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CYCLIC-PARALLEL REAL HYPERSURFACES OF A COMPLEX SPACE FORM

By

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Introduction.

In 1973 Takagi [14] classified homogeneous hypersurfaces of a complex projective space P_nC by proving that all of them could be divided into six types, and he [15], [16] showed also that if a real hypersurface M has two or three distinct constant principal curvatures, then M is congruent to one of the homogeneous hypersurfaces of type A_1, A_2 and B among these ones. This result is generalized by Kimura [6], who gives the complete classification that a real hypersurface M of P_nC has constant principal curvatures and FC is principal if and only if M is congruent to one of homogeneous examples, where C denotes the unit normal and F is the almost complex structure. The study of real hypersurfaces of type A_1, A_2 and B of P_nC was originated by Cecil and Ryan [1], Kimura [7], Kon [8], Maeda [10], Okumura [13] and so on.

Real hypersurfaces with cyclic-parallel Ricci tensor of a complex space form $M^n(c)$ have recently been classified by Kwon and Nakagawa [9] in the case where FC is principal. They also gave another characterization of real hypersurfaces of type A_1 and A_2 of P_nC.

On the other hand, many subjects for real hypersurfaces of a complex hyperbolic space H_nC were investigated from different points of view ([2], [3], [11], [12] etc.) one of which, done by Chen, Ludden and Montiel [3], asserts that a real hypersurface M of H_nC is of cyclic-parallel if and only if the structure tensor J induced on M and the shape operator A derived from the unit normal commute each other, that is, $JA=AJ$. In particular, real hypersurfaces of H_nC, which are said to be of type A, similar to those of type A_1 and A_2 of P_nC, were treated by Montiel and Romero [12].

The purpose of the present paper is to show that a real hypersurface of a complex space form $M^n(c)$, $c\neq 0$, is of cyclic-parallel if and only if $JA=AJ$, and to give a complete classification of such hypersurfaces by using those examples constructed in [9], [12] and [15].

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1. Preliminaries.

We begin by recalling fundamental properties on real hypersurfaces of a Kaehlerian manifold. Let N be a real $2n$-dimensional Kaehlerian manifold equipped with a parallel almost complex structure F and a Riemannian metric tensor G which is F-Hermitian, and covered by a system of coordinate neighborhoods $\{U; x^A\}$. Let M be a real hypersurface of N covered by a system of coordinate neighborhoods $\{V; y^h\}$ and immersed isometrically in N by the immersion $i: M \to N$. Throughout the present paper the following convention on the range of indices are used, unless otherwise stated:

$$A, B, \cdots = 1, 2, \cdots, 2n; \qquad i, j, \cdots = 1, 2, \cdots, 2n-1.$$

The summation convention will be used with respect to those system of indices. When the argument is local, M need not be distinguished from $i(M)$. Thus, for simplicity, a point p in M may be identified with the point $i(p)$ and a tangent vector X at p may also be identified with the tangent vector $i_*(X)$ at $i(p)$ via the differential i_* of i. We represent the immersion i locally by $x^A = x^A(y^h)$ and $B_j = (B_j{}^A)$ are also $(2n-1)$-linearly independent local tangent vectors of M, where $B_j{}^A = \partial_j x^A$ and $\partial_j = \partial/\partial y^j$. A unit normal C to M may then be chosen. The induced Riemannian metric g with components g_{ji} on M is given by $g_{ji} = G(B_j, B_i)$ because the immersion is isometric.

For the unit normal C to M, the following representation are obtained in each coordinate neighborhood:

$$FB_i = J_i{}^h B_h + P_i C, \qquad FC = -P^i B_i, \tag{1.1}$$

where we have put $J_{ji} = G(FB_j, B_i)$ and $P_i = G(FB_i, C)$, P^h being components of a vector field P associated with P_i and $J_{ji} = J_j{}^r g_{ri}$. By the properties of the almost Hermitian structure F, it is clear that J_{ji} is skew-symmetric. A tensor field of type (1,1) with components $J_i{}^h$ will be denoted by J. By the properties of the almost complex structure F, the following relations are then given:

$$J_i{}^r J_r{}^h = -\delta_i{}^h + p_i p^h, \quad p^r J_r{}^h = 0, \quad p_r J_i{}^r = 0, \quad p_i p^i = 1,$$

that is, the aggregate (J, g, P) defines an almost contact metric structure. Denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation formed with g_{ji}, equations of Gauss and Weingarten for M are respectively obtained:

