of M over R. If at least one $\sigma(g)$, $1 \neq g \in M$, is of locally finite order, then $R \cap J(R*M)$ is a nil ideal of R.*

PROOF. Assume that $0 \neq r \in R \cap J(R*M)$ and that $1 \neq g \in M$ is such that $\sigma(g)$ is of locally finite order. Put $S = \{g\}$ and $L = J(R*M)$. Then, by the definition of $i_L(S, g)$, we have $i_L(S, g) = \{a \in R \mid a\bar{g} \in J(R*M)\}$. By hypothesis, $x = r\bar{g}$ is an element in $J(R*M)$ of minimal positive length with *Supp* $x = S$ and with $1 \notin$ *Supp* x. Hence, by Theorem 6.4 (ii), $i_L(S, g)$ is a $\sigma(g)$-nil ideal of R. Since $\sigma(g)$ is of locally finite order, we deduce that $i_L(S, g)$ is a nil ideal of R. Since $r \in i_L(S, g)$, the result follows. ∎

Note that in general $R \cap J(R*M)$ need not be nil even in the simplest case where M is a free monoid on one generator. For a corresponding example we refer to Bedi and Ram (1980).

COROLLARY 6.6. *Let M be a free monoid (or a free commutative monoid) freely generated by an infinite set X. Then, for an arbitrary ring R, $J(RM)$ is a nil ideal.*

PROOF. Let $a \in J(RM)$. Then there exist $x_1, x_2, \cdots, x_n$ in X, $n \geqq 1$, such that $a \in RM'$ where $M' = \langle x_1, \cdots, x_n \rangle$. Since X is infinite, we may choose $x \in X$ with $x \neq x_i$, $1 \leqq i \leqq n$. Put $X_1 = X - \{x\}$ and $X_2 = \{x\}$ and let M_i be generated by X_i, $i = 1, 2$. Since $M' \subseteqq X_1$, we have $a \in RM_1$. Put $S = RM_1$ and let $f: RM \to SM_2$ be the surjective homomorphism described in Lemma 3.10. Since $a \in J(RM) \cap RM_1$, we have $f(a) \in J(SM_2) \cap S$. By Corollary 6.5, $J(SM_2) \cap S$ is nil and so $f(a)$ is nilpotent. Hence, by Lemma 3.10, a is also nilpotent and the result follows. ∎

Let $R*M$ be a crossed product of M ever R and let S be a subset of R. In what follows, we put

$$S*M = \{\sum_{x \in M} a_x \bar{x} \mid a_x \in S\}$$

We say that S is *M-invariant* if $\sigma(g)(S)=S$ for all $g\in M$. For example, if $R*M$ is a twisted monoid ring of M over R (i.e. if $\sigma(g)=1$ for all $g\in M$), then any subset S of R is M-invariant.

LEMMA 6.7. *Let $R*M$ be a crossed product of M over R and let I be an M-invariant ideal of R. Then*

(i) *$I*M$ is an ideal of $R*M$ such that $R*M/I*M$ is a crossed product of M over R/I. Furthermore, if $R*M$ is a twisted monoid ring (respectively, skew monoid ring) of M over R, then $R*M/I*M$ is a twisted monoid ring (respectively, skew monoid ring) of M over R/I.*

(ii) *$(I*M)^n=I^n*M$ for all $n\geqq 1$.*

(iii) *If $R*M$ is a twisted monoid ring of M over R and if I is a locally nilpotent ideal of R, then $I*M$ is a nil ideal of $R*M$ and, in particular, $I*M\subseteq J(R*M)$.*

PROOF. (i) It is clear that $I*M$ is an additive subgroup of $R*M$. Fix $a\in I$, $x\in M$ and $g\in M$. Then $(a\bar{x})\bar{g}=a\alpha(x, y)\overline{xg}\in I*M$. Also

$$\bar{g}(a\bar{x})={}^{g}a\alpha(g, x)\overline{gx}\in I*M$$

since ${}^{g}a\in I$ by the assumption that I is M-invariant. This demonstrates that $I*M$ is an ideal of $R*M$.

For each $g\in G$, put $\tilde{g}=\bar{g}+I*M$. Then $R*M/I*M$ is a free left R/I-module freely generated by $\tilde{g}$, $g\in G$. Define

$$\tilde{\alpha}: M\times M\longrightarrow U(R/I)$$

by $\tilde{\alpha}(x, y)=\alpha(x, y)+I$. Then, for all $x, y\in M$,

$$\begin{aligned}\tilde{x}\tilde{y}&=(\bar{x}+I*M)(\bar{y}+I*M)=\alpha(x, y)\overline{xy}+I*M\\&=(\alpha(x, y)+I)(\overline{xy}+I*M)\\&=\tilde{\alpha}(x, y)\widetilde{xy}\end{aligned}$$

