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ON COMPLEX TORI WITH MANY ENDOMORPHISMS

By

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The endomorphism ring of a complex torus $\boldsymbol{T}$ of dimension n is a free module of rank$\leqq 2n^2$ as a $\boldsymbol{Z}$-module. When $\boldsymbol{T}$ is an abelian variety it is well-known that if the rank is equal to $2n^2$, $\boldsymbol{T}$ is isogenous to the direct sum of n copies of an elliptic curve with complex multiplication. We will prove a similar result in a more general form, that is, let $\boldsymbol{T}$ and $\boldsymbol{T}'$ be two complex tori of dimension n and n' respectively, and if the $\boldsymbol{Z}$-module of all homomorphisms of $\boldsymbol{T}$ into $\boldsymbol{T}'$ is of rank $2nn'$, then $\boldsymbol{T}$ and $\boldsymbol{T}'$ are isogenous to the direct sums of n and n' copies of an elliptic curve (Theorem 1-3). Next let $\boldsymbol{T}$ be a complex torus of dimension 2 and put $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})=\mathrm{End}(\boldsymbol{T})\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$. Then using the types of $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})$ we will classify all $\boldsymbol{T}$'s with a non-trivial endomorphism ring. The result is given in the last part of § 4. A complex torus $\boldsymbol{T}$ of dimension 2 which is not simple is an abelian variety, if and only if $\boldsymbol{T}$ is isogenous to the direct sum of two elliptic curves. On the other hand a simple torus $\boldsymbol{T}$ of dimension 2 such that $\mathrm{End}(\boldsymbol{T})$ is not isomorphic to $\boldsymbol{Z}$ is an abelian variety if and only if $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})$ contains some real quadratic field over $\boldsymbol{Q}$. This is proved in § 5.

NOTATIONS. We denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$, respectively, the ring of rational integers, the field of rational numbers, real numbers and complex numbers. For a ring R, $M(n\times m, R)$ denotes the R-module composed of all matrices with n rows and m columns with coefficients in R. When $n=m$, it is the R-algebra of all square matrices of size n. We simply denote it by $M(n, R)$. The group of all invertible elements of $M(n, R)$ is denoted by $GL(n, R)$.

Let $\boldsymbol{T}$ and $\boldsymbol{T}'$ be two complex tori. We denote by $\mathrm{Hom}(\boldsymbol{T}, \boldsymbol{T}')$ the set of all homomorphisms of $\boldsymbol{T}$ into $\boldsymbol{T}'$ and put $\mathrm{End}(\boldsymbol{T})=\mathrm{Hom}(\boldsymbol{T}, \boldsymbol{T})$. We put $\mathrm{Hom}^{\boldsymbol{Q}}(\boldsymbol{T}, \boldsymbol{T}')=\mathrm{Hom}(\boldsymbol{T}, \boldsymbol{T}')\otimes\boldsymbol{Q}$ and $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})=\mathrm{End}(\boldsymbol{T})\otimes\boldsymbol{Q}$. $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})$ is naturally considered as an algebra over $\boldsymbol{Q}$. $\boldsymbol{T}$ and $\boldsymbol{T}'$ are called isogenous and denoted by $\boldsymbol{T}\sim\boldsymbol{T}'$ if they are of the same dimension and there exists a homomorphism λ of the one onto the other; such a λ is called an isogeny. "$\sim$" is an equivalence relation. If $\boldsymbol{T}_1$ and $\boldsymbol{T}_1'$ are complex tori which are isogenous $\boldsymbol{T}$ and $\boldsymbol{T}'$ respectively, then $\mathrm{Hom}^{\boldsymbol{Q}}(\boldsymbol{T}_1, \boldsymbol{T}_1')$ is isomorphic to $\mathrm{Hom}^{\boldsymbol{Q}}(\boldsymbol{T}, \boldsymbol{T}')$ and $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T}_1)$ is

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isomorphic to $\mathrm{End}^{Q}(\boldsymbol{T})$ as a $\boldsymbol{Q}$-algebra.

Let $\boldsymbol{G}$ be a lattice subgroup of $\boldsymbol{C}^n$ and $(g_1, \cdots, g_{2n})$ its base. Then the matrix $G=(g_1, \cdots, g_{2n})\in M(n\times 2n, \boldsymbol{C})$ is called the period matrix of the complex torus $\boldsymbol{C}^n/\boldsymbol{G}$. We shall often denoted by $\boldsymbol{C}^n/G$ the complex torus $\boldsymbol{C}^n/\boldsymbol{G}$.

§1. Complex tori with endomorphism rings of the maximal rank.

Let $\boldsymbol{T}$ and $\boldsymbol{T}'$ be two complex tori of dimension n and n' respectively.

THEOREM 1-1. Hom$(\boldsymbol{T}, \boldsymbol{T}')$ *is a free abelian group whose rank is at most* $2nn'$.

PROOF. We put $\boldsymbol{T}=\boldsymbol{E}/\boldsymbol{G}$ and $\boldsymbol{T}'=\boldsymbol{E}'/\boldsymbol{G}'$, where E, E' are complex linear spaces and G, G' are respectively their lattice subgroups. Take a $\boldsymbol{C}$-base $(g_1, \cdots, g_n)$ of $\boldsymbol{E}$ which is also a part of a $\boldsymbol{Z}$-base of $\boldsymbol{G}$ and let H_1 the subgroup of $\boldsymbol{G}$ generated by $g_1, \cdots, g_n$. If λ is an element of Hom$(\boldsymbol{T}, \boldsymbol{T}')$, λ naturally induces a linear map L_λ of $\boldsymbol{E}$ to $\boldsymbol{E}'$. Then making correspond to λ the homomorphism of H_1 into $\boldsymbol{G}'$ which maps $(g_1, \cdots, g_n)$ to $(L_\lambda(g_1), \cdots, L_\lambda(g_n))$, we get an injective homomorphism of Hom$(\boldsymbol{T}, \boldsymbol{T}')$ into Hom$(H_1, \boldsymbol{G}')$. Since Hom$(H_1, \boldsymbol{G}')$ is a free abelian group of rank $2nn'$, Hom$(\boldsymbol{T}, \boldsymbol{T}')$ which is isomorphic to a subgroup of Hom$(H_1, \boldsymbol{G}')$ is a free abelian group whose rank is at most $2nn'$. (q. e. d.)

