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SYMMETRIC SUBMANIFOLDS AND GENERALIZED GAUSS MAPS

Dedicated to Professor Shingo Murakami on his sixtieth birthday

By

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Introduction.

Let (M, g) be the n-dimensional unit sphere of $\boldsymbol{R}^{n+1}$ and S an r-dimensional connected submanifold of (M, g). Regarding S as a submanifold of $\boldsymbol{R}^{n+1}$, we can associate the Gauss map with it. It is a smooth mapping of S to the Grassmannian manifold G_r^{n+1} of the r-dimensional linear subspaces in $\boldsymbol{R}^{n+1}$, defined as follows; $S \ni q \to T_q S \in G_r^{n+1}$. The target space G_r^{n+1} is a riemannian symmetric space with a suitable metric. If the second fundamental form of S is parallel, the Gauss map is a totally geodesic immersion by a result in Vilms [10]. Here we note that if such a submanifold S is complete, it is characterized as a symmetric submanifold, namely a submanifold preserved by the reflections with respect to all the normal spaces, and moreover the latter submanifold is analougously defined for the case that the ambient space is a riemannian symmetric space. The purpose of this paper is to extend the above result for a symmetric submanifold of a simply connected riemannian symmetric space without Euclidean factor.

We will first consider certain submanifold classes of such a riemannian symmetric space which contain the symmetric submanifolds, and then define a generalization of Gauss map for each submanifold class. The target space of this generalization is generally a pseudo-riemannian symmetric space, and moreover if the ambient riemannian symmetric space is compact, it is a compact riemannian symmetric space. We will next show that for a symmetric submanifold our generalized Gauss map is a totally geodesic immersion, and it is moreover isometric if and only if the submanifold is totally geodesic. Last we will give the list of the target spaces of the generalized Gauss maps for our considerable submanifold classes of the simply connected irreducible riemannian symmetric spaces.

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§ 1. Submanifolds of riemannian symmetric spaces.

Let (M, g) be a riemannian symmetric space. Denote by R the curvature tensor of (M, g). A vector subspace V of a tangent space T_pM is said to be *strongly curvature-invariant* if it holds that

$$(1.1) \qquad (1)\ R_p(V, V)V \subset V \quad \text{and} \quad (2)\ R_p(V^{\perp}, V^{\perp})V^{\perp} \subset V^{\perp},$$

where $V^{\perp}$ denotes the orthogonal complement of V. Obviously the subspace $V^{\perp}$ is also strongly curvature-invariant. Let V, W be strongly curvature-invariant subspaces of T_pM, T_qM, respectively. Then they are said to be *equivalent to each other* if there exists an isometry ϕ of (M, g) such that $\phi(p)=q$, $\phi_*(V)=W$. Denote by $[V]$ the equivalence class of a strongly curvature-invariant subspace V and by $\mathcal{S}(M, g)$ the set of all the equivalence classes. For $\mathcal{V} \in \mathcal{S}(M, g)$ a connected submanifold S of M is called a *$\mathcal{V}$-submanifold* if it holds that $[T_pS]=\mathcal{V}$ for any point $p \in S$.

LEMMA 1.1. *For each $\mathcal{V} \in \mathcal{S}(M, g)$ there exists a complete connected totally geodesic $\mathcal{V}$-submanifold uniquely except the difference of congruence.*

PROOF. Take a strongly curvature-invariant subspace V of a tangent space T_pM which represents the equivalence class $\mathcal{V}$. By (1.1), (1) there exists a unique complete connected totally geodesic submanifold N such that $p \in N$ and $T_pN=V$ (cf. [3]).

We first show that N is a $\mathcal{V}$-submanifold. Let q be another point of N and join q to p by a geodesic $\gamma(t)$ in N. Then $\gamma(t)$ is also a geodesic in M and moreover T_qN is translated to T_pN by the parallel translation of M along $\gamma(t)$. Since the parallel translation along a geodesic in a riemannian symmetric space equals the differential of an isometry (cf. [3]), the subspace T_qN is equivalent to $T_pN=V$. Hence N is a $\mathcal{V}$-submanifold.

Next let S be a complete connected totally geodesic $\mathcal{V}$-submanifold and take a point $q \in S$. Since $[T_qS]=\mathcal{V}$, there exists an isometry ϕ of (M, g) such that $\phi(q)=p$ and $\phi_*(T_qS)=V$. Then, since N, S are both complete connected totally geodesic, it follows that $\phi(S)=N$. Hence S is congruent to N. Q.E.D.

Let S be a connected (regular) submanifold of M. Then S is called a *symmetric submanifold* if for any point $p \in S$ there exists an *extrinsic symmetry* t_p, i.e., a unique isometry of (M, g) which preserves S and satisfies that

$$(1.2) \qquad \begin{cases} t_p(p)=p, \\ (t_p)_*x = \begin{cases} -x & \text{for } x \in T_pS, \\ x & \text{for } x \in N_pS, \end{cases} \end{cases}$$

where N_pS denotes the normal space at p. The tangent spaces of such S are strongly curvature-invariant.

LEMMA 1.2. *A symmetric submanifold S of a riemannian symmetric space (M, g) is a complete $\mathcal{V}$-submanifold for some $\mathcal{V}\in\mathcal{S}(M, g)$. Next assume that (M, g) is simply connected. Then a connected submanifold S of (M, g) is a complete totally geodesic $\mathcal{V}$-submanifold for some $\mathcal{V}\in S(M, g)$ if and only if it is a totally geodesic symmetric submanifold.*

PROOF. Since a symmetric submanifold is a riemannian symmetric spce with respect to the induced metric, it is complete. Hence, to show the first claim, we may see that S is a $\mathcal{V}$-submanifold for some $\mathcal{V}\in\mathcal{S}(M, g)$, namely the tangent spaces of S are equivalent to each other. This follows by the following fact; The subgroup of isometries generated by the extrinsic symmetries t_p, $p\in S$, acts transitively on S.

The second claim easily follows by the first claim and the characterization (Corollary 1.4, [6]) of a symmetric submanifold. Q.E.D.

Now we give concrete examples of $\mathcal{V}$-submanifolds of simply connected compact riemannian symmetric spaces of rank one.

EXAMPLE 1. Let (M, g) be the n-dimensional sphere S^n of positive constant sectional curvature. Then any subspace $V\subset T_pM$ is strongly curvature-invariant, and moreover two subspaces $V\subset T_pM$, $W\subset T_qM$ are equivalent to each other if and only if they have the same dimension. Hence the set $\mathcal{S}(M, g)$ are exhaused by the equivalence classes $\mathcal{V}^r$, $0\leqq r\leqq n$, of r-dimensional subspaces. Then a connected submanifold is a $\mathcal{V}^r$-submanifold if and only if it is r-dimensional, and in this case N given in Lemma 1.1 is the r-dimensional totally geodesic sphere.

EXAMPLE 2. Let (M, g) be the n-dimensional complex projective space $\boldsymbol{C}P_n$ of positive constant holomorphic sectional curvature. Denote by J the complex structure on M. In this case a subspace $V\subset T_pM$ is strongly curvature-invariant if and only if it is one of the following cases (1), (2);

(1) V is an r-dimensional complex subspace, where $0\leqq r\leqq n$.

(2) V is an n-dimensional totally real subspace, i.e., $JV=V^{\perp}$.

Moreover strongly curvature-invariant subspaces $V\subset T_pM$, $W\subset T_qM$ are equivalent to each other if and only if they are either complex subspaces with the same dimension or totally real subspaces with the dimension n. Hence the set $\mathcal{S}(M, g)$ are exhausted by the equivalence classes $\mathcal{V}_C^r$, $0\leqq r\leqq n$, of r-dimensional complex subspaces and the equivalence class $\mathcal{V}_{\boldsymbol{R}}$ of n-dimensional totally real subspaces. Then a connected submanifold is a $\mathcal{V}_C^r$-submanifoid (resp. $\mathcal{V}_{\boldsymbol{R}}$-submanifold) if and only if it is an r-dimersional Kähler submanifold (resp. n-dimensional totally real submanifold), and in this case N given in Lemma 1.1 is the r-dimensional totally geodesic complex projective space $\boldsymbol{C}P_r$ (resp. the n-dimensional totally geodesic real projective space $\boldsymbol{R}P_n$).

EXAMPLE 3. Let (M, g) be the n-dimensional quaternion projective space $\boldsymbol{H}P_n$ with the metric of riemannian symmetric space. Denote by $Q\subset \mathrm{Hom}(TM, TM)$ the quaternionic structure on M. In this case a subspace $V\subset T_pM$ is strongly curvature-invariant if and only if it is one of the following cases (1), (2);

(1) V is an r-dimensional invariant subspace, where $0\leqq r\leqq n$, i.e., $\dim_{\boldsymbol{R}} V=4r$ and $FV=V$ for $F\in Q$.