Since I is M-invariant, the map

$$\tilde{\sigma}: M\longrightarrow Aut(R/I)$$

given by

$$\tilde{\sigma}(g)(r+I)=\sigma(g)(r)+I \qquad \text{for all} \quad r\in R,\ g\in M$$

is well-defined. Furthermore, given $x\in M$ and $r\in R$, we have

$$\begin{aligned}\tilde{x}(r+I)&=(\bar{x}+I*M)(r+I*M)\\&=\sigma(x)(r)\bar{x}+I*M=(\sigma(x)(r)+I)\tilde{x}\\&=\tilde{\sigma}(x)(r+I)\tilde{x}\end{aligned}$$

Using associativity of the multiplication, it is now immediate to verify that $(M, R/I, \tilde{\sigma}, \tilde{\alpha})$ is a crossed system. Hence $R*M/I*M$ is a crossed product of M over R/I. Furthermore, if $\sigma(g)=1$ for all $g\in M$ (respectively, if $\alpha(x, y)=1$ for all $x, y\in M$), then $\tilde{\sigma}(g)=1$ for all $g\in M$ (respectively, $\tilde{\alpha}(x, y)=1$ for all $x, y\in M$), proving the second assertion.

(ii) It suffices to show that for all $a_i\in I$, $x_i\in M$, $1\leqq i\leqq n$,

$$(a_1\bar{x}_1)(a_2\bar{x}_2)\cdots(a_n\bar{x}_n)\in I^n*M$$

Since the latter is a consequence of the assumption that I is M-invariant, the required assertion follows.

(iii) Let $x=r_1\bar{g}_1+\cdots+r_n\bar{g}_n$, $0\neq r_i\in I$, $g_i\in M$, be a nonzero element of $I*M$ and let $S=\{r_1, \cdots, r_n\}$. Since I is locally nilpotent, $S^m=0$ for some $m\geqq 1$. We claim that $x^m=0$. It suffices to show that

$$(\lambda_1\bar{x}_1)(\lambda_2\bar{x}_2)\cdots(\lambda_m\bar{x}_m)=0$$

for all $\lambda_i\in S$, $x_i\in\{g_1, \cdots, g_n\}$. But

$$(\lambda_1\bar{x}_1)(\lambda_2\bar{x}_2)\cdots(\lambda_m\bar{x}_m)=(\lambda_1\cdots\lambda_m)\bar{x}_1\bar{x}_2\cdots\bar{x}_m$$

and $\lambda_1\cdots\lambda_m=0$, since $S^m=0$, hence the result follows. ■

The following assertion is contained implicitly in the work of Ram (1984).

LEMMA 6.8. *Let λ be an automorphism of a ring R and let R satisfy the ascending chain condition on left annihilators. If R has a nonzero λ-nil ideal, then R has a nonzero nilpotent ideal.*

PROOF. Let I be a nonzero λ-nil ideal of R. We claim that R has a nonzero right nil ideal. Since R satisfies the ascending chain condition on left annihilators, a standard argument will show that R has a nonzero nilpotent ideal. If I is nil, then there is nothing to prove. Hence we may assume that $r\in I$ is not nilpotent. We assert that $r\lambda^n(r)\neq 0$ for some $n\geqq 1$. Assume by way of contradiction that $r\lambda^n(r)=0$ for all $n\geqq 1$. For any $m\geqq 1$, put

$$I_m=\lambda^m(r)R+\lambda^{m+1}(r)R+\cdots+\cdots$$

It is clear that

$$I_1\supseteqq I_2\supseteqq\cdots\supseteqq\cdots$$

Hence

$$l(I_1)\subseteqq l(I_2)\subseteqq\cdots\subseteqq\cdots$$

where $l(I_i)$ is the left annihilator of I_i in R. By hypothesis, we have $l(I_k)=l(I_{k+1})$ for some $k\geqq 1$. Since for any $n\geqq 1$,

$$\lambda^k(r)\lambda^{k+n}(r)=\lambda^k(r\lambda^n(r))=0$$

it follows that

$$\lambda^k(r)\in l(I_{k+1})=l(I_k)$$

Thus $\lambda^k(r)\lambda^k(r)=0$, so $r^2=0$ a contradiction.

By the foregoing, we may choose $n\geqq 1$ such that $r\lambda^n(r)\neq 0$. Since $r\in I$ is a λ-nil ideal of R, there exists a positive integer t such that

$$r\lambda^n(r)\lambda^{2n}(r)\cdots\lambda^{(t-1)n}(r)=0 \tag{4}$$

Since $r\lambda^n(r)\neq 0$, $t>2$. Now choose t minimal such that (4) holds and put

$$s=r\lambda^n(r)\cdots\lambda^{(t-2)n}(r) \tag{5}$$

If the right ideal sR is not nil, sa is not nilpotent for some $a\in R$. Put $r_1=r$ and $r_2=sa$. We claim that $l(r_1)\subset l(r_2)$. Indeed, by (4) and (5), $\lambda^{-n}(r)s=0$ and so $\lambda^{-n}(r)\in l(r_2)$. But $\lambda^{-n}(r)\notin l(r_1)$ since $r\lambda^n(r)\neq 0$. Hence $l(r_1)\subset l(r_2)$. If r_2R is not nil, then arguing as before we get $r_3\in R$ such that $l(r_1)\subset l(r_2)\subset l(r_3)$. Continuing in this fashion, we will obtain a desired nonzero right nil ideal. ■

To take advantage of the above lemma, we need the following observation.