Let $\boldsymbol{T}$ and $\boldsymbol{T}'$ be the direct sums of r and r' complex tori $\boldsymbol{T}_1, \cdots, \boldsymbol{T}_r$ and $\boldsymbol{T}'_1, \cdots, \boldsymbol{T}'_{r'}$ respectively. Then, Hom$(\boldsymbol{T}, \boldsymbol{T}')$ is isomorphic to the direct sum of all Hom$(\boldsymbol{T}_i, \boldsymbol{T}'_{i'})$'s ($i=1, 2, \cdots, r$ and $i'=1, 2, \cdots, r'$). If $\boldsymbol{T}=\boldsymbol{T}'$, they are isomorphic as rings, where for two elements $(\lambda_{ii'})$, $(\mu_{ii'})$ of $\bigoplus_{i,i'}$Hom$(\boldsymbol{T}_i, \boldsymbol{T}_{i'})$ ($\lambda_{ii'}$ and $\mu_{ii'}$ are elements of Hom$(\boldsymbol{T}_i, \boldsymbol{T}_{i'})$.), we define the product of them by $(\sum_{j=1}^{r}\lambda_{ji'}\circ\mu_{ij})\in\bigoplus_{i,i'}Hom(\boldsymbol{T}_i, \boldsymbol{T}_{i'})$. Especially when $\boldsymbol{T}_1=\boldsymbol{T}_2=\cdots=\boldsymbol{T}_r$, End$(\boldsymbol{T})$ is isomorphic to $M(r, \mathrm{End}(\boldsymbol{T}_1))$.

Let C be an elliptic curve with complex multiplication, that is, complex torus of dimension 1 with an endomorphism ring of rank 2, and let $\boldsymbol{T}$ and $\boldsymbol{T}'$ be complex tori which are isogenous to the direct sums of n and n' copies of C respectively. Then the rank of Hom$(\boldsymbol{T}, \boldsymbol{T}')$ is clearly $2nn'$. We shall prove the converse is true.

THEOREM 1-2. *Let* $\boldsymbol{T}$ *and* $\boldsymbol{T}'$ *be complex tori of dimension* n *and* n' *respectively. If the rank of* Hom$(\boldsymbol{T}, \boldsymbol{T}')$ *is* $2nn'$, $\boldsymbol{T}$ *and* $\boldsymbol{T}'$ *are respectively isogenous to the direct sums of* n *and* n' *copices of an elliptic curve* C *with complex multiplication.*

PROOF. Notation being as in the proof of Theorem 1-1; choose a proper $\boldsymbol{C}$-base of $\boldsymbol{E}$ and a proper $\boldsymbol{Z}$-base of $\boldsymbol{G}$, and we may assume that the period matrix of $\boldsymbol{T}$ is $(1_n, T)$ where 1_n is the unit matrix of size n and T is an element of $M(n, \boldsymbol{C})$ such that the imaginary part of T is a regular matrix. Similarly we may assume that the period matrix of $\boldsymbol{T}'$ is $(1_{n'}, T')$ for some matrix $\boldsymbol{T}'$ of size n' which satisfies the same condition.

Now considering $\mathrm{Hom}(\boldsymbol{T}, \boldsymbol{T}')$ to be a subgroup of $\mathrm{Hom}(H_1, \boldsymbol{G}')$, since they are of the same rank, there exists an integer λ such that $\lambda(\mathrm{Hom}(H_1, \boldsymbol{G}'))\subset \mathrm{Hom}(\boldsymbol{T}, \boldsymbol{T}')$. In other words, for any $S\in M(2n'\times n, \boldsymbol{Z})$ there exist $\omega\in M(n'\times n, \boldsymbol{C})$ and $\Omega\in M(2n'\times 2n, \boldsymbol{Z})$ such that

$$\omega 1_n=(1_{n'}\ T')\lambda S \quad \text{and} \quad \omega(1_n\ T)=(1_{n'}\ T')\Omega\,.$$

For any $\alpha\in M(n'\times n, \boldsymbol{Z})$, putting $S=\begin{pmatrix}\alpha\\ 0\end{pmatrix}$, there exists $\Omega=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ $(A, B, C, D\in M(n'\times n, \boldsymbol{Z}))$ such that

$$\lambda\alpha(1_n, T)=(1_{n'}, T')\Omega=(A+T'C, B+T'D)\,,$$

and especially $\lambda\alpha T=B+T'D$. If we denote by $\mathbf{Im}\,T$ and $\mathbf{Re}\,T$ the imaginary part of T and the real part of T respectively, we have i) $\lambda\alpha(\mathbf{Im}\,T)=(\mathbf{Im}\,T')D$ and ii) $\lambda\alpha(\mathbf{Re}\,T)=B+(\mathbf{Re}\,T')D$. Therefore for any element α of $M(n'\times n, \boldsymbol{Z})$ we have

i′) $(\mathbf{Im}\,T')^{-1}(\lambda\alpha)(\mathbf{Im}\,T)\in M(n'\times n, \boldsymbol{Z})$

ii′) $(\lambda\alpha)(\mathbf{Re}\,T)-(\mathbf{Re}\,T')(\mathbf{Im}\,T')^{-1}(\lambda\alpha)(\mathbf{Im}\,T)\in M(n'\times n, \boldsymbol{Z})$.

Put $(\mathbf{Im}\,T')^{-1}=(\beta_{pr})$, $\alpha=(\alpha_{rs})$, $\mathbf{Im}\,T=(a_{sr})$, and i′) implies

$$\lambda\sum_{r=1}^{n'}\sum_{s=1}^{n}\beta_{pr}\alpha_{rs}a_{sq}\in\boldsymbol{Z}$$

for any p, q $(p=1, \cdots, n', q=1, \cdots, n)$. If we put α to be the matrix whose (r, s)-component is 1 and the others are all 0, we have $\lambda\beta_{pr}a_{sq}\in Z$ for any p, q, r, s. Especially putting $p=r=1$, we have $\lambda\beta_{11}a_{sq}\in\boldsymbol{Z}$ for any s, q. Therefore there exist a real number a_1 which is independent of s, q and integers a_{sq}^* $(s, q=1, 2, \cdots, n)$ such that $a_{sq}=a_1a_{sq}^*$. Put $T_1=(a_{sq}^*)\in M(n, \boldsymbol{Z})$, and we have $\mathbf{Im}\,T=a_1T_1$, where $a_1\neq 0$ and $\det T_1\neq 0$. Similarly there exist $b'\in\boldsymbol{R}$ and $T_0'\in M(n', \boldsymbol{Z})$ such that $(\mathbf{Im}\,T')^{-1}=b'T_0'$. Putting $a_1'=b'^{-1}(\det T_0')^{-1}$ and $T_1'=(\det T_0')T_0'^{-1}$, we have $\mathbf{Im}\,T'=a_1'T_1'$ where a_1' is a real number T_1' is an element of $M(n', \boldsymbol{Z})$. Now we have $T=\mathbf{Re}\,T+\sqrt{-1}\,a_1T_1$. Considering the isogeny whose rational representation is $\begin{pmatrix}1_n & 0\\ 0 & T_1^{-1}\end{pmatrix}$, we can see that $\boldsymbol{T}$ is isogenous to $\boldsymbol{C}^n/(1_n, (\mathbf{Re}\,T)T_1^{-1}+\sqrt{-1}\,a_11_n)$. So we may assume that $\mathbf{Im}\,T=a_11_n$. And similarly we may assume that $\mathbf{Im}\, T'=a_1'1_{n'}$. Put $\mu=a_1a_1'^{-1}\lambda$, and we have by ii')