(2) V is a $2n$-dimensional totally complex subspace, i.e., there exist endomorphisms I, J, $K\in Q$ satisfying $I^2=J^2=K^2=-1$, $IJ=-JI=K$, $JK=-KJ=I$, $KI=-IK=J$, and moreover $IV=V$, $JV=V^{\perp}$, $KV=V^{\perp}$. Here -1 denotes the minus identity map of T_pM.

Moreover strongly curvature-invariant subspaces $V\subset T_pM$, $W\subset T_qM$ are equivalent to each other if and only if they are either invariant subspaces with the same dimension or totally complex subspaces with the dimension $2n$. Hence the set $\mathcal{S}(M, g)$ are exhausted by the equivalence classes $\mathcal{V}_{\boldsymbol{H}}^r$, $0\leqq r\leqq n$, of r-dimensional invariant subspaces and the equivalence class $\mathcal{V}_C$ of $2n$-dimensional totally complex subspaces. Then a connected submanifold is a $\mathcal{V}_{\boldsymbol{H}}^r$-submanifold (resp. $\mathcal{V}_C$-submanifold) if and only if it is an r-dimensional invariant submanifold (resp. $2n$-dimensional totally complex submanifold), and in this case N given in Lemma 1.1 is the r-dimensional totally geodesic quaternion projective space $\boldsymbol{H}P_r$ (resp. the n-dimensional totally geodesic complex projective space $\boldsymbol{C}P_n$).

EXAMPLE 4. Let (M, g) be the Cayley projective plane $\boldsymbol{Ca}P_2$ with the metric of riemannian symmetric space. Then the set $\mathcal{S}(M, g)$ consists of two equivalence classes $\mathcal{V}_1$, $\mathcal{V}_2$. The equivalence class $\mathcal{V}_1$ is represented by a tangent space of the projective line S^8 of $\boldsymbol{Ca}P_2$ and the other equivalence class $\mathcal{V}_2$ is represented by a tangent space of the totally geodesic submanifold $\boldsymbol{H}P_2\subset\boldsymbol{Ca}P_2$ which is induced by the natural inclusion of the quaternion number field $\boldsymbol{H}$ to the Cayley number field $\boldsymbol{Ca}$.

Here we refer [5], [8], [9] to Examples 2, 3, 4, respectively.

§ 2. Generalized Gauss maps and symmetric submanifolds.

In this section we first assume that (M, g) is a riemannian symmetric space whose universal covering space does not contain any Euclidean factor, namely the identity component $I^0(M, g)$ of the isometry group of (M, g) is a semi-simple Lie group.

Fix an equivalence class $\mathcal{V} \in \mathcal{S}(M, g)$ and let S be a connected $\mathcal{V}$-submanifold of M. To define the "Gauss map" associated with S we first construct the target space.

Fix a point o of S and put $V = T_o S$. Let F_S be the set of the strongly curvature-invariant subspaces which are equivalent to V by isometries in $I^0(M, g)$. Hereafter denote by G the identity component $I^0(M, g)$. Then the Lie group G acts transitively on the set F_S by the following way; $\phi \cdot W = \phi_*(W)$ for $\phi \in G$, $W \in F_S$. We next define a relation $\sim$ on the set F_S. Let $W \subset T_p M$, $U \subset T_q M$ be subspaces in F_S. Then it holds that $W \sim U$ if and only if there exists a complete connected totally geodesic submanifold $N^\perp$ of M such that $p, q \in N^\perp$ and $T_p N^\perp = W^\perp$, $T_q N^\perp = U^\perp$. This relation is an equivalence relation since $N^\perp$ is uniquely determined by any one point in it and the tangent space. Denote by $\langle W \rangle$ the equivalence class of $W \in F_S$ with respect to this relation and by M^* the set of the equivalence classes. Since the action of G on F_S preserves the relation $\sim$, the Lie group G also acts transitively on the set M^* by the following way; $\phi \cdot \langle W \rangle = \langle \phi_* W \rangle$ for $\phi \in G$, $\langle W \rangle \in M^*$.

We first define a differentiable structure on M^*. Thus M^* is a smooth manifold. Let p_* be a point of M^* and denote by $K^*(p_*)$ the stabilizer in G of p_*. Moreover set $p_* = \langle W \rangle$ where $W \in F_S$ and $W \subset T_p M$, and denote by $N^\perp(p_*)$ a unique complete connected totally geodesic submanifold of M such that $p \in N^\perp(p_*)$ and $T_p N^\perp(p_*) = W^\perp$. Then this $N^\perp(p_*)$ is independent of selecting the representative W of p_* and it characterizes $K^*(p_*)$ as follows.

LEMMA 2.1. *It holds that*

$$K^*(p_*) = \{\phi \in G \,;\, \phi(N^\perp(p_*)) = N^\perp(p_*)\}\,. \tag{2.1}$$

Particularly, if (M, g) is simply connected, $K^*(p_*)$ is a closed subgroup of G.

PROOF. Let $\phi \in K^*(p_*)$. Since $\phi_* W \sim W$, it follows that $\phi(p) \in N^\perp(p_*)$ and $(\phi_* W)^\perp = T_{\phi(p)} N^\perp(p_*)$. Moreover since $(\phi_* W)^\perp = \phi_*(W^\perp)$, it follows that $\phi(N^\perp(p_*)) = N^\perp(p_*)$. Conversely assume that $\phi \in G$ satisfies $\phi(N^\perp(p_*)) = N^\perp(p_*)$. Then it follows that $\phi(p) \in N^\perp(p_*)$ and $\phi_*(W)^\perp = \phi_*(W^\perp) = T_{\phi(p)} N^\perp(p_*)$. Hence it holds that $\phi_* W \sim W$, which implies that $\phi \in K^*(p_*)$.

We next show that the subgroup $K^*(p_*)$ of G is closed. Define a linear isometry Φ of T_pM as follows;

$$\Phi(x)=\begin{cases} -x & \text{for} \quad x\in W, \\ x & \text{for} \quad x\in W^{\perp}. \end{cases}$$

Then it preserves the curvature tensor R_p at p by (1.1). Since (M, g) is a simply connected riemannian symmetric space, Φ is uniquely extended to an isometry ϕ of (M, g) such that $\phi(p)=p$ and $\phi_{*_p}=\Phi$ (cf. [3]). Here we can easily see that the totally geodesic submanifold $N^{\perp}(p_*)$ is a connected component of the fixed point set of ϕ, which containes p. Hence $N^{\perp}(p_*)$ is closed and so $K(p_*)$ is closed. Q.E.D.

By this lemma, if (M, g) is simply connected, the set M^* is, as set, bijective to the homogeneous space $G/K^*(p_*)$ for any point $p_*\in M^*$. Then, since G acts transitively on M^*, there exists a unique smooth structure on M^* such that M^* is diffeomorphic to $G/K^*(p_*)$ for any point $p_*\in M^*$. We regard M^* as a smooth manifold with this smooth structure.

We next define a pseudo-riemannian structure on M^*. Denote by $\mathfrak{g}$ the Lie algebra of the Killing vector fields on (M, g). Fix a point $p\in M$ and denote by s_p an involutive isometry defined by the geodesic symmetry at p. Then it induces an involutive automorphism σ of G by the following way; $\sigma(\phi)=s_p\circ\phi\circ s_p$ for $\phi\in G$. The differential of σ is also an involutive automorphism of $\mathfrak{g}$. Denote the differential by the same notation σ and let $\mathfrak{g}_{\pm 1}$ be its ± 1-eigenspaces. Then the vector space $\mathfrak{g}_{-1}$ is identified with the tangent space T_pM by the correspondence: $\mathfrak{g}_{-1}\ni X\rightarrow X_p\in T_pM$. Under this identification the adjoint representation $\mathrm{ad}_{\mathfrak{g}_{-1}}(\mathfrak{g}_1)$ of $\mathfrak{g}_1$ on $\mathfrak{g}_{-1}$ is identified with the Lie algebra spanned over $\boldsymbol{R}$ by the endomorphisms $R(x, y)$, $x, y\in T_pM$ (cf. [3]). Hence the metric g_p on T_pM induces an inner product $\langle\ ,\ \rangle_{\mathfrak{g}_{-1}}$ on $\mathfrak{g}_{-1}$ such that the endomorphisms $\mathrm{ad}_{\mathfrak{g}_{-1}}(X)$, $X\in\mathfrak{g}_1$, are skew symmetric. Since $\mathfrak{g}$ is semi-simple and $\mathrm{ad}_{\mathfrak{g}_{-1}}$ is faithful, the inner product $\langle\ ,\ \rangle_{\mathfrak{g}_{-1}}$ is uniquely extended to a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$ satisfying the following conditions (a), (b) (cf. [2]);

(a) The endomorphisms $\mathrm{ad}(X)$, $X\in\mathfrak{g}$, of $\mathfrak{g}$ are skew symmetric with respect to $\langle\ ,\ \rangle$.