LEMMA 6.9. *Let λ be an automorphism of a ring R, let I be a λ-nil ideal of R and let K be any ideal of R with $\lambda(K)=K$. If μ is the automorphism of R/K induced by λ, then $(I+K)/K$ is a μ-nil ideal of R/K.*

PROOF. By definition of μ, we have $\mu^k(r+K)=\lambda^k(r)+K$ for all $r\in R$ and all $k\geqq 1$. Now fixr $r\in I$ and a positive integer n. Sincer r is λ-nilpotent, there exists $n\geqq 1$ such that (3) holds. But then

$$(r+K)\mu^n(r+K)\mu^{2n}(r+K)\cdots\mu^{(m-1)n}(r+K)=0$$

as required. ■

COROLLARY 6.10. (C. Jordan (1975)). *Let R be a noetherian ring and let λ be an automorphism of R. Then every λ-nil ideal of R is nilpotent.*

PROOF. Let I be a λ-nil ideal of R. If R is semiprime, then by Lemma 6.8, $I=0$. Suppose that R is not semiprime. Let $rad(R)$ be the prime radical of R. Since R is noetherian, $rad(R)$ is nilpotent. Since $\lambda(rad(R))=rad(R)$, λ induces an automorphism μ of $R/rad(R)$. Hence, by Lemma 6.9, $(I+rad(R))/rad(R)$ is a μ-nil ideal of the noetherian semiprime ring $R/rad(R)$. By the foregoing, $I\subseteqq rad(R)$ and so I is nilpotent. ■

In wheat follows, $rad(R)$ and $N(R)$ denote the prime and upper nil radicals of R, respectively.

THEOREM 6.11. *Let $M\neq 1$ be a t.u.p.-monoid and let $R*M$ be a crossed product of M over a ring R.*

(i) *If $R/rad(R)$ satisfies the ascending chain condition on left annihilators, then $J(R*M)\subseteq rad(R)*M$.*

(ii) *If each automorphism $\tilde{\sigma}(g)$, $g\in M$, of $R/N(R)$ induced by $\sigma(g)$ is of locally finite order, then $J(R*M)\subseteq N(R)*M$.*

(iii) *If R is noetherian, then $J(R*M)$ is a nilpotent ideal such that $J(R*M)=rad(R)*M$*

PROOF. (i) It is clear that $rad(R)$ is an M-invariant ideal of R. Hence, by Lemma 6.7 (i), $rad(R)*M$ is an ideal of $R*M$ such that $R*M/rad(R)*M$ is a crossed product of M over $R/rad(R)$. Owing to Lemma 6.8, $R/rad(R)$ contains no nonzero λ-nil ideals where λ is any automorphism of $R/rad(R)$. Hence, by Theorem 6.4 (ii),

$$J(R*M/rad(R)*M)=0$$

as required.

(ii) As in (i), $R*M/N(R)*M$ is a crossed product of M over $R/N(R)$. The assumption on $\tilde{\sigma}(g)$, $g\in M$, guarantees that every $\tilde{\sigma}(g)$-nil ideal of $R/N(R)$ is a nil ideal. Since $R/N(R)$ contains no nonzero nil ideals, it follows that $R/N(R)$ contains no nonzero $\tilde{\sigma}(g)$-nil ideals for all $g\in M$. Hence, by Theorem 6.4 (ii),

$$J(R*M/N(R)*M)=0$$

as desired.

(iii) Since R is noetherian, $rad(R)$ is a nilpotent M-invariant ideal of $R*M$. Hence, by Lemma 6.7 (ii), $rad(R)*M$ is a nilpotent ideal of $R*M$. Since $R/rad(R)$ is noetherian, the result follows by virtue of (i). ■

COROLLARY 6.12. *Let $M\neq 1$ be a t.u.p.-monoid and let $R*M$ be a twisted monoid ring of M over R. Denote by $L(R)$ and $N(R)$ the Levitzki and upper nil radicals of R, respectively. Then*

$$L(R)*M\subseteq J(R*M)\subseteq N(R)*M$$

In particular, if every nil ideal of R is locally nilpotent, then $J(R*M)$ is a nil ideal of $R*M$ such that $J(R*M)=L(R)*M$.

PROOF. By Theorem 6.11 (ii), $J(R*M)\subseteq N(R)*M$. Since $L(R)$ is a locally nilpotent ideal of R, it follows from Lemma 6.7 (iii) that $L(R)*M$ is a nil ideal of $R*M$ (in particular, $L(R)*M\subseteq J(R*M)$). Finally, if every nil ideal of R is locally nilpotent, then $L(R)=N(R)$ and the required assertion follows. ■

A special case of Corollary 6.12 in which $R*M$ is a monoid ring of M over R and $N(R)=0$ is due to Schneider and Weissglass (1967). As a further application of Theorem 6.4, we also record the following result.