$$(\lambda\alpha)(\mathbf{Re}\, T)-\mu(\mathbf{Re}\, T')\alpha\in M(n'\times n, \boldsymbol{Z})$$

for any α. If we put $\mathbf{Re}\, T=(c_{sq})$, $\mathbf{Re}\, T'=(d_{pr})$ and $\alpha=(\alpha_{rs})$, we have

$$\lambda\sum_{s=1}^{n}\alpha_{ps}c_{sq}-\mu\sum_{r=1}^{n'}d_{pr}\alpha_{rq}\in\boldsymbol{Z}$$

for $p=1, \cdots, n$, $s=1, \cdots, n'$. Again putting α to be the matrix whose (r, q)-component is 1 and the others are all 0, we have A) $\lambda c_{sq}\in Z$, if $s\neq q$, B) $\mu d_{pr}\in Z$, if $p\neq r$, and C) $\lambda c_{ss}-\mu d_{rr}\in\boldsymbol{Z}$, for any p, q, r, s. Therefore we have $\lambda(c_{sq})-\mu d_{11}1_n\in M(n, \boldsymbol{Z})$ and $\mu(d_{pr})-\lambda c_{11}1_{n'}\in M(n', \boldsymbol{Z})$. Put $T_2=\lambda(c_{sq})-\mu d_{11}1_n$ and $c=\mu d_{11}$, and we have $\mathbf{Re}\, T=\lambda^{-1}(c1_2+T_2)$. So putting $z=\lambda^{-1}c+\sqrt{-1}\, a_1$, we have $T=z1_n+\lambda^{-1}T_2$. Consider the isogeny whose rational representation is $\begin{pmatrix}1_n & -\lambda^{-1}T_2\\ 0 & 1_n\end{pmatrix}$, and we can see that $\boldsymbol{T}$ is isogenous to $\boldsymbol{C}^n/(1_n, z1_n)$ which is clearly isogenous to the direct sum of n copies of $C=\boldsymbol{C}/(1, z)$. Similarly $\boldsymbol{T}'$ is isogenous to the direct sum of n' copies of some complex torus C' of dimension 1. Since $\mathrm{Hom}(\boldsymbol{T}, \boldsymbol{T}')$ is isomorphic to the direct sum of nn' copies of $\mathrm{Hom}(C, C')$, the rank of $\mathrm{Hom}(C, C')$ is 2, hence C is an elliptic curve with complex multiplication which is isomorphic to C'. (q. e. d.)

§ 2. Period matrices of complex tori with many endomorphisms.

Let $\boldsymbol{T}$ be a complex torus whose $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})$ contains a division sub-algebra D which contains $\boldsymbol{Q}$ properly. Let Z be the center of D and K one of the maximal commutative subfields of D and denote the dimensions of the vector spaces D, K and Z over $\boldsymbol{Q}$ by d, e and f respectively. Then we have $d/f=(e/f)^2$, in other words $df=e^2$. On the other hand, considering a rational representation of D, the linear space $\boldsymbol{Q}^{2n}$ can be regarded as a D-module. Since D is a division algebra, a D-module is always free, hence denoting by r the rank of the module over D, we have $rd=2n$. Now the following theorem has been proved.

THEOREM 2-1. *Let D be a division algebra contained in* $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})$. *If we donote by d, e and f, respectively, the dimensions over $\boldsymbol{Q}$ of D, one of the maximal subfield of D and the center of D, we have*

i) $df=e^2$

ii) $f\,|\,e\,|\,d\,|\,2n$ (where $a|b$ means a divides b.)

COROLLARY 2-2. *Let n be a positive odd integer which is square-free, and $\boldsymbol{T}$ a complex torus of dimension n. Then any division algebra which is contained in*

$\mathrm{End}^{Q}(\boldsymbol{T})$ *is commutative.*

PROOF. Notations being as in Theorem 2-1, $(e/f)^2=d/f$ divides $2n$. Hence $e/f=1$, that is, D is commutative. (q. e. d.)

Next we shall inquire into the period matrix of $\boldsymbol{T}$.

THEOREM 2-3. *Let $\boldsymbol{T}=\boldsymbol{E}/\boldsymbol{G}$ be a complex torus of dimension n such that $\mathrm{End}^{Q}(\boldsymbol{T})$ contains a division algebra D which contains $\boldsymbol{Q}$ properly. Take any element ϕ of D which is not contained in $\boldsymbol{Q}$. Choosing an adequate $\boldsymbol{C}$-base of $\boldsymbol{C}$-vector space $\boldsymbol{E}$, the analytic representation of ϕ is a diagonal matrix*

$$\begin{pmatrix}\alpha_1 & & 0\\ & \ddots & \\ 0 & & \alpha_n\end{pmatrix}$$

where α_i is the image of ϕ by an isomorphism of $\boldsymbol{Q}(\phi)$ into $\boldsymbol{C}$ $(i=1, 2, \cdots, n)$.

And put $h=[\boldsymbol{Q}(\phi):\boldsymbol{Q}]$, $s=2n/h$ and

$$\Phi=\begin{pmatrix}1 & \alpha_1 & \alpha_1^2\cdots\alpha_1^{h-1}\\ \vdots & \vdots & \vdots\\ 1 & \alpha_n & \alpha_n^2\cdots\alpha_n^{h-1}\end{pmatrix}\in M(n\times h, \boldsymbol{C})\,.$$

And put

$$G(g_{ij})=\left(\begin{pmatrix}g_{11} & & 0\\ & \ddots & \\ 0 & & g_{1n}\end{pmatrix}\Phi\ \begin{pmatrix}g_{21} & & 0\\ & \ddots & \\ 0 & & g_{2n}\end{pmatrix}\Phi\cdots\begin{pmatrix}g_{s1} & & 0\\ & \ddots & \\ 0 & & g_{sn}\end{pmatrix}\Phi\right)$$

where g_{ij} $(i=1, \cdots, s,\ j=1, \cdots, n)$ are somd given complex numbers. Then there exists ns complex numbers g_{ij} such that $\boldsymbol{T}$ is isogenous to the complex torus $\boldsymbol{T}(g_{ij})$ whose period matrix is $G(g_{ij})$.