(b) The involutive automorphism σ preserves $\langle\ ,\ \rangle$.

We here note that this bilinear from $\langle\ ,\ \rangle$ is independent of taking the fixed point p of M. This fact follows by the condition (a) and the uniqueness of the extension $\langle\ ,\ \rangle$ of $\langle\ ,\ \rangle_{\mathfrak{g}_{-1}}$. Now the Lie algebra $\mathfrak{g}$ is isomorphic to that of G, and so the bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$ moreover induces a bi-invariant pseudo-riemannian metric on G by virtue of the condition (a). This metric is also denoted by $\langle\ ,\ \rangle$. If (M, g) is of compact type, i.e., G is a semi-simple Lie group of compact type, the pseudo-riemannian metric is riemannian.

Assume that (M, g) is simply connected. Again fix a point $p_* \in M^*$ and let $p_* = \langle W \rangle$ where $W \subset T_p M$. Moreover let N be a complete connected totally geodesic submanifold of M such that $p \in N$ and $T_p N = W$. Then N is a symmetric submanifold by Lemma 1.2. Let t_p be the extrinsic symmetry of N at p. Similarly as s_p, it also induces an involutive automorphism τ of G and thus $\mathfrak{g}$. This involution τ of $\mathfrak{g}$ has the following properties (1), (2);

(1) $[\tau, \sigma]=0$.

(2) τ preserves the bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$.

The property (1) follows since $[t_p, s_p]=0$ and the property (2) follows since τ preserves $\langle\ ,\ \rangle_{\mathfrak{g}_{-1}}$. Now denote by $\mathfrak{k}^*(p_*)$ the Lie algebra of $K^*(p_*)$. Then it is characterized by τ as follows.

LEMMA 2.2. *It holds that*

$$\mathfrak{k}^*(p_*) = \{X \in \mathfrak{g}\ ;\ \tau(X) = X\}.$$

PROOF. We first recall that $N^\perp(p_*)$ is a connected component of the fixed point set of t_p, which contains p. Hence, for any point $q \in N^\perp(p_*)$, it follows that $t_p(q)=q$ and $(t_p)_*(x)=x$ or $-x$ according as $x \in T_q N^\perp(p_*)$ or $x \in N_q N^\perp(p_*)$.

Let $X \in \mathfrak{k}^*(p_*)$ and $t \in \boldsymbol{R}$. Then it holds that $(\exp tX)(N^\perp(p_*)) = N^\perp(p_*)$ by Lemma 2.1. Hence, by the above remark, it follows that $(t_p \circ \exp tX \circ t_p)(q) = (\exp tX)(q)$ and $(t_p \circ \exp tX \circ t_p)_{*_q} = (\exp tX)_{*_q}$. Since $t_p \circ \exp tX \circ t_p$ and $\exp tX$ are both isometries of (M, g), it holds that $t_p \circ \exp tX \circ t_p = \exp tX$, and thus $\tau(X)=X$.

Conversely assume that $\tau(X)=X$ where $X \in \mathfrak{g}$, i.e., $t_p \circ \exp tX \circ t_p = \exp tX$ for $t \in \boldsymbol{R}$. Again by the above remark it follows that $t_p((\exp tX)(q)) = (\exp tX)(q)$ for $q \in N^\perp(p_*)$. Hence it holds that $(\exp tX)(q) \in N^\perp(p_*)$ and so $(\exp tX)(N^\perp(p_*)) = N^\perp(p_*)$. By Lemma 2.1 it follows that $\exp tX \in K^*(p_*)$ and thus $X \in \mathfrak{k}^*(p_*)$.

Q. E. D.

Denote by $\mathfrak{p}^*(p_*)$ the (-1)-eigenspace of τ. By Lemma 2.1 it is identified with the tangent space at the origin $K^*(p^*)$ of the homogeneous space $G/K^*(p_*)$. Since $\langle\ ,\ \rangle$ is preserved by τ, its restriction to $\mathfrak{p}^*(p_*)$ is nondegenerate. Hence the bi-invariant metric $\langle\ ,\ \rangle$ on G induces a pseudo-riemannian metric g^* on $G/K^*(p_*)$. This metric g^* moreover induces a pseudo-riemannian metric on M^* such that G acts isometrically on M^*. This is also denoted by g^*. We here note that the metric g^* on M^* is independent of taking the fixed point p_* of M^* since the metric $\langle\ ,\ \rangle$ on G is bi-invariant.

We next show that this pseudo-riemannian homogeneous space (M^*, g^*) is independent of selecting a connected $\mathcal{V}$-submanifold S and a fixed point $o\in S$. Namely let (M'^*, g'^*) be the pseudo-riemannian homogeneous space constructed above from another connected $\mathcal{V}$-submanifold S' and another fixed point $o'\in S'$. Then it holds that (M'^*, g'^*) is isometric to (M^*, g^*). In fact, since $[T_{o'}S']=[T_oS]=\mathcal{V}$, there exists an isometry ϕ of (M, g) such that $\phi(o')=o$ and $\phi_*(T_{o'}S')=T_oS$. This isometry induces a bijection of $F_{S'}$ onto F_S since $\phi\circ G\circ\phi^{-1}=G$, and the bijection moreover induces a bijection $\phi^\#$ of M'^* onto M^* since it preserves the equivalence relation $\sim$. Identify M'^*, M^* with the homogeneous spaces $G/K'^*(o'_*)$, $G/K^*(o_*)$, where $o'_*=\langle T_{o'}S'\rangle$, $o_*=\langle T_oS\rangle$. Then we can easily see that the bijection $\phi^\#$ is identified with a smooth mapping of $G/K'^*(o'_*)$ to $G/K^*(o_*)$ induced from the following isomorphism $\hat{\phi}$ of G: $\hat{\phi}(\psi)=\phi\circ\psi\circ\phi^{-1}$ for $\psi\in G$. Here, noting that $\hat{\phi}$ preserves the metric $\langle\ ,\ \rangle$ on G, we can moreover see that $\phi^\#$ is an isometry of (M'^*, g'^*) onto (M^*, g^*).

We call this pseudo-riemannian homogeneous space (M^*, g^*) the *target space associated with* the equivalence class $\mathcal{V}$.

THEOREM 2.3. *Let (M, g) be a simply connected riemannian symmetric space without Euclidean factor and let $\mathcal{V}\in\mathcal{S}(M, g)$. Then the target space (M^*, g^*) associated with $\mathcal{V}$ is a pseudo-riemannian symmetric space.*

Moreover if (M, g) is compact, the target space (M^*, g^*) is a compact riemannian symmetric space.

PROOF. Fix a point $o_*\in M^*$ and set $o_*=\langle W\rangle$ where $W\subset T_oM$. Moreover let t_o be the extrinsic symmetry at o of the totally geodesic symmetric submanifold N such that $o\in N$ and $T_oN=W$. Then, similarly as the above arguement, the isometry t_o induces an involutive isometry $t_o^\#$ of $M^*=G/K^*(o_*)$ and it moreover holds that $t_o^\#(o_*)=o_*$ and $\hat{t}_o=\tau$. Obviously this isometry $t_o^\#$ defines the geodesic symmetry at o_*. Moreover since (M^*, g^*) is a pseudo-riemannian homogeneous space, it is a pseudo-riemannian symmetric space.

Next assume that (M, g) is compact. Then the Lie group G is compact and the metric $\langle\ ,\ \rangle$ on G is riemannian. Hence (M^*, g^*) is a compact riemannian symmetric space. Q.E.D.

Now we define a "generalized Gauss map" associated with a connected $\mathcal{V}$-submanifold of M. Assume that (M, g) is a simply connected riemannian symmetric space without Euclidean factor and let $\mathcal{V}\in\mathcal{S}(M, g)$. Let S be a connected $\mathcal{V}$-submanifold and fix a point $o\in S$. Then the target space (M^*, g^*) is constructed from S and o. We define a smooth mapping κ of S to M^* in the following way. For a point $p\in S$ the tangent space T_pS is contained in the set F_S by the connectedness of S. Then we put $\kappa(p)=\langle T_pS\rangle\in M^*$. We call this mapping κ the *generalized Gauss map* associated with the $\mathcal{V}$-submanifold S.

We first remark the followings (a), (b), (c);

(a) The generalized Gauss map κ is independent of taking the fixed point $o\in S$ since the set F_S is so.

(b) The generalized Gauss map κ only depends on the congruence class of S. Namely let S' be another connected $\mathcal{V}$-submanifold of M which is congruent to S by an isometry ϕ of (M, g). Then there exists an isometry ψ of (M'^*, g'^*) onto (M^*, g^*) such that $\kappa'\circ\phi=\psi\circ\kappa$, where (M'^*, g'^*), κ' denote the target space, the generalized Gauss map associated with S'. In fact, this isometry ψ is given by the isometry $\phi^\#$ constructed above by ϕ.