COROLLARY 6.13. *Let $M\neq 1$ be a t.u.p.-monoid and let $R*M$ be a crossed product of M over R. If R has no zero divisors, then $J(R*M)=0$.*

PROOF. Let λ be an automorphism of R. If $r\in R$ is λ-nilpotent, then $r=0$ since R has no zero divisors. Hence R contains no nonzero λ-nil ideals. The desired assertion is therefore a consequence of Theorem 6.4 (ii). ■

Our next theorem requires the following two preliminary results.

LEMMA 6.14. *Let I be a left or right ideal of a ring R. If $I^n\subseteq J(R)$ for some $n\geqq 1$, then $I\subseteq J(R)$.*

PROOF. Since $J(R/J(R))=0$, $R/J(R)$ contains no nonzero (left or right) nil ideals. But, by hypothesis, $(I+J(R))/J(R)$ is nil, hence $I+J(R)=J(R)$ as required. ■

LEMMA 6.15. *Let M be a monoid such that $gM=Mg$ for all $g\in M$ and let $R*M$ be a crossed product of M over R. Then*

(i) *$(R*M)\bar{g}=\bar{g}(R*M)$ for all $g\in M$.*

(ii) *For any $r\in R$, $g\in G$ and $n\geqq 1$,*

$$[r\bar{g}(R*M)]^n\subseteq r\overline{g^n}(R*M)$$

PROOF. (i) We will demonstrate that $(R*M)\bar{g}\subseteq\bar{g}(R*M)$. A similar argument will establish the opposite containment. Since $R*M=\bigoplus_{x\in M}R\bar{x}$. it suffices to show that for any given $x\in M$, $R\bar{x}\bar{g}\subseteq\bar{g}(R*M)$. Now $\bar{g}r={}^{g}r\bar{g}$ for all $r\in R$, hence $\bar{g}R=R\bar{g}$. Also $xg=gy$ for some $y\in M$, by the hypothesis on M. Accordingly,

$$\begin{aligned} R\bar{x}\bar{g} &= R\alpha(x, y)\overline{xy}=R\overline{gy}=R\alpha(g, y)^{-1}\bar{g}\bar{y} \\ &= R\bar{g}\bar{y}=\bar{g}R\bar{y}\subseteq\bar{g}(R*M) \end{aligned}$$

as required.

(ii) The case $n=1$ being trivial, we argue by induction on n. So assume that

$$[r\bar{g}(R*M)]^n \subseteqq r\overline{g^n}(R*M)$$

Then

$$\begin{aligned}[r\bar{g}(R*M)]^{n+1} &\subseteqq r\overline{g^n}(R*M)r\bar{g}(R*M)\\ &\subseteqq r\overline{g^n}\bar{g}(R*M) \qquad \text{(by (i))}\\ &\subseteqq r\overline{g^{n+1}}(R*M)\end{aligned}$$

as desired. ■

Turning our attention to a special class of t.u.p.-monoids, we now prove the following theorem.

THEOREM 6.16. *Let $M \neq 1$ be a free monoid or a free commutative monoid, and let $R*M$ be a crossed product of M over an arbitrary ring R. For each $1 \neq g \in M$, put*

$$I_g = \{a \in R \mid a\bar{g} \in J(R*M)\}$$

Then

(i) $J(R*M) = (R \cap J(R*M)) + \sum_{1 \neq g \in M} I_g \bar{g}$.

(ii) *$R \cap J(R*M)$ is a $\sigma(t)$-nil ideal of R for all $1 \neq t \in M$, while each I_g is a $\sigma(g)$-nil ideal of R.*

(iii) *If M is a free commutative monoid, then for each $1 \neq g \in M$ and each $n \geqq 1$, $I_g = I_{g^n}$.*

PROOF. (i) Owing to Corollary 5.7, $J(R*M)$ is a graded ideal of $R*M$. Hence, if $a \in J(R*M)$, then there exist $r \in J(R*M) \cap R$, $r_i \in R$, $1 \leqq i \leqq n$, and some nonidentity $g_1, \cdots, g_n$ in M such that

$$a = r + r_1\bar{g}_1 + \cdots + r_n\bar{g}_n \quad \text{and} \quad r_i\bar{g}_i \in J(R*M) \qquad (1 \leqq i \leqq n)$$

This demonstrates that

$$J(R*M) \subseteqq (R \cap J(R*M)) + \sum_{1 \neq g \in G} I_g \bar{g}$$

The opposite inclusion being obvious, the required assertion follows.

(ii) Fix $1 \neq g \in M$ and put $S = \{g\}$, $L = J(R*M)$. Then $i_L(S, g) = I_g$ by the definition of the ideal $i_L(S, g)$ of R (see Lemma 6.1). We may, of course, assume that $I_g \neq 0$. Choose any $0 \neq a \in I_g$ and observe that $x = a\bar{g}$ is an element of $J(R*M)$ of minimal positive length and with $1 \notin Supp\, x$. Hence, by Theorem 6.4 (ii), I_g is a $\sigma(g)$-nil ideal of of R.