PROOF. Let ω be an analytic representation of ϕ and Ω a rational representation. Since the minimal polynomial f of Ω is also the minimal polynomial of ϕ when $\boldsymbol{Q}(\phi)$ is regarded as an algebraic field over $\boldsymbol{Q}$, f is irreducible. Clearly $f(\omega)=0$, so that the minimal polynomial of ω has no multiple root. Here choosing an adequate $\boldsymbol{C}$-base of $\boldsymbol{E}$,

$$\omega=\begin{pmatrix}\alpha_1 & & 0\\ & \ddots & \\ 0 & & \alpha_n\end{pmatrix}$$

where $\alpha_1, \cdots, \alpha_n$ are roots of the algebraic equation $f(x)=0$. On the other hand the characteristic polynomial F of Ω is s-th power of f. Therefore if we consider Ω to be a linear transformation on $\boldsymbol{Q}^{2n}$, there exists an element P of $GL(2n, \boldsymbol{Q})\cap M(2n, \boldsymbol{Z})$ such that

$$P^{-1}\Omega P=\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_s \end{pmatrix}$$

where
$$A_1=A_2=\cdots=A_s=\begin{pmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & \ddots & \vdots & -a_1 \\ 0 & \ddots & 0 & \vdots \\ \vdots & \ddots & & \vdots \\ 0 & 0 & 1 & -a_{h-1} \end{pmatrix}\in GL(h, \boldsymbol{Q}),$$

and $f(x)=x^h+a_{h-1}x^{h-1}+\cdots+a_0$. Considering the isogeny whose rational representation is P, we may assume that the analytic representation ω of ϕ is a diagonal matrix $\begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$ and the rational representation Ω of ϕ is $\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_s \end{pmatrix}$. Then let G be the period matrix, and we have $\omega G=G\Omega$. We only have to compare each component of ωG with the corresponding component of $G\Omega$ to complete the proof. (q. e. d.)

Conversely suppose complex numbers $\{g_{ij}\}$ are given. Is $G(g_{ij})$ the period matrix of some complex torus? Since $\begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix}\begin{pmatrix} G \\ \bar{G} \end{pmatrix}=\begin{pmatrix} G \\ \bar{G} \end{pmatrix}\Omega$, $\alpha_1, \cdots, \alpha_n$ have to satisfy the following condition (♯);
(♯) the image of ϕ by any isomorphism of $\boldsymbol{Q}(\phi)$ into $\boldsymbol{C}$ appears just s times in $\alpha_1, \cdots, \alpha_n, \bar{\alpha}_1, \cdots, \bar{\alpha}_n$ (where $\bar{\alpha}$ means the complex conjugate of α).

THEOREM 2-4. *We assume $\alpha_1, \cdots, \alpha_n$ satisfy the condition (♯). Then if g_{ij} $(i=1, \cdots, s, j=1, \cdots, n)$ are generally given, $G(g_{ij})$ is the period matrix of some complex torus. (That is, the subset in $\boldsymbol{C}^{sn}$ composed of all $\{g_{ij}\}$ such that $G(g_{ij})$ is a period matrix is open dense in $\boldsymbol{C}^{sn}$.)*

PROOF. Let X_{ij} $(i=1, \cdots, s, j=1, \cdots n)$ be ns variables, and we only have to prove that $\det\begin{pmatrix} G(X_{ij}) \\ \overline{G(X_{ij})} \end{pmatrix}=0$ is a non-trivial equation. Let $\phi_1, \cdots, \phi_h$ be the images of ϕ by all the isomorphisms of $\boldsymbol{Q}(\phi)$ into $\boldsymbol{C}$, and put

$$\Phi=\begin{pmatrix} 1 & \phi_1 \cdots \phi_1^{h-1} \\ \vdots & \vdots \quad\quad \vdots \\ 1 & \phi_n \cdots \phi_h^{h-1} \end{pmatrix}.$$

Then we have

$$\det\begin{pmatrix} G(X_{ij}) \\ \overline{G(X_{ij})} \end{pmatrix}=\begin{vmatrix} X_{11}^*\Phi & \cdots & X_{1s}^*\Phi \\ \vdots & & \vdots \\ X_{s1}^*\Phi & \cdots & X_{ss}^*\Phi \end{vmatrix}=\begin{vmatrix} X_{11}^* & \cdots & X_{1s}^* \\ \vdots & & \vdots \\ X_{s1}^* & \cdots & X_{ss}^* \end{vmatrix}(\det\Phi)^s$$

where X_{ij}^* $(i, j=1, 2, \cdots, s)$ are diagonal matrices such that all X_{ij} and all $\bar{X}_{ij}$ appear once and only once in their diagonal components. Since $\det\Phi\neq 0$, we only have to prove the following lemma to complete the proof.

LEMMA 2-5. *Let $f(x_1, \cdots, x_m, y_1, \cdots, y_m)$ be a polynomial of $2m$ variables $x_1, \cdots, x_m, y_1, \cdots, y_m$ with coefficients in $\boldsymbol{C}$. If $f(z_1, \cdots z_m, \bar{z}_1, \cdots, \bar{z}_m)=0$ for any m complex numbers $z_1, \cdots, z_m$, then $f=0$ as a polynomial.*

PROOF. It is easily seen that we may assume $m=1$. Put $f(x, y)=F_p(x)y^p+\cdots+F_0(x)$. If $f(z, \bar{z})=0$, $\bar{z}$ is a root of the algebraic equation $F_p(z)y^p+\cdots+F_0(z)=0$ with an unknown y. If $p>0$, $\bar{z}$ is locally a holomorphic function of z on an open subset in $\boldsymbol{C}$. That is a contradiction. Therefore $p=0$. Then it is clear that $f=0$ since $F_0(z)=0$ for any z. (q. e. d.)

§ 3. Invariant subtori.

Let $\boldsymbol{T}$ be a complex torus and $\boldsymbol{T}'$ its subtorus. We call $\boldsymbol{T}'$ invariant throughout this paper if the image of $\boldsymbol{T}'$ by any endomorphism of $\boldsymbol{T}$ is contained in $\boldsymbol{T}'$. Of course $\boldsymbol{T}$ itself and $\{0\}$ are invariant subtori. We call each of them a trivial invariant subtorus.