(c) Let (M, g) be the n-dimensional unit sphere S^n of the Euclidean space $\boldsymbol{R}^{n+1}$ and S an r-dimensional connected submanifold of S^n. Then the generalized Gauss map associated with S is, so is called, the "usual" Gauss map since the target space M^* is identified with the Grassmannian manifold G_r^{n+1}.

We next show the following theorem, which is a generalization of the result by Vilms [10] described in Introduction.

THEOREM 2.4. *Let (M, g) be simply connected riemannian symmetric space without Euclidean factor and let $\mathcal{V}\in\mathcal{S}(M, g)$. If S is a symmetric $\mathcal{V}$-submanifold of M, then the generalized Gauss map is a totally geodesic immersion of S to (M^*, g^*).*

Moreover it is isometric if and only if S is a totally geodesic submanifold of M.

Before proving this theorem we prepare the following lemma. Let (M, g) be a riemannian symmetric space and S a symmetric submanifold of M. Let $\gamma(t)$ be a complete geodesic of S and denote by t_t the extrinsic symmetry of S at $\gamma(t)$. Moreover set $T(t)=t_{(t/2)}\circ t_0$ for $t\in\boldsymbol{R}$.

LEMMA ([6]). *The curve $T(t)$ is a one-parameter subgroup of $I^0(M, g)$ satisfying the following conditions;*

(1) $T(t)(S)=S$ *for* $t\in\boldsymbol{R}$.

(2) $t_0\circ T(t)\circ t_0=T(-t)$ *for* $t\in\boldsymbol{R}$.

PROOF OF THEOREM 2.4. Fix a point $o\in S$ and let (M^*, g^*) be the target space constructed from S and o. Let $\gamma(t)$ be a complete geodesic of S such that $\gamma(0)=o$. By the condition (1) of the lemma it holds that $T_{\gamma(t)}S=(T(t))_*T_oS$.

Hence it follows that

$$T(t)(o_*)=T(t)(\langle T_oS\rangle)=\langle T_{\gamma(t)}S\rangle$$
$$=\kappa(\gamma(t))$$

for $t\in\boldsymbol{R}$, where $o_*=\langle T_oS\rangle$. Here set $T(t)=\exp tX$ where $X\in\mathfrak{g}$. By the condition (2) of the lemma it holds that $\tau(X)=-X$ and thus $X\in\mathfrak{p}^*(o_*)$. This implies that $T(t)(o_*)$ is a geodesic of (M^*, g^*) by the general theory of symmetric space. Hence the generalized Gauss map κ transposes a geodesic of S to a geodesic of (M^*, g^*).

We next show that the differential κ_{*_o} at o is injective. Assume that $T(t)(o_*)=o_*$ for all $t\in\boldsymbol{R}$. Then it holds that $\langle T(t)_*T_oS\rangle=\langle T_oS\rangle$ and thus $\langle T_{\gamma(t)}S\rangle=\langle T_oS\rangle$. Since the geodesic $\gamma(t)$ intersects the totally geodesic submanifold $N^\perp(o_*)$ orthogonally at o, it follows that $\gamma(t)=o$ for $t\in\boldsymbol{R}$. This implies that κ_{*_o} is injective.

Hence κ is a totally geodesic immersion. We show the second claim. We first remark that $d\gamma/dt(0)=X_o$ since $\gamma(t)=T(t)(o)=(\exp tX)(o)$. Decompose the Killing vector field X into the sum of a Killing vector field X_1 in $\mathfrak{g}_1$ and a Killing vector field X_{-1} in $\mathfrak{g}_{-1}$. Then it holds that $X_o=(X_{-1})_o$. Hence it follows that $g(d\gamma/dt(0), d\gamma/dt(0))=\langle X_{-1}, X_{-1}\rangle$, while it holds that $g^*(d\kappa\circ\gamma/dt(0), d\kappa\circ\gamma/dt(0))=\langle X, X\rangle$. Here it holds that $\langle X, X\rangle=\langle X_{-1}, X_{-1}\rangle$ if and only if $X\in\mathfrak{g}_{-1}$, equivalently, $\gamma(t)$ is a geodesic of M. Hence κ is isometric if and only if S is totally geodesic in M. Q. E. D.

REMARK. In Theorem 2.3 and Theorem 2.4 we may change the simply connectedness of M for the following condition (#) with respect to an equivalence class $\mathcal{V}$.

(#) The unique complete connected totally geodesic $\mathcal{V}$-submanifold is defined by a connected component of the fixed point set of an involutive isometry of (M, g).

In fact, the arguements in this section are valid under this assumption (#). Moreover the classification of such $\mathcal{V}$-submanifolds has been studied in Nagano [4].

§ 3. Target spaces and the local expressions as symmetric space.

In this section we assume that (M, g) is a simply connected irreducible riemannian symmetric space and then express the target spaces M^* locally and concretely as symmetric space. We note that in this case (M, g) is of compact type or of noncompact type.

Let $\mathfrak{g}$ be a semi-simple Lie algebra and σ an involutive automorphism of $\mathfrak{g}$. Moreover let $\mathfrak{g}_{\pm 1}$ be the (± 1)-eigenspaces of σ. Then a pair $(\mathfrak{g}, \sigma)$ is called an *(effective) symmetric Lie algebra* if the adjoint representation $\mathrm{ad}_{\mathfrak{g}_{-1}}(\mathfrak{g}_1)$ of $\mathfrak{g}_1$ onto $\mathfrak{g}_{-1}$ is faithful, and it is moreover said to be *irreducible* if it is not decomposed into any sum of proper factors. Here the direct sum of symmetric Lie algebras is defined naturally. Also, a symmetric Lie algebra $(\mathfrak{g}, \sigma)$ is said to be *of compact type* if $\mathfrak{g}$ is a semi-simple Lie algebra of compact type, while it is said to be *of noncompact type* if $\mathfrak{g}_1$ does not contain any compact simple ideal of $\mathfrak{g}$.

Next assume that $\mathfrak{g}$ is a semi-simple Lie algebra of compact type. Let σ, τ be involutive automorphisms of $\mathfrak{g}$ and $\langle , \rangle$ a nondegenerate symmetric bilinear form on $\mathfrak{g}$. Then a triple $(\mathfrak{g}, \sigma, \tau)$ is called a *pairwise symmetric Lie algebra* if the pairs $(\mathfrak{g}, \sigma)$, $(\mathfrak{g}, \tau)$ are symmetric Lie algebras such that $[\sigma, \tau]=0$, and moreover a quadruple $(\mathfrak{g}, \sigma, \tau, \langle,\rangle)$ associated with a pairwise symmetric Lie algebra is called an *orthogonal pairwise symmetric Lie algebra* if the bilinear form $\langle,\rangle$ is preserved by σ, τ and the endomorphisms $\mathrm{ad}(X)$, $X \in \mathfrak{g}$, of $\mathfrak{g}$ are skew symmetric with respect to $\langle,\rangle$.

We note that for these objects $(\mathfrak{g}, \sigma)$, $(\mathfrak{g}, \sigma, \tau)$, $(\mathfrak{g}, \sigma, \tau, \langle,\rangle)$ the equivalences are naturally defined respectively. Next let (M, g) be a simply connected compact riemannian symmetric space and N a symmetric submanifold of M. Then N is called *substantial* if N is not contained in any proper product factor of (M, g). We have the following two correspondences.

LEMMA 3.1 ([6]). *The congruence classes $((M, g), N)$ of the simply connected connected compact riemannian symmetric spaces (M, g) and the totally geodesic substantial symmetric submanifolds N of M with* $\dim N \geqq 1$ *bijectively correspond the equivalence classes of the orthogonal pairwise symmetric Lie algebras* $(\mathfrak{g}, \sigma, \tau, \langle,\rangle)$.

Moreover the equivalence classes of the pairwise symmetric Lie algebras $(\mathfrak{g}, \sigma, \tau)$ bijectively correspond the equivalence classes of the symmetric Lie algebras $(\hat{\mathfrak{g}}, \hat{\tau})$ of noncompact type.

Here the correspondence: $((M, g), N) \mapsto (\mathfrak{g}, \sigma, \tau, \langle,\rangle)$ is given by the following way. Let $\mathfrak{g}$ be the Lie algebra of the Killing vector fields on (M, g). Fix a point $p \in N$ and let s_p, t_p be the geodesic symmetry of (M, g) at p and the extrinsic symmetry of N at p. Then σ, τ are the involutive automorphisms defined from s_p, t_p, and $\langle,\rangle$ is the bilinear form defined from g_p on T_pM. (See §2 for these constructions.) We note that the object $(\mathfrak{g}, \sigma, \langle,\rangle)$ only depends on the ambient space (M, g). Next the correspondence: $(\mathfrak{g}, \sigma, \tau) \mapsto (\hat{\mathfrak{g}}, \hat{\tau})$ is given by the following way. Decompose $\mathfrak{g}$ to the sum of the (± 1)-eigenspaces $\mathfrak{g}_{\pm 1}$ by σ and put $\hat{\mathfrak{g}} = \mathfrak{g}_1 + \sqrt{-1}\,\mathfrak{g}_{-1}$. Then $\hat{\mathfrak{g}}$ has a semi-simple Lie algebra structure of noncompact type, and τ induces the involutive automorphism $\hat{\tau}$ of $\mathfrak{g}$ since it holds that $[\sigma, \tau]=0$.