(1.2) $$\nabla_j B_i = h_{ji} C, \qquad \nabla_j C = -h_j{}^r B_r,$$

where h_{ji} are components of a second fundamental from σ, $A=(h_j{}^k)$ which is related by $h_{ji}=h_j{}^r g_{ri}$ being the shape operator derived form C. We notice here that h_{ji} is symmetric. By means of (1.1) and (1.2) the covariant derivatives of the structure tensors are yielded:

(1.3) $$\nabla_j J_{ih} = -h_{ji} p_h + h_{jh} p_i, \qquad \nabla_j p_i = -h_{jr} J_i{}^r.$$

In the sequel, the ambient Kaehlerian manifold N is assumed to be of constant holomorphic sectional curvature c and real dimension $2n$, which is called a complex space form and denoted by $M^n(c)$. Then the curvature tensor K of $M^n(c)$ takes the following form:

$$K_{DCBA} = \frac{c}{4}(G_{DA}G_{CB} - G_{DB}G_{CA} + F_{DA}F_{CB} - F_{DB}F_{CA} - 2F_{DC}F_{BA}).$$

Thus, equations of Gauss and Codazzi for M are respectively obtained:

(1.4) $$R_{kjih} = \frac{c}{4}(g_{kh}g_{ji} - g_{jh}g_{ki} + J_{kh}J_{ji} - J_{jh}J_{ki} - 2J_{kj}J_{ih}) + h_{kh}h_{ji} - h_{jh}h_{ki},$$

(1.5) $$\nabla_k h_{ji} - \nabla_j h_{ki} = \frac{c}{4} A_{kji}, \qquad A_{kji} = p_k J_{ji} - p_j J_{ki} - 2p_i J_{kj},$$

where R_{kjih} are components of the Riemannian curvature tensor R of M. Let S_{ji} be components of the Ricci tensor S of M, then the Gauss equation implies

(1.6) $$S_{ji} = \frac{c}{4}\{(2n+1)g_{ji} - 3p_j p_i\} + h h_{ji} - h_{ji}{}^2,$$

where h denotes the trace of the shape operator A and $h_{ji}{}^2 = h_{jr} h_i{}^r$.

2. Cylic-parallel hypersurfaces.

Let M be a real hypersurface of a complex space form $M^n(c)$. The hypersurface M is called *cyclic-parallel* if the cyclic sum of $\nabla\sigma$ vanishes identically, namely

(2.1) $$\nabla_k h_{ji} + \nabla_j h_{ik} + \nabla_i h_{kj} = 0.$$

It was proved in [4] that geodesic hypersurfaces of a complex space form $M^n(c)$, $c \neq 0$, are cyclic-parallel and not parallel. Throughout the present paper we only consider the case where the holomorphic sectional curvature c is not zero.

From now on we suppose that M is of cyclic-parallel. Then we have from (1.5)

$$2\nabla_k h_{ji} = -\nabla_i h_{kj} + \frac{c}{4} A_{kji},$$

or equivalently $3\nabla_k h_{ji}=c/4(A_{kji}-A_{ikj})$. By the second equation of (1.5), it follows that

$$\nabla_k h_{ji}=\frac{c}{4}(p_j J_{ik}+p_i J_{jk}). \tag{2.2}$$

Differentiating this covariantly along M and making use of (1.3), we find

$$\nabla_m \nabla_k h_{ji}=\frac{c}{4}\{(\nabla_m p_j)J_{ik}+(\nabla_m p_i)J_{jk}-h_{mi}p_j p_k-h_{mj}p_k p_i+2h_{mk}p_j p_i\}. \tag{2.3}$$