THEOREM 3-1. *If a complex torus $\boldsymbol{T}$ has no non-trivial invariant subtorus. Then $\boldsymbol{T}$ is isogenous to the direct sum of some copies of a simple torus. (A complex torus is called simple if it has no subtorus but itself and $\{0\}$.)*

PROOF. Let $\boldsymbol{T}'$ be a simple subtorus which is not $\{0\}$. The set $\Lambda=\{\lambda(\boldsymbol{T}') \mid \lambda\in\mathrm{End}(\boldsymbol{T})\}$ is a finite set. In fact, since any $\lambda(\boldsymbol{T}')$ is simple, if $\Lambda'=\{\lambda_1(\boldsymbol{T}'), \cdots, \lambda_m(\boldsymbol{T}')\}$ be a subset of Λ ($\lambda_i(\boldsymbol{T}')\neq\lambda_j(\boldsymbol{T}')$ if $i\neq j$), $T_0=\lambda_1(\boldsymbol{T}')+\cdots+\lambda_m(\boldsymbol{T}')$ is isogenous to the direct sum $\lambda_1(\boldsymbol{T}')\oplus\cdots\oplus\lambda_m(\boldsymbol{T}')$ which is isogenous to the direct sum of m copies of $\boldsymbol{T}'$. So Λ is a finite set. Put $\Lambda'=\Lambda$ especially, and $\boldsymbol{T}_0=\lambda_1(\boldsymbol{T}')+\cdots+\lambda_m(\boldsymbol{T}')$ is an invariant subtorus which is not $\{0\}$. Therefore $\boldsymbol{T}_0=\boldsymbol{T}$, that is, $\boldsymbol{T}$ is isogenous to the direct sum of m copies of a simple subtorus $\boldsymbol{T}'$. (q. e. d.)

THEOREM 3-2. *Let $\boldsymbol{T}'$ be an invariant svbtorus of a complex torus $\boldsymbol{T}$. Then we have*

i) $\mathrm{rank}_{\boldsymbol{Z}}\mathrm{End}(\boldsymbol{T})\leqq\mathrm{rank}_{\boldsymbol{Z}}\mathrm{End}(\boldsymbol{T}/\boldsymbol{T}')+\mathrm{rank}_{\boldsymbol{Z}}\mathrm{Hom}(\boldsymbol{T}, \boldsymbol{T}')$

ii) $\mathrm{rank}_{\boldsymbol{Z}}\mathrm{End}(\boldsymbol{T})\leqq\mathrm{rank}_{\boldsymbol{Z}}\mathrm{End}(\boldsymbol{T}')+\mathrm{rank}_{\boldsymbol{Z}}\mathrm{Hom}(\boldsymbol{T}/\boldsymbol{T}', \boldsymbol{T})$.

PROOF. We define an homomorphism $\Phi:\mathrm{End}(\boldsymbol{T})\to\mathrm{End}(\boldsymbol{T}')$ by the natural restriction. It is clear that the kernel of Φ can be considered to be a subset of $\mathrm{Hom}(\boldsymbol{T}/\boldsymbol{T}', \boldsymbol{T})$, so ii) is proved. Considering similarly the natural homomorphism

$\Phi': \mathrm{End}(\boldsymbol{T}) \to \mathrm{End}(\boldsymbol{T}/\boldsymbol{T}')$, we have i). (q. e. d.)

COROLLARY 3-3. *Let* $\boldsymbol{T}$ *be a complex torus of dimension* n. *If* $\mathrm{rank}_{\boldsymbol{Z}}\mathrm{End}(\boldsymbol{T}) > 2n^2-2n+2$, *there exists an integer* $m>1$ *such that* $\boldsymbol{T}$ *is isogenous to the direct sum of* m *copies of a simple torus.*

PROOF. Let $\boldsymbol{T}'$ be an invariant subtorus and k its dimension. By ii) we have $2n^2-2n+2<\mathrm{rank}_{\boldsymbol{Z}}\mathrm{End}(\boldsymbol{T}) \leqq \mathrm{rank}_{\boldsymbol{Z}}\mathrm{End}(\boldsymbol{T}')+\mathrm{rank}_{\boldsymbol{Z}}\mathrm{Hom}(\boldsymbol{T}/\boldsymbol{T}', \boldsymbol{T}) \leqq 2k^2+2(n-k)n$. So we have $k=0$ or n. On the other hand if T is simple, $\mathrm{rank}_{\boldsymbol{Z}}\mathrm{End}(\boldsymbol{T}) \leqq 2n$. Therefore $\boldsymbol{T}$ is isogenous to the direct sum of m copies of a simple torus for some $m>1$. (q. e. d.)

We will use the corollary to prove the following proposition which is a special case of Theorem 1-2

PROPOSITION. *Let* $\boldsymbol{T}$ *be complex torus of dimension* n. *If the rank of* $\mathrm{End}(\boldsymbol{T})$ *is* $2n^2$, $\boldsymbol{T}$ *is isogenous to the direct sum of* n *copies of an elliptic curve* C *with complex multiplication.*

PROOF. We may assume $n>1$. Then since $\mathrm{rank}_{\boldsymbol{Z}}\mathrm{End}(\boldsymbol{T})=2n^2>2n^2-2n-2$, $\boldsymbol{T}$ is isogenous to the direct sum of some copies of a simple torus $\boldsymbol{T}'$. Let r be the dimension of $\boldsymbol{T}'$, and $\mathrm{rank}_{\boldsymbol{Z}}\mathrm{End}(\boldsymbol{T})=\mathrm{rank}_{\boldsymbol{Z}}M(n/r, \mathrm{End}(\boldsymbol{T}'))$, therefore $2n^2 \leqq (n/r)^2(2r)=2n^2/r$. So $r=1$ and $\mathrm{rank}_{\boldsymbol{Z}}\mathrm{End}(\boldsymbol{T}')=2$. (q. e. d.)

REMARK. Let $\boldsymbol{T}$ and $\boldsymbol{T}_1$ be two complex tori and $\boldsymbol{T}'$ and $\boldsymbol{T}_1'$ their subtori respectively. We call the pair $(\boldsymbol{T}', \boldsymbol{T}_1')$ I-pair if the image of $\boldsymbol{T}'$ by any homomorphism of $\boldsymbol{T}$ into $\boldsymbol{T}_1$ is contained in $\boldsymbol{T}_1'$. If $\boldsymbol{T}$ and $\boldsymbol{T}_1$ have no non-trivial I-pair, $\boldsymbol{T}_1$ is isogenous to the direct sum of copies of a simple torus. And we have equations which are similar to i) and ii) in Theorem 3-2. Therefore if $\mathrm{Hom}(\boldsymbol{T}, \boldsymbol{T}_1)$ is of the maximal rank, $\boldsymbol{T}_1$ is isogenous to the direct sum of copies of an elliptic curve. Considering dual tori, we can see that $\boldsymbol{T}$ is also isogenous to the direct sum of copies of an elliptic curve. Thus Theorem 1-2 itself can be proved.