Finally, let $0 \neq r \in R \cap J(R*M)$ and let $1 \neq t \in M$. Then $r\bar{t} \in J(R*M)$ and so $r \in I_t$. Hence, by the above, r is $\sigma(t)$-nilpotent and therefore $R \cap J(R*M)$ is $\sigma(t)$-nil as required.

(iii) Assume that M is a free commutative monoid, and let $r \in I_{g^n}$ for some

$1\neq g\in G$ and some $n\geqq 1$. Owing to Lemma 6.15 (ii), we have

$$[r\bar{g}(R*M)]^n\subseteq r\overline{g^n}(R*M)\subseteq J(R*M)$$

Hence, by Lemma 6.14, $r\bar{g}\in J(R*M)$. Thus $r\in I_g$ and so $I_{g^n}\subseteq I_g$. The opposite containment being obvious, the result follows. ■

Our final aim is to improve the main result of Zalesskii (1965) concerning the Jacobson radical of $R*G$ where G is a right-ordered group. As a preliminary to the next lemma, let us make the following useful observations in which G denotes a right-ordered group.

(a) If $g_1<g_2<g_3<\cdots<g_n$ is a chain of elements in G with $n\geqq 3$, then

$$g_1g_2^{-1}<1<g_3g_2^{-1}<\cdots<g_ng_2^{-1}$$

Thus if x is an element of $J(R*G)$ of minimal positive length $n\geqq 3$, then there is another such element y (replace x by $x\bar{g}^{-1}$ for suitable $g\in G$) for which

$$1\in Supp\, y=\{g_1, g_2, \cdots, g_n\}$$

and (6)

$$g_1<1=g_2<g_3<\cdots<g_n$$

(b) If S is any finite nonempty subset of G and $g_1, \cdots, g_n$, $n\geqq 3$, are as in (6), then there exist two elements $s, t\in S$ such that g_1s and g_nt are uniquely representable elements of $\{g_1, \cdots, g_n\}\cdot S$.

The following lemma is due to Zalesskii (1965).

LEMMA 6.17. *Let $R*G$ be a crossed product of a right-ordered group G over a ring R and let $x\in J(R*G)$ be of length $n\geqq 3$ and of the form*

$$x=r_1\bar{g}_1+r_2\bar{g}_2+r_3\bar{g}_3+\cdots+r_n\bar{g}_n \qquad (r_i\in R) \tag{7}$$

where $g_1, \cdots, g_n$ satisfy (6). Assume that for all $\lambda, \mu\in R$ either $\lambda x\mu=0$ or $\lambda x\mu$ is of length n. Then r_2 is a nilpotent element.

PROOF. Let $y=\sum y_h\bar{h}$, $y_h\in R$, $h\in G$ be such that $(1-x)y=1$ and let $S=$ *Supp* y. By property (b), we may choose $s, t\in$ *Supp* y such that g_1s and g_nt are uniquely representable elements of (*Supp* x) (*Supp* y). Observe also that $g_1, g_n\in Supp(1-x)\subseteq Supp\, x$. Hence $r_1\bar{g}_1y_s\bar{s}=0$ or $r_n\bar{g}_ny_t\bar{t}=0$, which implies $r_1\bar{g}_1y_s=0$ or $r_n\bar{g}_ny_t=0$. If $r_1\bar{g}_1y_s=0$, then $1\cdot x\cdot y_s$ is an element of length$<n$; hence by hypothesis $xy_s=0$ which in turn implies that $r_2y_s=0$. Similarly, if $r_n\bar{g}_ny_t\bar{t}=0$ then $xy_t=0$ and $r_2y_t=0$. Let $S'=\{h\in Supp\, y\,|\, xy_h=0\}$ and let $S''=S-S'$. Then $S''\subset S$ and

$$1=(1-x)(\sum_{h\in S''} y_h\bar{h}+\sum_{h\in S'} y_h\bar{h})$$
$$=(1-x)(\sum_{h\in S''} y_h\bar{h})+\sum_{h\in S'} y_h\bar{h}$$

Multiplying both sides on the left by r_2, we obtain

$$r_2(1-x)(\sum_{h\in S''} y_h\bar{h})=r_2 \tag{8}$$

If $r_2\neq 0$, then by hypothesis r_2x is of length n, i.e. $Supp\, r_2x=\{g_1,\cdots,g_n\}$. Applying the above argument to $r_2(1-x)$ instead of $1-x$, we see that there exists $h\in S''$ such that $r_2xy_h=0$, $r_2^2y_h=0$. Multiplying both sides of (8) on the left by r_2, we again obtain a relation of type (8) with respect to a proper subset of S''. Hence, after finitely many steps, we obtain $r_2^m=0$ for some $m\geqq 1$, as desired. ■

COROLLARY 6.18. (Zalesskii (1965)). *Let $R*G$ be a crossed product of a right-ordered group G over a ring R which has no nonzero nil ideals. Then*