Since equation (2.2) tells us that $\nabla_k h_j{}^k=0$, the Ricci formula for h_{ji} gives rise to

$$\nabla_k \nabla_j h_i{}^k=S_{jr}h_i{}^r-R_{kjih}h^{kh}.$$

If we substitute (1.4), (1.6) and (2.3) into the last equation and take account of (1.3), we get

$$\begin{aligned} hh_{ji}{}^2=&\left\{h_2-\frac{c}{2}(n+1)\right\}h_{ji}+ch_{rs}J_j{}^r J_i{}^s \\ &+\frac{c}{2}\{(h_{jr}p^r)p_i+(h_{ir}p^r)p_j\}+\frac{c}{4}h(g_{ji}-p_j p_i), \end{aligned} \tag{2.4}$$

where $h_2=h_{ji}h^{ji}$, which yields

$$hh_{jr}{}^2 p^r=\left(h_2-\frac{c}{2}n\right)h_{jr}p^r+\frac{c}{2}\alpha p_j, \tag{2.5}$$

where we have have defined $\alpha=h_{rs}p^r p^s$. Thus, it follows that

$$h\beta=\left\{h_2-\frac{c}{2}(n-1)\right\}\alpha, \qquad \beta=h_{ji}{}^2 p^j p^i. \tag{2.6}$$

On the other hand, if we substitute (1.4) and (2.3) into the Ricci formula, which is given by

$$\nabla_m \nabla_k h_{ji}-\nabla_k \nabla_m h_{ji}=-R_{mkjr}h_i{}^r-R_{mkir}h_j{}^r,$$

then we have

$$\begin{aligned} & h_{ik}{}^2 h_{mj}-h_{im}{}^2 h_{kj}+h_{jk}{}^2 h_{im}-h_{jm}{}^2 h_{ik} \\ &=\frac{c}{4}\{h_{mi}(g_{kj}-p_k p_j)-h_{ki}(g_{mj}-p_m p_j)+h_{jm}(g_{ki}-p_k p_i)-h_{jk}(g_{mi}-p_m p_i) \\ &\quad +J_{jk}(\nabla_m p_i+\nabla_i p_m)-J_{jm}(\nabla_k p_i+\nabla_i p_k)+J_{ik}(\nabla_m p_j+\nabla_j p_m) \\ &\quad -J_{im}(\nabla_k p_j+\nabla_j p_k)+2J_{mk}(\nabla_j p_i+\nabla_i p_j)\}, \end{aligned} \tag{2.7}$$

where we have used the second equation of (1.3). By transvecting (2.7) with J^{ik} and $p^j p^i p^k$ respectively and making use of the fact that properties of the almost contact metric structure (J, g, P), we can see that

$$J^{sr}(h_{ms}h_{jr}{}^2+h_{js}h_{mr}{}^2) \tag{2.8}$$
$$=\frac{1}{4}(2n+1)c(\nabla_j p_m+\nabla_m p_j)-\frac{1}{4}c\{(p^r\nabla_r p_j)p_m+(p^r\nabla_r p_m)P_j\},$$

$$\alpha h_{mr}{}^2 p^r=\beta h_{mr}p^r. \tag{2.9}$$

Combining (2.5) and (2.6) with (2.9), it follows that $\alpha(h_{jr}p^r-\alpha p_j)=0$ and hence $\alpha(\beta-\alpha^2)=0$.

Let M_1 be a set consisting of points of M at which the function $\beta-\alpha^2$ does not vanish. Suppose that M_1 is not empty. We then have $\alpha=0$ and thus $\beta h_{mr}p^r=0$ because of (2.9). By transvecting $h_s{}^m p^s$, it follows that $\beta^2=0$ and hence β vanishes on M_1. Therefore the assumption of M_1 will produce a contradiction. Accordingly we have $\beta=\alpha^2$ on M, which means that P is the principal curvature vector corresponding to α, that is,

$$h_{jr}p^r=\alpha p_j. \tag{2.10}$$

Applying p^m to (2.8) and summing up m, we obtain

$$p^r\nabla_r p_j=0 \tag{2.11}$$

because of the fact that $c\neq 0$. By means of (2.2), (2.10), (2.11) and the definition of α, we can easily see that α is constant everywhere. Thus, differentiating (2.10) covariantly along M, we find