Now let $\boldsymbol{T}$ be a complex torus such that a division algebra D is contained in $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})$ as a subalgebra. If $\boldsymbol{T}'$ is a non-trivial invariant subtorus, Φ and Φ' in the proof of Theorem 3-2 induce the following $\boldsymbol{Q}$-algebra homomorphisms;

$$\Phi^{\boldsymbol{Q}}: \mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T}) \to \mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T}')$$

$$\Phi'^{\boldsymbol{Q}}: \mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T}) \to \mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T}/\boldsymbol{T}') .$$

Φ^{Q} is injective on D. In fact, if not, there exists an element of D such that $\Phi^{Q}(\phi)=0$ then $\phi(\boldsymbol{T}')=\{0\}$. But such a ϕ cannot be an isogeny. Similarly Φ'^{Q} is injective on D, too. Hence we may consider D a subalgebra of $\mathrm{End}^{Q}(\boldsymbol{T}')$ and $\mathrm{End}^{Q}(\boldsymbol{T}/\boldsymbol{T}')$.

THEOREM 3-3. *Let $\boldsymbol{T}$ be a complex torus of dimension n. If $\mathrm{End}^{Q}(\boldsymbol{T})$ contains a division algebra of dimension $2n$ as a $\boldsymbol{Q}$-vector space, $\boldsymbol{T}$ is isogenous to the direct sum of some copies of a simple torus.*

PROOF. If $\boldsymbol{T}$ has a non-trivial invariant subtorus $\boldsymbol{T}'$, $\mathrm{End}^{Q}(\boldsymbol{T}')$ contains a division algebra of dimension $2n$. But this is impossible. Hence $\boldsymbol{T}$ has no non-trivial invariant subtorus, so that, by theorem 3-1, $\boldsymbol{T}$ is isogenous to the direct sum of some copies of a simple torus. (q. e. d.)

§4. Complex tori of dimension 2.

Throughout this section $\boldsymbol{T}$ will denote a complex torus of dimension 2. In this section we will study the structure of $\mathrm{End}^{Q}(\boldsymbol{T})$.

(1) The case that $\boldsymbol{T}$ is simple.

If $\boldsymbol{T}$ is simple any endomorphism is an isogeny, so $\mathrm{End}^{Q}(\boldsymbol{T})$ is a division algebra. Let K be one of the maximal commutative subfields of $\mathrm{End}^{Q}(\boldsymbol{T})$ and d its degree over $\boldsymbol{Q}$, and d divides 4, so $d=1$, 2 or 4. If $d=1$, $\mathrm{End}^{Q}(\boldsymbol{T})=\boldsymbol{Q}$.

a) The case of $d=4$.

In this case $\mathrm{End}^{Q}(\boldsymbol{T})=K$ is isomorphic to a quartic field $\boldsymbol{Q}[X]/(f(X))$ over $\boldsymbol{Q}$ where $f(X)$ is an irreducible polynomial of degree 4. By Theorem 2-3, there exist complex numbers ζ, ξ such that $\{\zeta, \xi, \bar{\zeta}, \bar{\xi}\}$ is the set of all roots of the equation $f(X)=0$ and $\boldsymbol{T}$ is isogenous to

$$\boldsymbol{T}'(\zeta, \xi)=\boldsymbol{C}^{2}/\begin{pmatrix}1 & \zeta & \zeta^{2} & \zeta^{3}\\ 1 & \xi & \xi^{2} & \xi^{3}\end{pmatrix}.$$

Conversely let $f(X)$ be an irreducible polynomial of degree 4 and ζ, ξ two complex numbers such that $\{\zeta, \xi, \bar{\zeta}, \bar{\xi}\}$ is the set of all roots of the equation $f(X)=0$. Then $\boldsymbol{T}'(\zeta, \xi)$ is a complex torus such that $\mathrm{End}^{Q}(\boldsymbol{T}'(\zeta, \xi))$ contains a division algebra $\boldsymbol{Q}(\zeta)$ of dimension 4. If $\boldsymbol{T}'(\zeta, \xi)$ is not simple, by Theorems 3-3, $\boldsymbol{T}'(\zeta, \xi)$ is isogenous to the direct sum of two copies of an eliptic curve $C=\boldsymbol{C}/(1, z)$. In other words there exist $\omega\in GL(2, \boldsymbol{C})$ and $\Omega\in GL(4, \boldsymbol{Q})$ such that

$$\begin{pmatrix}1 & \zeta & \zeta^{2} & \zeta^{3}\\ 1 & \xi & \xi^{2} & \xi^{3}\end{pmatrix}\Omega=\omega\begin{pmatrix}1 & z & 0 & 0\\ 0 & 0 & 1 & z\end{pmatrix}. \tag{1}$$

Let F be the minimal Galois extension of $\boldsymbol{Q}$ containing $\boldsymbol{Q}(\zeta)$, G^{*} its Galois group and σ one of elements of G^* such that $\zeta^\sigma=\xi$. Put $\omega=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ and (1) implies that $\alpha, \beta, \alpha z$ and βz are all contained in $\boldsymbol{Q}(\zeta)$ and $\gamma, \delta, \gamma z$ and δz are in $\boldsymbol{Q}(\xi)$ and moreover $\alpha^\sigma=\gamma$, $(\alpha z)^\sigma=\gamma z$, $\beta^\sigma=\delta$, $(\beta z)^\sigma=\delta z$. So z is contained in both $\boldsymbol{Q}(\zeta)$ and $\boldsymbol{Q}(\xi)$, and $z^\sigma=z$. We put $K'=\boldsymbol{Q}(z)$, then $\boldsymbol{Q}(\zeta)$ is a quadratic extension of K' and ξ is the conjugate of ζ over K'. Therefore $\boldsymbol{Q}(\zeta)=\boldsymbol{Q}(\xi)$ and $\boldsymbol{Q}(\bar{\zeta})=\boldsymbol{Q}(\bar{\xi})$. By the way there exist only four distinct elements in all ζ^ρ $(\rho\in G^*)$, and there exist at most two elements ρ of G^* such that $\zeta^\rho=\zeta$. In fact if $\zeta^\rho=\zeta$, $\xi^\rho=\xi$, so $\bar{\zeta}^\rho$ must be $\bar{\zeta}$ or $\bar{\xi}$. Hence the order of G^* is 4 or 8. Making $\zeta, \xi, \bar{\zeta}, \bar{\xi}$ correspond to 1, 2, 3, 4 respectively we consider G^* to be a subgroup of the symmetric group S_4. Then $G^*=V_4=\{id, (12)(34), (13)(23), (14)(23)\}$ or $G^*=V_4\cup(12)V_4=\{id, (12), (12)(34), (34), (13)(24), (1423), (1324), (14)(23)\}$ where "id" means the unit element of the group.