Now let (M, g) be a simply connected compact irreducible riemannian symmetric space and $(\mathfrak{g}, \sigma, \langle,\rangle)$ the object associated above with (M, g). Let $\mathcal{V}_0 \in \mathcal{S}(M, g)$ be the trivial equivalence class of a 0-dimensional subspace. Then, by Lemma 1.1 the set $\mathcal{S}(M, g) - \{\mathcal{V}_0\}$ is bijective to the congruence classes of the totally geodesic symmetric submanifolds N of M with $\dim N \geqq 1$. Since in this case a symmetric submanifold of M is necessarily substantial, by the first correspondence of Lemma 3.1 the set $\mathcal{S}(M, g) - \{\mathcal{V}_0\}$ is moreover bijective to the equivalence classes $(\mathfrak{g}, \sigma, \tau, \langle,\rangle)$ by the automorphisms which preserve the object $(\mathfrak{g}, \sigma, \langle,\rangle)$. Let $\hat{\mathfrak{g}}$ be the Lie algebra constructed from $(\mathfrak{g}, \sigma)$ in the second correspondence of Lemma 3.1. It is a simple Lie algebra of noncompact type since $(\mathfrak{g}, \sigma)$ is irreducible. Then, by the second correspondence, the equivalence classes $(\mathfrak{g}, \sigma, \tau)$ underlying the above equivalence classes $(\mathfrak{g}, \sigma, \tau, \langle,\rangle)$ is moreover bijective to the equivalence classes $(\hat{\mathfrak{g}}, \hat{\tau})$ by the automorphisms which preserve $\hat{\mathfrak{g}}$. The latter equivalence classes are classified in Berger [1]. Hence, using this classification, we can decide the local structures of the target spaces M^*.

We start with an irreducible symmetric Lie algebra $(\hat{\mathfrak{g}}, \hat{\tau})$ of noncompact type and with simple Lie algebra $\hat{\mathfrak{g}}$, and next find the symmetric Lie algebra $(\hat{\mathfrak{g}}, \hat{\rho})$ associated as follows with $(\hat{\mathfrak{g}}, \hat{\tau})$. Let $(\mathfrak{g}, \sigma, \tau)$ be the pairwise symmetric Lie algebra corresponding $(\hat{\mathfrak{g}}, \hat{\tau})$. Then $(\hat{\mathfrak{g}}, \hat{\rho})$ is the symmetric Lie algebra corresponding the pairwise symmetric Lie algebra $(\mathfrak{g}, \sigma, \sigma\tau)$. Here we note that $(\hat{\mathfrak{g}}, \hat{\rho})$ is not always effective. In fact, this occurs if and only if $\sigma = \tau$, and then the totally geodesic symmetric submanifold N coincides with the ambient space M. Hence this case is out of our consideration. Now, using these symmetric Lie algebras $(\hat{\mathfrak{g}}, \hat{\tau})$, $(\hat{\mathfrak{g}}, \hat{\rho})$, we clarify the local structures of M, N, M^* associated with $(\hat{\mathfrak{g}}, \hat{\tau})$. First the local structure of M is given by $(\mathfrak{g}, \sigma)$. Here $\mathfrak{g}$ is the compact form of $\hat{\mathfrak{g}}$ and the subalgebra $\mathfrak{g}_1$, the set of fixed points of σ, is the maximal compact subalgebra of $\hat{\mathfrak{g}}$. Next the local structure of M^* is given by $(\mathfrak{g}, \tau)$. Hence let $\mathfrak{k}^*$, $\hat{\mathfrak{k}}^*$ be the subalgebras of the fixed points of τ, $\hat{\tau}$ respectively. Then $\mathfrak{k}^*$ is the compact form of $\hat{\mathfrak{k}}^*$. Lastly let $\mathfrak{g}_{\pm 1}$ be the (± 1)-eigenspaces by σ and decompose $\mathfrak{g}_1$, $\mathfrak{g}_{-1}$ to the (± 1)-eigenspaces $\mathfrak{g}_{1\pm 1}$, $\mathfrak{g}_{-1\pm 1}$ by τ respectively. Then the subalgebra $\hat{\mathfrak{k}}$ of the fixed points of $\hat{\tau}$ is given by $\mathfrak{g}_{11} + \sqrt{-1}\,\mathfrak{g}_{-1-1}$. Since N is the totally geodesic submanifold of M defined by the Lie triple system $\mathfrak{g}_{-1-1}$, the local structure is given by the quotient space

$\mathfrak{g}_{11}+\mathfrak{g}_{-1-1}/\mathfrak{g}_{11}$. Here $\mathfrak{g}_{11}+\mathfrak{g}_{-1-1}$ is the compact form of $\hat{\mathfrak{k}}$ and $\mathfrak{g}_{11}$ is the maximal compact subalgebra of $\hat{\mathfrak{k}}$.

Next let $(\hat{M}, \hat{g})$ be the irreducible riemannian symmetric space of noncompact type which is dual to (M, g). Then the local structure of $\hat{M}$ is given by the quotient space $\hat{\mathfrak{g}}/\mathfrak{g}_1$. Let $\hat{N}$ be the totally geodesic symmetric submanifold of $\hat{M}$ defined by the Lie triple system $\sqrt{-1}\,\mathfrak{g}_{-1-1}$. Then the local structure of $\hat{N}$ is given by the quatient space $\hat{\mathfrak{k}}/\mathfrak{g}_{11}$. We here note that a totally geodesic symmetric submanifold of $\hat{M}$ is obtained in this way from a totally geodesic symmetric submanifold of M. Let $\hat{M}^*$ be the target space associated with the congruence class $((\hat{M}, \hat{g}), \hat{N})$. The local structure of $\hat{M}^*$ is given by $(\hat{\mathfrak{g}}, \hat{\tau})$.

Lastly we list up the local structures of M, N, M^* and $\hat{M}$, $\hat{N}$, $\hat{M}^*$ in the form of quotient space. The local structures of $\hat{M}$, $\hat{N}$ are the noncompact duals of the local structures of M, N. Hence we do not describe the local local structures of $\hat{M}$, $\hat{N}$ in the following tables. Moreover we assume that N is neither M nor one point of M.

Table I. (The case that M is of classical type and not of group type.)

No.	symmetric Lie algebra $(\hat{\mathfrak{g}}, \hat{\tau})$ (local structure of $\hat{M}^*$)		symmetric Lie algebra $(\hat{\mathfrak{g}}, \hat{\rho})$ associated with $(\hat{\mathfrak{g}}, \hat{\tau})$
	M	N	M^*
1	$sl(2n, \boldsymbol{R})/sl(n, \boldsymbol{C})+\boldsymbol{T}$		$sl(2n, \boldsymbol{R})/sp(n, \boldsymbol{R})$
	$su(2n)/so(2n)$	$sp(n)/u(n)$	$su(2n)/s(u(n)+u(n))$
2	$sl(2n, \boldsymbol{R})/sp(n, \boldsymbol{R})$		$sl(2n, \boldsymbol{R})/sl(n, \boldsymbol{C})+\boldsymbol{T}$
	$su(2n)/so(2n)$	$su(n)$	$su(2n)/sp(2n)$
3	$sl(n, \boldsymbol{R})/sl(k, \boldsymbol{R})+sl(n-k, \boldsymbol{R})+\boldsymbol{R}$		$sl(n, \boldsymbol{R})/so^k(n)$
	$su(n)/so(n)$	$so(n)/so(k)+so(n-k)$	$su(n)/s(u(k)+u(n-k))$
4	$sl(n, \boldsymbol{R})/so^k(n)$		$sl(n, \boldsymbol{R})/sl(k, \boldsymbol{R})+sl(n-k, \boldsymbol{R})+\boldsymbol{R}$
	$su(n)/so(n)$	$su(k)/so(k)\oplus$ $su(n-k)/so(n-k)\oplus\boldsymbol{T}$	$su(n)/so(n)$
5	$su^*(2n)/su^*(2i)+su^*(2n-2i)+\boldsymbol{R}$		$su^*(2n)/sp^i(n)$
	$su(2n)/sp(n)$	$sp(n)/sp(i)+sp(n-i)$	$su(2n)/$ $s(u(2i)+u(2n-2i))$
6	$su^*(2n)/sp^i(n)$		$su^*(2n)/su^*(2i)+su^*(2n-2i)+\boldsymbol{R}$
	$su(2n)/sp(n)$	$su(2i)/sp(i)\oplus$ $su(2n-2i)/sp(n-i)\oplus\boldsymbol{T}$	$su(2n)/sp(n)$