$$(\nabla_k h_{jr})p^r+h_{jr}\nabla_k p^r=\alpha\nabla_k p_j,$$

which together with (1.3) and (2.2) yield

$$\frac{c}{4}J_{jk}-h_{jr}h_{ks}J^{rs}=\alpha\nabla_k p_j. \tag{2.12}$$

If we take the symmetric part of this, then we obtain $\nabla_k p_j+\nabla_j p_k=0$ provided that $\alpha\neq 0$. But, if $\alpha=0$, then (2.12) implies $h_{jr}h_{is}{}^2J^{rs}=-(c/4)\nabla_i p_j$ with the aid of (1.3), which together with (2.8) and (2.11) give $\nabla_j p_m+\nabla_m p_j=0$. Consequently we see in any case that $h_j{}^r J_r{}^k=J_j{}^r h_r{}^k$. Thus we have the following fact:

LEMMA 1. *Let M be a cyclic-parallel real hypersurfaces of $M^n(c)$, $c\neq 0$. Then the shape operator and the induced structure tensor commute each other, that is,*

$$AJ=JA. \tag{2.13}$$

REMARK 1. Chen, Ludden and Montiel [3] proved this lemma for the case where $c<0$. The converse assertion of Lemma 1 is well known. The proof was used the theory of Riemann fibre bundles (cf. [3], [8]). But, we introduce here the other simple proof. The method is similar to that used in the previous paper [5].

From (2.13), it is easy to see that

$$h_{jr}p^r = \alpha p_j \tag{2.14}$$

by means of the properties of the almost contact metric structure. Differentiating (2.14) covariantly and taking account of (1.3), we obtain

$$(\nabla_k h_{jr})p^r - h_{jr}h_{ks}J^{rs} = \alpha_k p_j - \alpha h_{kr}J_j{}^r, \tag{2.15}$$

where $\alpha_k = \nabla_k \alpha$, which together with equations of Codazzi and (2.13) give

$$\frac{c}{2}J_{jk} + 2h_{jr}h_s{}^r J_k{}^s = \alpha_k p_j - \alpha_j p_k + 2\alpha h_{jr}J_k{}^r. \tag{2.16}$$

It means that $\alpha_k = Bp_k$ for some function B. It is easy to see that α is constant everywhere. Thus, the last equation reduces to

$$h_{ji}^2 = \alpha h_{ji} + \frac{c}{4}(g_{ji} - p_j p_i) \tag{2.17}$$

because of (2.13) and the properties of (J, g, P). Accordingly (2.15) becomes

$$(\nabla_k h_{jr})p^r = \frac{c}{4}J_{jk}. \tag{2.18}$$

LEMMA 2. *Let M be a real hypersurface satisfying* (2.13) *of* $M^n(c)$, $c \neq 0$. *Then M is of cyclic-parallel provided that* $\alpha^2 + c = 0$.

PROOF. Since we have $\alpha^2 + c = 0$, the relationships (2.14) and (2.17) tell us that M has at most two constant principal curvatures α and $\alpha/2$. Their multiplicities are denoted respectively by r and $2n-1-r$. Thus, the trace of the shape operator is given by

$$h = \frac{\alpha}{2}(2n-1+r) \tag{2.19}$$

and that of A^2 is given by

$$h_2 = \frac{\alpha^2}{4}(2n-1+3r). \tag{2.20}$$

On the other hand, it is seen from (2.17) that $h_2 = \alpha h - (\alpha^2/2)(n-1)$. Therefore, the last three equations imply that $r=1$ because of $\alpha^2 + c = 0$ and $c \neq 0$. Accordingly (2.19) and (2.20) reduces respectively to

$$h = n\alpha, \qquad h_2 = \frac{1}{2}(n+1)\alpha^2. \tag{2.21}$$

We also have the followings:

$$h_3 = \frac{1}{4}(n+3)\alpha^3, \qquad h_4 = \frac{1}{8}(n+7)\alpha^4, \tag{2.22}$$

where h_3 and h_4 denote the trace of A^3 and A^4 respectively. By using (2.21) and (2.22), it is not hard to see that

$$h_{ji}^2=\frac{3}{2}\alpha h_{ji}-\frac{\alpha^2}{2}g_{ji},$$

which together with (2.17) implies that $h_{ji}=(1/2)\alpha(g_{ji}+p_jp_i)$ because of $\alpha\neq 0$. Differentiating this covariantly, we find

$$\nabla_k h_{ji}=\frac{1}{2}\alpha\{(\nabla_k p_j)p_i+(\nabla_k p_i)p_j\}.$$

Therefore, by means of (1.3) and (2.13) we can verify that M is of cyclic-parallel. This completes the proof.