(i) *Every element of $J(R*G)$ is of length $\leqq 2$.*

(ii) *If for any $g\in G$, $J(R*\langle g\rangle)=0$ then $J(R*G)=0$.*

PROOF. (i) Assume by way of contradiction that $x\in J(R*G)$ is of length $n\geqq 3$. By observation (a), we may assume that x satisfies (6) and (7). Now fix λ_i, $\mu_i\in R$, $1\leqq i\leqq m$. Then $\sum_{i=1}^m \lambda_i x\mu_i=0$ or is of length $n\geqq 3$. In the latter case

$$Supp\Big(\sum_{i=1}^m \lambda_i x\mu_i\Big)=Supp\ x$$

and $\sum_{i=1}^m \lambda_i x\mu_i$ satisfies the hypothesis of Lemma 6.17. Hence $\sum_{i=1}^m \lambda_i r_2\mu_i$, the coefficient of 1 in $\sum_{i=1}^m \lambda_i x\mu_i$, must be nilpotent. This shows that Rr_2R is a nonzero nil ideal of R, a contradiction.

(ii) Given $x\in J(R*G)$, it follows from (i) that x is of length $\leqq 2$, say $x=\lambda_1\bar{g}_1+\lambda_2\bar{g}_2$ for some λ_1, $\lambda_2\in R$, g_1, $g_2\in G$. Multiplying on the right by $\bar{g}_2^{-1}$, we we may assume that $x\in R*\langle g\rangle$ for some $g\in G$. Hence, by Lemma 3.3 (ii),

$$x\in R*\langle g\rangle\cap J(R*G)\subseteq J(R*\langle g\rangle)=0$$

as desired. ■

The following result was established by Zalesskii (1965) under either of the following hypotheses:

(i) For any $r\in R$, $2r=0$ implies $r=0$.

(ii) R is commutative.

We close by demonstrating that these assumptions are redundant.

THEOREM 6.19. *Let R be an arbitrary ring such that $J(R)$ contains no zero divisors and let $G\neq 1$ be a right-ordered group. Then, for any crossed product $R*G$ of G over R, $J(R*G)=0$*

PROOF. Let $x\in J(R*G)$ be either zero or an element of minimal positive length n. By assumption, R contains no nonzero nil ideals. Hence, by Corollary 6.18 (i), $n\leqq 2$. To prove that $x=0$, we may assume that $x=r_1\cdot\bar{1}+r_2\bar{g}$ for some $r_1, r_2\in R$, $1\neq g\in G$. Then

$$x\in R*\langle g\rangle\cap J(R*G)\subseteqq J(R*\langle g\rangle)$$

by Lemma 3.3 (ii). Hence, by Corollary 5.6 and Lemma 3.3 (ii),

$$r_i\in R\cap J(R*\langle g\rangle)\subseteqq J(R) \qquad (i=1, 2)$$

Let λ be the automorphism of R corresponding to g. Since $r_i\bar{g}\in J(R*\langle g\rangle)$ and $g\neq 1$, it follows from Theorem 4.9 that $r_i\bar{g}$ is nilpotent. Hence, by Lemma 6.2 (ii),

$$r_i\lambda(r_i)\lambda^2(r_i)\cdots\lambda^{m-1}(r_i)=0$$

for some $m\geqq 1$. Since $\lambda^k(r_i)\in J(R)$, $0\leqq k\leqq m-1$, and $J(R)$ has no zero divisors, it follows that $r_i=0$ as required. ■

7. Applications.

In this section we shall demonstrate that most of what is known concerning the Jacobson radical of polynomial rings and skew polynomial rings is an easy consequence of our results. Throughout, R denotes an arbitrary ring.

For any cardinal α, let X_α denote a set of cardinality α and let $[X_\alpha]$ be the free commutative monoid freely generated by X_α. Then the monoid ring $R[X_\alpha]$ of $[X_\alpha]$ over R is the polynomial ring over R in α commuting indeterminates $x\in X_\alpha$. If X_α is a finite set, say $X_\alpha=\{x_1, \cdots, x_\alpha\}$, then we write $R[x_1, \cdots, x_\alpha]$ instead of $R[X_\alpha]$.

In what follows, $L(R)$ and $N(R)$ denote the Levitzki and upper nil radicals of R, respectively. Given any cardinal α, we put

$$J_\alpha(R)=R\cap J(R[X_\alpha])$$

It will also be convenient to define $J_\infty(R)$ by

$$J_\infty(R)=\bigcap_{n=1}^{\infty} J_n(R)$$

THEOREM 7.1. (Amitsur (1956)). *Let R be an arbitrary ring and let α be any cardinal. Then*

(i) $J(R[X_\alpha])=J_\alpha(R)[X_\alpha]$.

(ii) $L(R)\subseteqq J_\alpha(R)\subseteqq N(R)$.

(iii) $J(R)\supseteqq J_1(R)\supseteqq J_2(R)\supseteqq \cdots \supseteqq J_\infty(R)$.

(iv) *If α is an infinite cardinal, then $J(R[X_\alpha])$ is nil and*

$$J_\alpha(R)=J_\infty(R)$$

(v) $J_\alpha(R/J_\alpha(R))=0$.