<table>
<tr><td rowspan="2">7</td><td colspan="2">$su^*(2n)/sl(n, C)+\boldsymbol{T}$</td><td colspan="2">$su^*(2n)/so^*(2n)$</td></tr>
<tr><td>$su(2n)/sp(n)$</td><td colspan="2">$so(2n)/u(n)$</td><td>$su(2n)/s(u(n)+u(n))$</td></tr>
<tr><td rowspan="2">8</td><td colspan="2">$su^*(2n)/so^*(2n)$</td><td colspan="2">$su^*(2n)/sl(n, \boldsymbol{C})+\boldsymbol{T}$</td></tr>
<tr><td>$su(2n)/sp(n)$</td><td colspan="2">$su(n)$</td><td>$su(2n)/so(2n)$</td></tr>
<tr><td rowspan="2">9</td><td colspan="2">$su^i(n)/$
$su^k(k+h)+su^{i-k}(n-k-h)+\boldsymbol{T}$</td><td colspan="2">$su^i(n)/su^k(n-h-i+k)$
$+su^{i-k}(h+i-k)+\boldsymbol{T}$</td></tr>
<tr><td>$su(n)/s(u(i)+u(n-i))$</td><td colspan="2">$su(n-h-i+k)/$
$s(u(k)+u(n-h-i))\oplus$
$su(h+i-k)/$
$s(u(i-k)+u(h))$</td><td>$su(n)/$
$s(u(k+h)+u(n-k-h))$</td></tr>
<tr><td rowspan="2">10</td><td colspan="2">$su^i(n)/so^i(n)$</td><td colspan="2">$su^i(n)/so^i(n)$</td></tr>
<tr><td>$su(n)/s(u(i)+u(n-i))$</td><td colspan="2">$so(n)/so(i)+so(n-i)$</td><td>$su(n)/so(n)$</td></tr>
<tr><td rowspan="2">11</td><td colspan="2">$su^{2i}(2n)/sp^i(n)$</td><td colspan="2">$su^{2i}(2n)/sp^i(n)$</td></tr>
<tr><td>$su(2n)/$
$s(u(2i)+u(2n-2i))$</td><td colspan="2">$sp(n)/sp(i)+sp(n-i)$</td><td>$su(2n)/sp(n)$</td></tr>
<tr><td rowspan="2">12</td><td colspan="2">$su^n(2n)/so^*(2n)$</td><td colspan="2">$su^n(2n)/sp(n, \boldsymbol{R})$</td></tr>
<tr><td>$su(2n)/s(u(n)+u(n))$</td><td colspan="2">$sp(n)/u(n)$</td><td>$su(2n)/so(2n)$</td></tr>
<tr><td rowspan="2">13</td><td colspan="2">$su^n(2n)/sp(n, \boldsymbol{R})$</td><td colspan="2">$su^n(2n)/so^*(2n)$</td></tr>
<tr><td>$su(2n)/s(u(n)+u(n))$</td><td colspan="2">$so(2n)/u(n)$</td><td>$su(2n)/sp(n)$</td></tr>
<tr><td rowspan="2">14</td><td colspan="2">$so^i(n)/so^k(k+h)+so^{i-k}(n-k-h)$</td><td colspan="2">$so^i(n)/$
$so^k(n-h-i+k)+so^{i-k}(h+i-k)$</td></tr>
<tr><td>$so(n)/so(i)+so(n-i)$</td><td colspan="2">$so(n-h-i+k)/$
$so(k)+so(n-h-i)\oplus$
$so(h+i-k)/$
$so(i-k)+so(h)$</td><td>$so(n)/$
$so(k+h)+so(n-k-h)$</td></tr>
<tr><td rowspan="2">15</td><td colspan="2">$so^{2i}(2n)/su^i(n)+\boldsymbol{T}$</td><td colspan="2">$so^{2i}(2n)/su^i(n)+\boldsymbol{T}$</td></tr>
<tr><td>$so(2n)/so(2i)+so(2n-2i)$</td><td colspan="2">$su(n)/s(u(i)+u(n-i))$</td><td>$so(2n)/u(n)$</td></tr>
<tr><td rowspan="2">16</td><td colspan="2">$so^n(2n)/so(n, \boldsymbol{C})$</td><td colspan="2">$so^n(2n)/sl(n, \boldsymbol{R})+\boldsymbol{R}$</td></tr>
<tr><td>$so(2n)/so(n)+so(n)$</td><td colspan="2">$u(n)/so(n)$</td><td>$so(2n)/so(n)+so(n)$</td></tr>
<tr><td rowspan="2">17</td><td colspan="2">$so^n(2n)/sl(n, \boldsymbol{R})+\boldsymbol{R}$</td><td colspan="2">$so^n(2n)/so(n, \boldsymbol{C})$</td></tr>
<tr><td>$so(2n)/so(n)+so(n)$</td><td colspan="2">$su(n)$</td><td>$so(2n)/u(n)$</td></tr>
<tr><td rowspan="2">18</td><td colspan="2">$so^*(2n)/so^*(2i)+so^*(2n-2i)$</td><td colspan="2">$so^*(2n)/su^i(n)+\boldsymbol{T}$</td></tr>
<tr><td>$so(2n)/u(n)$</td><td colspan="2">$su(n)/s(u(i)+u(n-i))$</td><td>$so(2n)/$
$so(2i)+so(2n-2i)$</td></tr>
</table>

19	$so^*(2n)/su^i(n)+\boldsymbol{T}$		$so^*(2n)/so^*(2i)+so^*(2n-2i)$
	$so(2n)/u(n)$	$so(2i)/u(i)\oplus so(2n-2i)/u(n-i)$	$so(2n)/u(n)$
20	$so^*(2n)/so(n, \boldsymbol{C})$		$so^*(2n)/so(n, \boldsymbol{C})$
	$so(2n)/u(n)$	$so(n)$	$so(2n)/so(n)+so(n)$
21	$so^*(4n)/su^*(2n)+\boldsymbol{R}$		$so^*(4n)/su^*(2n)+\boldsymbol{R}$
	$so(4n)/u(2n)$	$u(2n)/sp(n)$	$so(4n)/u(2n)$
22	$sp(n, \boldsymbol{R})/sp(i, \boldsymbol{R})+sp(n-i, \boldsymbol{R})$		$sp(n, \boldsymbol{R})/su^i(n)+\boldsymbol{T}$
	$sp(n)/u(n)$	$su(n)/s(u(i)+u(n-i))$	$sp(n)/sp(i)+sp(n-i)$
23	$sp(n, \boldsymbol{R})/su^i(n)+\boldsymbol{T}$		$sp(n, \boldsymbol{R})/sp(i, \boldsymbol{R})+sp(n-i, \boldsymbol{R})$
	$sp(n)/u(n)$	$sp(i)/u(i)\oplus sp(n-i)/u(n-i)$	$sp(n)/u(n)$
24	$sp(2n, \boldsymbol{R})/sp(n, \boldsymbol{C})$		$sp(2n, \boldsymbol{R})/sp(n, \boldsymbol{C})$
	$sp(2n)/u(2n)$	$sp(n)$	$sp(2n)/sp(n)+sp(n)$
25	$sp(n, \boldsymbol{R})/sl(n, \boldsymbol{R})+\boldsymbol{R}$		$sp(n, \boldsymbol{R})/sl(n, \boldsymbol{R})+\boldsymbol{R}$
	$sp(n)/u(n)$	$u(n)/so(n)$	$sp(n)/u(n)$
26	$sp^i(n)/sp^k(k+h)+sp^{i-k}(n-k-h)$		$sp^i(n)/sp^k(n-h-i+k)+sp^{i-k}(h+i-k)$
	$sp(n)/sp(i)+sp(n-i)$	$sp(n-h-i+k)/sp(k)+sp(n-h-i)\oplus sp(h+i-k)/sp(i-k)+sp(h)$	$sp(n)/sp(k+h)+sp(n-k-h)$
27	$sp^i(n)/su^i(n)+\boldsymbol{T}$		$sp^i(n)/su^i(n)+\boldsymbol{T}$
	$sp(n)/sp(i)+sp(n-i)$	$su(n)/s(u(i)+u(u-i))$	$sp(n)/u(n)$
28	$sp^n(2n)/su^*(2n)+\boldsymbol{R}$		$sp^n(2n)/sp(n, \boldsymbol{C})$
	$sp(2n)/sp(n)+sp(n)$	$sp(n)$	$sp(2n)/u(2n)$
29	$sp^n(2n)/sp(n, \boldsymbol{C})$		$sp^n(2n)/su^*(2n)+\boldsymbol{R}$
	$sp(2n)/sp(n)+sp(n)$	$u(2n)/sp(n)$	$sp(2n)/sp(n)+sp(n)$

Table II. (The case that M is of exceptional type and not of group type.)