Conversely if G^* is one of those subgroups, putting $z=\zeta+\xi$, it is easily seen that $\boldsymbol{T}'(\zeta, \xi)$ is not simple.

b) The case of $d=2$.

In this case K is isomorphic to a quadratic field $\boldsymbol{Q}(\sqrt{m})$ where m is a square-free integer. By Theorem 2-3 $\boldsymbol{T}$ is isogenous to

$$\boldsymbol{C}^2/\begin{pmatrix}a & \sqrt{m}\,a & b & \sqrt{m}\,b\\ c & \sqrt{m}\,c & d & \sqrt{m}\,d\end{pmatrix} \quad\text{or}\quad \boldsymbol{C}^2/\begin{pmatrix}a & \sqrt{m}\,a & b & \sqrt{m}\,b\\ c & -\sqrt{m}\,c & d & -\sqrt{m}\,d\end{pmatrix}$$

for some complex numbers a, b, c, d. Since $\boldsymbol{T}$ is simple, $abcd\neq 0$, so we may assume $a=c=1$. But $\begin{pmatrix}1 & \sqrt{m} & b & \sqrt{m}\,b\\ 1 & \sqrt{m} & d & \sqrt{m}\,d\end{pmatrix}$ cannot be a period matrix of a simple torus. Hence $\boldsymbol{T}$ is isogenous to a complex torus

$$\boldsymbol{T}_1(m\,;b, d)=\boldsymbol{C}^2/\begin{pmatrix}1 & \sqrt{m} & b & b\sqrt{m}\\ 1 & -\sqrt{m} & d & -d\sqrt{m}\end{pmatrix}$$

where b, d are complex numbers such that $b, d\notin\boldsymbol{R}$ if $m>0$ and $b\neq\bar{d}$ if $m<0$. Conversely if such m, b, d are given, $\begin{pmatrix}1 & \sqrt{m} & b & b\sqrt{m}\\ 1 & -\sqrt{m} & d & -d\sqrt{m}\end{pmatrix}$ is certainly a period matrix of some complex torus $\boldsymbol{T}_1(m\,;b, d)$.

LEMMA 4-1. *$\boldsymbol{T}_1(m\,;b, d)$ defined above is not simple if and only if the following condition i*) is satisfied.*

i) There exist rational numbers x, y and an element z of $\boldsymbol{Q}(\sqrt{m})$ with are not all zero and satisfy*

(†) $2xbd+zb+z^\sigma d+2y=0$ (*where z^σ means the conjugate of z*).

(††) $N(z/2)+xy\in N(\boldsymbol{Q}(\sqrt{m}))$ (*where $N(z)=zz^\sigma$ for $z\in\boldsymbol{Q}(\sqrt{m})$*).

PROOF. Let $x, y, z_1, z_2, b_1, b_2, b_3, b_4$ are given rational numbers such that

$(x, y, z_1, z_2) \neq (0, 0, 0, 0)$ and $(b_1, b_2, b_3, b_4) \neq (0, 0, 0, 0)$ and consider simultaneous equations with unknowns X_1, X_2, X_3, X_4,

$$(1) \qquad \begin{cases} x = b_3X_4 - b_4X_3 \\ y = b_1X_2 - b_2X_1 \\ z_1 = b_1X_4 - b_2X_3 - b_4X_1 + b_3X_2 \\ z_2 = b_1X_3 - mb_2X_4 - b_3X_1 + mb_4X_2 , \end{cases}$$

that is,

$$\begin{pmatrix} 0 & 0 & -b_4 & b_3 \\ -b_2 & b_1 & 0 & 0 \\ -b_4 & b_3 & -b_2 & b_1 \\ -b_3 & mb_4 & b_1 & -mb_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z_1 \\ z_2 \end{pmatrix}.$$

Put $z = z_1 + \sqrt{m}^{-1} z_2$. If x, y, z satisfy (†) and (1) has a solution $X_i = a_i$ $(i=1, 2, 3, 4)$, $\boldsymbol{T}_1(m; b, d)$ is not simple. In fact let Ω be an element of $GL(4, \boldsymbol{Q})$ such that

$$\Omega = \begin{pmatrix} a_1 & b_1 & \\ a_2 & b_2 & \\ a_3 & b_3 & * \\ a_4 & b_4 & \end{pmatrix}$$

and ω an element of $GL(2, \boldsymbol{C})$ such that

$$\omega = \begin{pmatrix} -\alpha & \beta \\ * & * \end{pmatrix}$$

where $\alpha = b_1 - b_2\sqrt{m} + b_3 d - b_4 d\sqrt{m}$, $\beta = b_1 + b_2\sqrt{m} + b_3 b + b_4 b\sqrt{m}$. Then we have by (1) and (†)

$$\omega \begin{pmatrix} 1 & \sqrt{m} & b & b\sqrt{m} \\ 1 & -\sqrt{m} & d & -d\sqrt{m} \end{pmatrix} \Omega = \begin{pmatrix} 0 & 0 & * & * \\ * & * & * & * \end{pmatrix}.$$

Conversely if $\boldsymbol{T}_1(m; b, d)$ is not simple, there exist such an ω and an Ω. Therefore there exist x, y, z which satisfy (†) and b_1, b_2, b_3, b_4 such that (1) has a solution.

On the other hand (1) has a solution if and only if

$$\operatorname{rank} \begin{pmatrix} 0 & 0 & -b_4 & b_3 & x \\ -b_2 & b_1 & 0 & 0 & y \\ -b_4 & b_3 & -b_2 & b_1 & z_1 \\ -b_3 & mb_4 & b_1 & -mb_2 & z_2 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} 0 & 0 & -b_4 & b_3 \\ -b_2 & b_1 & 0 & 0 \\ -b_4 & b_3 & -b_2 & b_1 \\ -b_3 & mb_4 & b_1 & -mb_2 \end{pmatrix}.$$

It is easily seen that this equation is equivalent to the following equation (2);

$$(2) \qquad x(b_1^2-mb_2^2)+y(b_3^2-mb_4^2)+z_2(b_1b_4-b_2b_3)-z_1(b_1b_3-mb_2b_4)=0\,.$$

Put $\varepsilon=b_1+\sqrt{m}\,b_2$ and $\eta=b_3+\sqrt{m}\,b_4$, and (2) implies

$$(3) \qquad \varepsilon\varepsilon^\sigma x+\eta\eta^\sigma y-(\varepsilon\eta^\sigma z+\varepsilon^\sigma\eta z^\sigma)/2=0\,.$$

There exist ε and η which are not both zero and satisfy (3) if and only if (††) is satisfied. In fact, put $\nu=2y\eta-z\varepsilon$, and (3) implies

$$(N(z/2)-xy)\varepsilon\varepsilon^\sigma=\nu\nu^\sigma/4\in N(\boldsymbol{Q}(\sqrt{m}))\,.$$

Hence the proof is completed.