PROOF. (i) By Theorem 6.15, it suffices to show that, for any given $r\in R$, $1\neq g\in[X_\alpha]$, if $rg\in J(R[X_\alpha])$, then $r\in J(R[X_\alpha])$. Write $g=x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}$, $x_i\in X_\alpha$, $n_i\geqq 1$ and let M be the monoid generated by $X_\alpha-\{x_k\}$. Then $rg\in J((RM)[x_k])$ and so, by induction, we may assume that $k=1$. Furthermore, by Theorem 6.15, we may assume that $g=x$ for some $x\in X_\alpha$. Since the map $x\mapsto x+1$ determines an automorphism of $R[X_\alpha]$, it follows that $r(x+1)\in J(R[x_\alpha])$ and so $r\in J(R[X_\alpha])$, as desired.

(ii) This is a direct consequence of (i) and Corollary 6.11.

(iii) It is an easy consequence of Lemma 3.9 (ii) and the definition of $J_n(R)$ that $J_n(R)\supseteqq J_{n+1}(R)$ for all $n\geqq 1$, as required.

(iv) Assume that α is an infinite cardinal. Then, by Corollary 6.6, $J(R[X_\alpha])$ is nil. Let $\{S_i|i\in I\}$ be the set of all finite subsets of X_α and let M_i be the submonoid of $[X_\alpha]$ generated by S_i, $i\in I$. Given $a\in R$, it follows from Lemma 3.9 (iii) that $a\in J_\alpha(R)$ if and only if $a\in J(RM_i)$ for all $i\in I$. Since α is an infinite cardinal, it follows that $a\in J_\alpha(R)$ if and only if $a\in J_n(R)$ for all $n\geqq 1$.

(v) Owing to (i), we have

$$R[X_\alpha]/J(R[X_\alpha])\cong(R/J_\alpha(R))[X_\alpha]$$

Hence $(R/J_\alpha(R))[X_\alpha]$ is semiprimitive and, by (i), $J_\alpha(R/J_\alpha(R))=0$. ■

Turning to skew polynomial rings, let λ be an automorphism of R. Recall that the corresponding skew polynomial ring $R^\lambda[x]$ is the skew monoid ring of the free commutative monoid $[x]$ generated by x with respect to λ. Thus each element of $R^\lambda[x]$ can be written uniquely in the form

$$\sum_{i\geqq 0} r_i x^i \qquad (r_i\in R)$$

with finitely many r_i distinct from zero. Addition is defined in the usual manner, while multiplication is determined by the rule

$$x^i r=\lambda^i(r)x^i \qquad \text{for all} \quad r\in R,\ i\geqq 0$$

It will also be convenient to consider the corresponding skew group ring $R^\lambda\langle x\rangle$, where $\langle x\rangle$ is an infinite cyclic group generated by x. In what follows, we put

$$I=\{r\in R \mid rx\in J(R^{\lambda}[x])\}, \qquad K=J(R^{\lambda}\langle x\rangle)\cap R \tag{1}$$

LEMMA 7.2. (i) *Both I and K are λ-invariant ideals of R.*

(ii) $I\cap J(R)=R\cap J(R_{\gamma}[x])$.

PROOF. The map $\lambda^*: R^{\lambda}[x]\to R^{\lambda}[x]$ induced by λ is obviously an automorphism of $R^{\lambda}[x]$. Hence, if $r\in I$, then $\lambda^*(rx)=\lambda(r)x\in J(R^{\lambda}[x])$ and so $\lambda(r)\in I$. This shows that $\lambda(I)\subseteq I$ and a similar argument shows that $\lambda^{-1}(I)\subseteq I$. The proof that K is λ-invariant is identical to the above proof.

(ii) By Lemma 3.3 (ii), $R\cap J(R^{\lambda}[x])\subseteq J(R)$ and hence $R\cap J(R^{\lambda}[x])\subseteq I\cap J(R)$. Conversely, let $r\in I\cap J(R)$. Then, for all $i\geqq 0$, $r_i\in R$, $r(r_ix^i)\in J(R^{\lambda}[x])$. Hence $1-rf$ is a unit of $R^{\lambda}[x]$ for all $f\in R^{\lambda}[x]$. Thus $r\in J(R^{\lambda}[x])$, as required. ■

THEOREM 7.3. (Bedi and Ram (1980)). *With the notation above, the following properties hold:*

(i) $J(R^{\lambda}[x])=I\cap J(R)+Ix+\cdots+Ix^n+\cdots$.

(ii) $J(R^{\lambda}\langle x\rangle)=K^{\lambda}\langle x\rangle\subseteq J(R)^{\lambda}\langle x\rangle$.

(iii) $K\subseteq I$ *and* $J(R^{\lambda}\langle x\rangle)\cap R^{\lambda}[x]\subseteq J(R^{\lambda}[x])$.

(iv) *If λ is of locally finite order, then I and K are nil ideals and* $J(R^{\lambda}[x])=I^{\lambda}[x]$.

(v) *If λ is of locally finite order and $J(R)$ is locally nilpotent, then*

(a) $J(R^{\lambda}[x])=J(R)^{\lambda}[x]$.