30	E_6^1/F_4^1		$E_6^1/su^*(6)+su(2)$
	$E_6/sp(4)$	$su(6)/sp(3)$	E_6/F_4

31	$E_6^1/su^*(6)+su(2)$	E_6^1/F_4^1	
	$E_6/sp(4)$	$F_4/sp(3)+su(2)$	$E_6/su(6)+su(2)$
32	$E_6^1/so^5(10)+\boldsymbol{R}$	$E_6^1/sp^2(4)$	
	$E_6/sp(4)$	$sp(4)/sp(2)+sp(2)$	$E_6/so(10)+\boldsymbol{T}$
33	$E_6^1/sp^2(4)$	$E_6^1/so^5(10)+\boldsymbol{T}$	
	$E_6/sp(4)$	$so(10)/so(5)+so(5)\oplus\boldsymbol{T}$	$E_6/sp(4)$
34	$E_6^1/sp(4, \boldsymbol{R})$	$E_6^1/sl(6, \boldsymbol{R})+sl(2, \boldsymbol{R})$	
	$E_6/sp(4)$	$su(6)/so(6)\oplus$ $su(2)/so(2)$	$E_6/sp(4)$
35	$E_6^1/sl(6, \boldsymbol{R})+sl(2, \boldsymbol{R})$	$E_6^1/sp(4, \boldsymbol{R})$	
	$E_6/sp(4)$	$sp(4)/u(4)$	$E_6/su(6)+su(2)$
36	$E_6^2/su^2(6)+su(2)$	$E_6^2/so^4(10)+\boldsymbol{T}$	
	$E_6/su(6)+su(2)$	$so(10)/so(4)+so(6)$	$E_6/su(6)+su(2)$
37	$E_6^2/so^4(10)+\boldsymbol{T}$	$E_6^2/su^2(6)+su(2)$	
	$E_6/su(6)+su(2)$	$su(6)/s(u(2)+u(4))$	$E_6/so(10)+\boldsymbol{T}$
38	$E_6^2/so^*(10)+\boldsymbol{T}$	$E_6^2/so^*(10)+\boldsymbol{T}$	
	$E_6/su(6)+su(2)$	$so(10)/u(5)$	$E_6/so(10)+\boldsymbol{T}$
39	$E_6^2/su^3(6)+sl(2, \boldsymbol{R})$	$E_6^2/su^3(6)+sl(2, \boldsymbol{R})$	
	$E_6/su(6)+su(2)$	$su(6)/s(u(3)+u(3))\oplus$ $su(2)/so(2)$	$E_6/su(6)+su(2)$
40	$E_6^2/sp^1(4)$	E_6^2/F_4^1	
	$E_6/su(6)+su(2)$	$F_4/sp(3)+su(2)$	$E_6/sp(4)$
41	E_6^2/F_4^1	$E_6^2/sp^1(4)$	
	$E_6/su(6)+su(2)$	$sp(4)/sp(1)+sp(3)$	E_6/F_4
42	$E_6^2/sp(4, \boldsymbol{R})$	$E_6^2/sp(4, \boldsymbol{R})$	
	$E_6/su(6)+su(2)$	$sp(4)/u(4)$	$E_6/sp(4)$
43	$E_6^3/su^1(6)+sl(2, \boldsymbol{R})$	$E_6^3/so^*(10)+\boldsymbol{T}$	
	$E_6/so(10)+\boldsymbol{T}$	$so(10)/u(5)$	$E_6/su(6)+su(2)$
44	$E_6^3/so^*(10)+\boldsymbol{T}$	$E_6^3/su^1(6)+sl(2, \boldsymbol{R})$	
	$E_6/so(10)+\boldsymbol{T}$	$su(6)/s(u(1)+u(5))\oplus$ $su(2)/so(2)$	$E_6/so(10)+\boldsymbol{T}$

<table>
<tr><td rowspan="2">45</td><td colspan="2">$E_6^3/so^2(10)+\boldsymbol{T}$</td><td colspan="2">$E_6^3/so^2(10)+\boldsymbol{T}$</td></tr>
<tr><td>$E_6/so(10)+\boldsymbol{T}$</td><td colspan="2">$so(10)/so(2)+so(8)$</td><td>$E_6/so(10)+\boldsymbol{T}$</td></tr>
<tr><td rowspan="2">46</td><td colspan="2">$E_6^3/su^2(6)+su(2)$</td><td colspan="2">$E_6^3/su^2(6)+su(2)$</td></tr>
<tr><td>$E_6/so(10)+\boldsymbol{T}$</td><td colspan="2">$su(6)/s(u(2)+u(4))$</td><td>$E_6/su(6)+su(2)$</td></tr>
<tr><td rowspan="2">47</td><td colspan="2">E_6^3/F_4^2</td><td colspan="2">E_6^3/F_4^2</td></tr>
<tr><td>$E_6/so(10)+\boldsymbol{T}$</td><td colspan="2">$F_4/so(9)$</td><td>$E_6/F_4$</td></tr>
<tr><td rowspan="2">48</td><td colspan="2">$E_6^3/sp^2(4)$</td><td colspan="2">$E_6^3/sp^2(4)$</td></tr>
<tr><td>$E_6/so(10)+\boldsymbol{T}$</td><td colspan="2">$sp(4)/sp(2)+sp(2)$</td><td>$E_6/sp(4)$</td></tr>
<tr><td rowspan="2">49</td><td colspan="2">$E_6^4/so^1(10)+\boldsymbol{R}$</td><td colspan="2">$E_6^4/F_4^2$</td></tr>
<tr><td>E_6/F_4</td><td colspan="2">$F_4/so(9)$</td><td>$E_6/so(10)+\boldsymbol{T}$</td></tr>
<tr><td rowspan="2">50</td><td colspan="2">E_6^4/F_4^2</td><td colspan="2">$E_6^4/so^1(10)+\boldsymbol{R}$</td></tr>
<tr><td>E_6/F_4</td><td colspan="2">$\boldsymbol{T}\oplus so(10)/so(9)$</td><td>$E_6/F_4$</td></tr>
<tr><td rowspan="2">51</td><td colspan="2">$E_6^4/sp^1(4)$</td><td colspan="2">$E_6^4/su^*(6)+su(2)$</td></tr>
<tr><td>E_6/F_4</td><td colspan="2">$su(6)/sp(3)$</td><td>$E_6/sp(4)$</td></tr>
<tr><td rowspan="2">52</td><td colspan="2">$E_6^4/su^*(6)+su(2)$</td><td colspan="2">$E_6^4/sp^1(4)$</td></tr>
<tr><td>E_6/F_4</td><td colspan="2">$sp(4)/sp(1)+sp(3)$</td><td>$E_6/su(6)+su(2)$</td></tr>
<tr><td rowspan="2">53</td><td colspan="2">$E_7^1/E_6^2+\boldsymbol{T}$</td><td colspan="2">$E_7^1/su(2)+so^*(12)$</td></tr>
<tr><td>$E_7/su(8)$</td><td colspan="2">$so(12)/u(6)$</td><td>$E_7/E_6+\boldsymbol{T}$</td></tr>
<tr><td rowspan="2">54</td><td colspan="2">$E_7^1/su(2)+so^*(12)$</td><td colspan="2">$E_7^1/E_6^2+\boldsymbol{T}$</td></tr>
<tr><td>$E_7/su(8)$</td><td colspan="2">$E_6/su(6)+su(2)$</td><td>$E_7/su(2)+so(12)$</td></tr>
<tr><td rowspan="2">55</td><td colspan="2">$E_7^1/so^6(12)+sl(2, \boldsymbol{R})$</td><td colspan="2">$E_7^1/su^4(8)$</td></tr>
<tr><td>$E_7/su(8)$</td><td colspan="2">$su(8)/s(u(4)+u(4))$</td><td>$E_7/so(12)+su(2)$</td></tr>
<tr><td rowspan="2">56</td><td colspan="2">$E_7^1/su^4(8)$</td><td colspan="2">$E_7^1/so^6(12)+sl(2, \boldsymbol{R})$</td></tr>
<tr><td>$E_7/su(8)$</td><td colspan="2">$so(12)/so(6)+so(6)\oplus$
$su(2)/so(2)$</td><td>$E_7/su(8)$</td></tr>
<tr><td rowspan="2">57</td><td colspan="2">$E_7^1/E_6^1+\boldsymbol{R}$</td><td colspan="2">$E_7^1/su^*(8)$</td></tr>
<tr><td>$E_7/su(8)$</td><td colspan="2">$su(8)/sp(4)$</td><td>$E_7/E_6+\boldsymbol{T}$</td></tr>
<tr><td rowspan="2">58</td><td colspan="2">$E_7^1/su^*(8)$</td><td colspan="2">$E_7^1/E_6^1+\boldsymbol{R}$</td></tr>
<tr><td>$E_7/su(8)$</td><td colspan="2">$\boldsymbol{T}\oplus E_6/sp(4)$</td><td>$E_7/su(8)$</td></tr>
</table>