Differentiation (2.17) covariantly and making use of (1.3), we get

$$(2.23)\qquad (\nabla_k h_{jr})h_i{}^r+(\nabla_k h_{ir})h_j{}^r=\alpha\nabla_k h_{ji}+\frac{c}{4}\{(h_{kr}J_j{}^r)p_i+(h_{kr}J_i{}^r)p_j\},$$

from which, taking the skew-symmetric part with respect to indices k and j and utilizing (2.13) and (2.14),

$$h_{jr}\nabla_k h_i{}^r-h_{kr}\nabla_j h_i{}^r=\frac{c}{4}\alpha(p_kJ_{ji}-p_jJ_{ki})+\frac{c}{2}p_i(h_{kr}J_j{}^r).$$

Thus, it follows that

$$h_j{}^r\nabla_k h_{ir}-h_i{}^r\nabla_k h_{jr}=\frac{c}{4}\{p_jh_{ir}J_k{}^r-p_ih_{jr}J_k{}^r+\alpha(p_jJ_{ik}-p_iJ_{jk})\},$$

where we have used (1.5), (2.13) and (2.14). From this and (2.23), it is seen that

$$(2.24)\qquad 2h_j{}^r\nabla_k h_{ir}-\alpha\nabla_k h_{ji}=\frac{c}{4}\{-2p_i(h_{jr}J_k{}^r)+\alpha(p_jJ_{ik}-p_iJ_{jk})\}.$$

Transforming this by $h_m{}^j$ and using (2.13), (2.17) and (2.18), we obtain

$$\alpha h_j{}^r\nabla_k h_{ir}+\frac{c}{2}\nabla_k h_{ji}=\frac{c}{4}\Big\{\Big(\alpha^2+\frac{c}{2}\Big)J_{ik}p_j-\frac{c}{2}J_{kj}p_i-\alpha p_i(h_{jr}J_k{}^r)\Big\}.$$

Combining this with (2.24), it follows that

$$(\alpha^2+c)\Big\{\nabla_k h_{ji}-\frac{c}{4}(p_jJ_{ik}+p_iJ_{jk})\Big\}=0,$$

which shows that M is of cyclic-parallel because of Lemma 2.

From this fact and Lemma 1 we have

THEOREM 3. *Let M be a real hypersurface of a complex space form $M^n(c)$, $c\neq 0$. Then M is of cyclic-parallel if and only if $AJ=JA$.*

REMARK 2. It is obvious that if M is of cyclic-parallel, then the Ricci tensor is cyclic-parallel because of (1.3), (1.6) and (2.10).

3. Homogeneous hypersurfaces.

It is known that the complete and simply connected complex space form $M^n(c)$ consists of a complex projective space P_nC, a complex Euclidean space C_n or a complex hyperbolic space H_nC, according as $c>0$, $c=0$ or $c<0$. Some standard examples given by [9], [12], [14] of real hypersurfaces $M^n(c)$, $c\neq 0$ whose second fundamental form are cyclic-parallel are introduced. In a complex Euclidean space C^{n+1} equipped with Hermitian form ϕ, the Euclidean metric of C^{n+1} which is identified with R^{2n+2} is given by $\mathrm{Re}\,\phi$. The unit sphere $S^{2n+1}=\{z\in C^{n+1}: \phi(z, z)=1\}$ is denoted.