(b) $J(R^{\lambda}\langle x\rangle)=J(R)^{\lambda}\langle x\rangle$.

(c) $J(R^{\lambda}[x])$ *and* $J(R^{\lambda}\langle x\rangle)$ *are locally nilpotent.*

PROOF. (i) This is direct consequence of Lemma 7.2 (ii) and Theorem 6.16.

(ii) Apply Corollary 5.6 to the special case where $N=1$.

(iii) Given $r\in K$, we have $rx\in J(R^{\lambda}\langle x\rangle)$. Let $f=\sum_{i=0}^{n} a_ix^i$ be a typical element of $R^{\lambda}[x]$. Since $rxf\in J(R^{\lambda}\langle x\rangle)$, there exists $\gamma=\sum_{i\in Z} b_ix^i\in R^{\lambda}\langle x\rangle$ such that

$$rx\Big(\sum_{i=0}^{n} a_ix^i\Big)+\sum_{i\in Z} b_ix^i+rx\Big(\sum_{i=0}^{n} a_ix^i\Big)\Big(\sum_{i\in Z} b_ix^i\Big)=0$$

It follows that $\gamma\in R^{\lambda}[x]$ and so $K\subseteq I$. The last assertion is a consequence of (ii) and the fact that $K\subseteq I$.

(iv) Assume that λ is of locally finite order. Then, by Theorem 6.16 (ii), I is a nil ideal. Since $K\subseteq I$, K is also a nil ideal. Since I is nil, $I\subseteq J(R)$ and so, by (i), $J(R^{\lambda}[x])=I^{\lambda}[x]$.

(v) Assume that λ is of locally finite order and $J(R)$ is locally nilpotent.

Since λ is of locally finite order, it follows from (iv) that $J(R^{\lambda}[x])\subseteq J(R)^{\lambda}[x]$. Also, by (ii), $J(R^{\lambda}\langle x\rangle)\subseteq J(R)^{\lambda}\langle x\rangle$. It will be shown that $J(R)^{\lambda}[x]$ is locally nilpotent, which will prove (a) and the first part of (c). A similar argument will show that $J(R^{\lambda}\langle x\rangle)$ is locally nipotent, which will prove (b) and the second part of (c).

Let S be any finite subset of $J(R)^{\lambda}[x]$, let B be the set of all coefficients of elements of S and let $C=\bigcup_{i\geq 0}\lambda^{i}(B)$. Since B is a finite set and λ is of locally finite order, C is a finite subset of $J(R)$. But $J(R)$ is locally nilpotent, hence $C^{m}=0$ for some $m\geqq 1$. Therefore $S^{m}=0$ and the result follows. ∎

In what follows, $rad(R)$ denotes the prime radical of R.

THEOREM 7.4. (Ram (1984)). *Let R be a ring satisfying the ascending chain condition on left annihilators and let λ be an automorphism of R. Then the following conditions are equivalent:*

(i) $J(R^{\lambda}\langle x\rangle)\neq 0$.

(ii) $J(R^{\lambda}[x])\neq 0$.

(iii) *R has a nonzero λ-invariant λ-nil ideal.*

(iv) *R has a nonzero right nil ideal.*

(v) $rad(R)\neq 0$.

(vi) $rad(R^{\lambda}\langle x\rangle)\neq 0$.

(vii) $rad(R^{\lambda}[x])\neq 0$.

PROOF. (i)$\Rightarrow$(ii): Owing to Theorem 7.3 (ii), (iii), $I\neq 0$ and so $J(R^{\lambda}[x])\neq 0$, by the definition of I.

(ii)$\Rightarrow$(iii): By Lemma 7.1 (i), I is λ-invariant, while by Theorem 6.16 (ii), I is λ-nil. Since $J(R^{\lambda}[x])\neq 0$, Theorem 7.3 (i) also tells us that $I\neq 0$, as required.

(iii)$\Rightarrow$(iv): This was established in the proof of Lemma 6.8.

(v)$\Rightarrow$(vi): Let $A=\{l(Rx)\mid xRx=0,\ x\neq 0\}$. Since $rad(R)\neq 0$, the set A is nonempty. Let $l(Rr)$ be a maximal element of A. Then a straighforward argument shows that $rR\lambda^{n}(r)=0$ for all $n\in\boldsymbol{Z}$. Hence $rR^{\lambda}\langle x\rangle r=0$ and so $rad(R^{\lambda}\langle x\rangle)\neq 0$.

(vi)$\Rightarrow$(i): Apply the inclusion $rad(R^{\lambda}\langle x\rangle)\subseteq J(R^{\lambda}\langle x\rangle)$.

(vi)$\Rightarrow$(vii): Let J be a nonzero nilpotent ideal of $R^{\lambda}\langle x\rangle$. Then $J\cap R^{\lambda}[x]$ is a nonzero nilpotent ideal of $R^{\lambda}[x]$. Hence $rad(R^{\lambda}[x])\neq 0$.

(vii)$\Rightarrow$(ii): This follows from the inclusion $rad(R^{\lambda}[x])\subseteq J(R^{\lambda}[x])$. ∎

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