59	$E_7^1/sl(8, \boldsymbol{R})$		$E_7^1/sl(8, \boldsymbol{R})$
	$E_7/su(8)$	$su(8)/so(8)$	$E_7/su(8)$
60	$E_7^2/so^*(12)+sl(2, \boldsymbol{R})$		$E_7^2/so^*(12)+sl(2, \boldsymbol{R})$
	$E_7/so(12)+su(2)$	$so(12)/u(6)$	$E_7/so(12)+su(2)$
61	$E_7^2/E_6^3+\boldsymbol{T}$		$E_7^2/E_6^3+\boldsymbol{T}$
	$E_7/so(12)+su(2)$	$E_6/so(10)+\boldsymbol{T}$	$E_7/E_6+\boldsymbol{T}$
62	$E_7^2/su^2(8)$		$E_7^2/E_6^2+\boldsymbol{T}$
	$E_7/so(12)+su(2)$	$E_6/su(6)+su(2)$	$E_7/su(8)$
63	$E_7^2/E_6^2+\boldsymbol{T}$		$E_7^2/su^2(8)$
	$E_7/so(12)+su(2)$	$su(8)/s(u(2)+u(6))$	$E_7/E_6+\boldsymbol{T}$
64	$E_7^2/so^4(12)+su(2)$		$E_7^2/so^4(12)+su(2))$
	$E_7/so(12)+su(2)$	$so(12)/so(4)+so(8)$	$E_7/so(12)+su(2)$
65	$E_7^2/su^4(8)$		$E_7^2/su^4(8)$
	$E_7/so(12)+su(2)$	$su(8)/s(u(4)+u(4))$	$E_7/su(8)$
66	$E_7^3/so^2(12)+sl(2, \boldsymbol{R})$		$E_7^3/E_6^3+\boldsymbol{T}$
	$E_7/E_6+\boldsymbol{T}$	$E_6/so(10)+\boldsymbol{T}$	$E_7/so(12)+su(2)$
67	$E_7^3/E_6^3+\boldsymbol{T}$		$E_7^3/so^2(12)+sl(2, \boldsymbol{R})$
	$E_7/E_6+\boldsymbol{T}$	$so(12)/so(2)+so(10)\oplus su(2)/so(2)$	$E_7/E_6+\boldsymbol{T}$
68	$E_7^3/su^2(8)$		$E_7^3/so^*(12)+su(2)$
	$E_7/E_6+\boldsymbol{T}$	$so(12)/u(6)$	$E_7/su(8)$
69	$E_7^3/so^*(12)+su(2)$		$E_7^3/su^2(8)$
	$E_7/E_6+\boldsymbol{T}$	$su(8)/s(u(2)+u(6))$	$E_7/so(12)+su(2)$
70	$E_7^3/E_6^4+\boldsymbol{R}$		$E_7^3/E_6^4+\boldsymbol{R}$
	$E_7/E_6+\boldsymbol{T}$	$\boldsymbol{T}\oplus E_6/F_4$	$E_7/E_6+\boldsymbol{T}$
71	$E_7^3/su^*(8)$		$E_7^3/su^*(8)$
	$E_7/E_6+\boldsymbol{T}$	$su(8)/sp(4)$	$E_7/su(8)$
72	$E_8^1/E_7^1+sl(2, \boldsymbol{R})$		$E_8^1/so^*(16)$
	$E_8/so(16)$	$so(16)/u(8)$	$E_8/E_7+su(2)$

73	$E^1_8/so^*(16)$	$E^1_8/E^1_7+sl(2, \boldsymbol{R})$	
	$E_8/so(16)$	$E_7/su(8)\oplus su(2)/so(2)$	$E_8/so(16)$
74	$E^1_8/so^8(16)$	$E^1_8/so^8(16)$	
	$E_8/so(16)$	$so(16)/so(8)+so(8)$	$E_8/so(16)$
75	$E^1_8/E^2_7+su(2)$	$E^1_8/E^2_7+su(2)$	
	$E_8/so(16)$	$E_7/so(12)+su(2)$	$E_8/E_7+su(2)$
76	$E^2_8/so^4(16)$	$E^2_8/E^2_7+su(2)$	
	$E_8/E_7+su(2)$	$E_7/so(12)+su(2)$	$E_8/so(16)$
77	$E^2_8/E^2_7+su(2)$	$E^2_8/so^4(16)$	
	$E_8/E_7+su(2)$	$so(16)/so(4)+so(12)$	$E_8/E_7+su(2)$
78	$E^2_8/E^3_7+sl(2, \boldsymbol{R})$	$E^2_8/E^3_7+sl(2, \boldsymbol{R})$	
	$E_8/E_7+su(2)$	$E_7/E_6+\boldsymbol{T}\oplus su(2)/so(2)$	$E_8/E_7+su(2)$
79	$E^2_8/so^*(16)$	$E^2_8/so^*(16)$	
	$E_8/E_7+su(2)$	$so(16)/u(8)$	$E_8/so(16)$
80	$F^1_4/sp(3, \boldsymbol{R})+sl(2, \boldsymbol{R})$	$F^1_4/sp(3, \boldsymbol{R})+sl(2, \boldsymbol{R})$	
	$F_4/sp(3)+su(2)$	$sp(3)/u(3)\oplus su(2)/so(2)$	$F_4/sp(3)+su(2)$
81	$F^1_4/so(9)$	$F^1_4/sp^1(3)+su(2)$	
	$F_4/sp(3)+su(2)$	$sp(3)/sp(1)+sp(2)$	$F_4/so(9)$
82	$F^1_4/sp^1(3)+su(2)$	$F^1_4/so^4(9)$	
	$F_4/sp(3)+su(2)$	$so(9)/so(4)+so(5)$	$F_4/sp(3)+su(2)$
83	$F^2_4/sp^1(3)+su(2)$	$F^2_4/sp^1(3)+su(2)$	
	$F_4/so(9)$	$sp(3)/sp(1)+sp(2)$	$F_4/sp(3)+su(2)$
84	$F^2_4/so^1(9)$	$F^2_4/so^1(9)$	
	$F_4/so(9)$	$so(9)/so(8)$	$F_4/so(9)$
85	$G^*_2/sl(2, \boldsymbol{R})+sl(2, \boldsymbol{R})$	$G^*_2/sl(2, \boldsymbol{R})+sl(2, \boldsymbol{R})$	
	$G_2/\mathrm{su}(2)+su(2)$	$su(2)/so(2)\oplus su(2)/so(2)$	$G_2/su(2)+su(2)$

Table III. (The case that M is of group type.)

Let $\mathfrak{l}$ be a simple Lie algebra of noncompact type whose complexification $\mathfrak{l}^C$ is also simple, and denote by α the conjugation of $\mathfrak{l}^C$ with respect to $\mathfrak{l}$. Moreover let $\mathfrak{h}$ be a maximal compact subalgebra of $\mathfrak{l}$ and denote by β the Cartan involution of $\mathfrak{l}$ with respect to $\mathfrak{h}$. The $\boldsymbol{R}$-linear extension of β to $\mathfrak{l}^C$ is also denoted by β and the complexification of $\mathfrak{h}$ is denoted by $\mathfrak{h}^C$. Then the pairs $(\mathfrak{l}^C, \alpha)$, $(\mathfrak{l}^C, \beta)$ are irreducible symmetric Lie algebras of noncompact type and they are associated with each other. These exhaust the case that M is of group type. Denote by $\mathfrak{l}_u$ the compact real form of $\mathfrak{l}$.

<table>
<tr><td rowspan="2">86</td><td colspan="2">$\mathfrak{l}^C/\mathfrak{l}$</td><td colspan="2">$\mathfrak{l}^C/\mathfrak{h}^C$</td></tr>
<tr><td>$\mathfrak{l}_u$</td><td colspan="2">$\mathfrak{h}$</td><td>$\mathfrak{l}_u$</td></tr>
<tr><td rowspan="2">87</td><td colspan="2">$\mathfrak{l}^C/\mathfrak{h}^C$</td><td colspan="2">$\mathfrak{l}^C/\mathfrak{l}$</td></tr>
<tr><td>$\mathfrak{l}_u$</td><td colspan="2">$\mathfrak{l}_u/\mathfrak{h}$</td><td>$\mathfrak{l}_u/\mathfrak{y} \oplus \mathfrak{l}_u/\mathfrak{h}$</td></tr>
</table>

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