First of all, examples of real hypersurfaces of P_nC are considered. For any positive number r a hypersurface $N_0(2n, r)$ of S^{2n+1} is defined by

$$N_0(2n, r)=\left\{(z_1, \cdots, z_{n+1})\in S^{2n+1}\subset C^{n+1}: \sum_{j=1}^{n}|z_j|^2=r|z_{n+1}|^2\right\}.$$

For an integer m $(2\leqq m\leqq n-1)$ and a positive number s, a hypersurface $N(2n, m, s)$ of S^{2n+1} is defined by

$$N(2n, m, s)=\left\{(z_1, \cdots, z_{n+1})\in S^{2n+1}\subset C^{n+1}: \sum_{j=1}^{m}|z_j|^2=s\sum_{j=m+1}^{n+1}|z_j|^2\right\}.$$

Then, for the projection π of the Hopf-fibration S^{2n+1} onto P_nC, $M_0(2n-1, r)=\pi(N_0(2n, r))$ and $M(2n-1, m, s)=\pi(N(2n, m, s))$ $(n\geqq 3)$ are examples of real hypersurfaces of P_nC whose shape operator and the induced structure tensor commute each other. It is known [14] that $M_0(2n-1, r)$ and $M(2n-1, m, s)$ are both compact connected real hypersurfaces of P_nC with constant two or three distinct principal curvatures respectively, which are said to be of type A_1 and A_2 respectively. In [13], it is proved that $M_0(2n-1, r)$ and $M(2n-1, m, s)$ are only hypersurfaces of P_nC satisfying $AJ=JA$.

In the next place, the example of real hypersurfaces of H_nC defined by Montiel [11] and Montiel and Romero [12] is introduced. In C^{n+1} with standard basis, a Hermitian form ϕ is defined by

$$\phi(z, w)=-z_0\bar{w}_0+\sum_{k=1}^{n}z_k\bar{w}_k.$$

where $z=(z_0, \cdots, z_n)$ and $w=(w_0, \cdots, w_n)$ are in C^{n+1}. Let H_1^{2n+1} be a real hypersurface of the Minkoski space C_1^{n+1} defined by

$$H_1^{2n+1}=\{z\in C_1^{n+1}: \phi(z, z)=-1\},$$

and let $\bar{G}$ be a semi-Riemannian metric of H_1^{2n+1} induced from the complex Lorentzian metric $\mathrm{Re}\,\phi$ of C_1^{n+1}. Then $(H_1^{2n+1}, \bar{G})$ is the Lorentzian manifold of constant curvature -1, which is called an anti-de Sitter space.

Let r and s be integers with $r+s=n-1$ and $t\in R$ with $0<t<1$. We consider a Lorentzian hypersurface $N_{r+s}(t)$ of H_1^{2n+1} defined by the following:

$$N_{r+s}(t)=\left\{(z_0, \cdots, z_n)\in H_1^{2n+1} : t(-|z_0|^2)+\sum_{j=1}^{r}|z_j|^2=-\sum_{k=r+1}^{n}|z_k|^2\right\}$$

and a Lorentzian hypersurface of H_1^{2n+1} is given by

$$N_n=\{(z_0, \cdots, z_n)\in H_1^{2n+1} : |z_0-z_1|=1\}.$$

Since it is known that H_1^{2n+1} is a principal S^1-bundle over a complex hyperbolic space with projection $\bar{\pi} : H_1^{2n+1}\to H_nC$, and $N_{r+s}(t)$ and N_n are S^1-invariant, we see that $M_{r+s}(t)=\pi(N_{r+s}(t))$ and $M_n=\pi(N_n)$ are real hypersurfaces of H_nC, where $\pi : N_{r+s}(t)\to M_{r+s}(t)$ and $\pi : N_n\to M_n$ are semi-Riemannian submersions which are compatible with S^1-fibration. It is seen that $M_{r+s}(t)$ and M_n are complete connected real hypersurfaces of H_nC with constant two or three distinct principal curvatures, which are said to be of type A ([9]). In [12], it is proved that $M_{r+s}(t)$ and M_n are only complete hypersurfaces of H_nC satisfying $AJ=JA$. Thus, by combining above facts and Theorem 3, we obtain the following classifications.

THEOREM 4. *$M_0(2n-1, r)$, $M(2n-1, m, s)$, $M_{r+s}(t)$ and M_n are only complete and connected cyclic-parallel real hypersurfaces of $M^n(c)$, $c\neq 0$.*

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