Let R be a commutative ring and α, β elements of R. We denote by $(\alpha, \beta)_R$ the quatenion over R which is generated as a R-module by $\{1, e_1, e_2, e_3\}$ where 1 is the unit and $e_1^2=\alpha$, $e_2^2=\beta$, $e_1e_2=-e_2e_1=e_3$.

We will call a complex torus of dimension 2 which is isogenous to $\boldsymbol{T}_1(m\,;b,d)$ such that there exist x, y, z which satisfy (†) but there exist no x, y, z which satisfy both (†) and (††) of a quatenion type. By the above lemma a complex torus of a quatenion type is simple.

THEOKEM 4-2. *Let $\boldsymbol{T}$ be a simple complex torus of dimension* 2. End$(\boldsymbol{T})$ *is a non-commutative ring of rank* 4 *if and only if $\boldsymbol{T}$ is of a quatenion type. In this case, $\boldsymbol{T}$ is isogenous to $\boldsymbol{T}_1(m\,;b,d)$ such that $bd=q$ is a rational number and* End$^{\boldsymbol{Q}}(\boldsymbol{T})$ *is isomorphic to* $(m, q)_{\boldsymbol{Q}}$.

PROOF. First assume that $\boldsymbol{T}$ is of a quatenion type. Then we may assume that $\boldsymbol{T}=\boldsymbol{T}_1(m\,;b,d)$ and there exist x, y, z such that $2xbd+zb+z^\sigma d+2y=0$. Since (††) is not satisfied, $xy\neq 0$ and we may assume $x=1$. If we put $b'=b-z^\sigma$, $d'=d-z$ and $q=zz^\sigma-y\in\boldsymbol{Q}$, then $b'd'=q$ and $\boldsymbol{T}=\boldsymbol{T}_1(m\,;b,d)$ is isogenous to $\boldsymbol{T}_1(m\,;b',d')$ by an isogeny the rational representation of which is

$$M\begin{pmatrix} 1 & 0 & -z_1 & mz_2 \\ 0 & 1 & z_2 & -z_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $z=z_1+z_2\sqrt{m}$ and M is an integer which is large enough to make coefficients integral. It can be easily seen that End$^{\boldsymbol{Q}}(\boldsymbol{T}_1(m\,;b',d'))$ is a quatenion generated as a $\boldsymbol{Q}$-module by four elements whose analytic representations are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \sqrt{m} & 0 \\ 0 & -\sqrt{m} \end{pmatrix}, \begin{pmatrix} 0 & b' \\ d' & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{m}\,b' \\ -\sqrt{m}\,d' & 0 \end{pmatrix}.$$

That implies the "if" part of the theorem, so we next prove the "only if" part of the theorem. If End($\boldsymbol{T}$) is a non-commutative ring of rank 4, $\boldsymbol{T}$ is clearly isogenous to $\boldsymbol{T}_1(m; b, d)$ for some complex numbers b, d, and we may assume that $\boldsymbol{T}=\boldsymbol{T}_1(m; b, d)$. We denote by ϕ the endomorphism whose analytic representation is $\begin{pmatrix}\sqrt{m} & 0\\ 0 & -\sqrt{m}\end{pmatrix}$. Let ψ be an endomorphism which is not commutative with ϕ and $\begin{pmatrix}s & u\\ v & t\end{pmatrix}$ its analytic representation. Since

$$\begin{pmatrix}\sqrt{m} & 0\\ 0 & -\sqrt{m}\end{pmatrix}\begin{pmatrix}s & u\\ v & t\end{pmatrix}\begin{pmatrix}\sqrt{m} & 0\\ 0 & -\sqrt{m}\end{pmatrix}^{-1}-\begin{pmatrix}s & u\\ v & t\end{pmatrix}=\begin{pmatrix}0 & -2u\\ -2v & 0\end{pmatrix},$$

There exists an endomorphism ψ' whose rational representation is $\begin{pmatrix}0 & u'\\ v' & 0\end{pmatrix}$ for some u', v'. Since End($\boldsymbol{T}$) is not commutative, the degree of ψ' over Q is 2, so there exist rational numbers a_1, a_2 such that $\psi'^2+a_1\psi'+a_2=0$. Hence

$$\begin{pmatrix}u'v' & 0\\ 0 & u'v'\end{pmatrix}+a_1\begin{pmatrix}0 & u'\\ v' & 0\end{pmatrix}+a_2=0$$

That implies $a_1=0$ and $u'v'$ is a rational number. Let $\Omega=(\Omega_{ij})$ be the rational representation of ψ', and

$$\begin{pmatrix}0 & u'\\ v' & 0\end{pmatrix}\begin{pmatrix}1 & \sqrt{m} & b & b\sqrt{m}\\ 1 & -\sqrt{m} & d & -d\sqrt{m}\end{pmatrix}=\begin{pmatrix}1 & \sqrt{m} & b & b\sqrt{m}\\ 1 & -\sqrt{m} & d & -d\sqrt{m}\end{pmatrix}\begin{pmatrix}\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14}\\ \Omega_{21} & \Omega_{22} & \Omega_{23} & \Omega_{24}\\ \Omega_{31} & \Omega_{32} & \Omega_{33} & \Omega_{34}\\ \Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44}\end{pmatrix}.$$

Put $\alpha_1=\Omega_{11}+\sqrt{m}\,\Omega_{21}$ and $\alpha_2=\Omega_{31}+\sqrt{m}\,\Omega_{41}$, and $u'=\alpha_1+b\alpha_2$ and $v'=\alpha_1^\sigma+d\alpha_2^\sigma$ where α_1 and α_2 are not both zero. Since $u'v'$ is a rational number, putting $x=\alpha_2\alpha_2^\sigma/2$, $y=(\alpha_1\alpha_1^\sigma-u'v')/2$ and $z=\alpha_2\alpha_2^\sigma$, the equation (†) is satisfied. In fact

$$0=(\alpha_1+b\alpha_2)(\alpha_1^\sigma+d\alpha_2^\sigma)-u'v'=\alpha_2\alpha_2^\sigma bd+\alpha_2\alpha_1^\sigma b+\alpha_2^\sigma\alpha_1 d+\alpha_1\alpha_1^\sigma-u'v'. \qquad \text{(q. e. d.)}$$

(2) The case that $\boldsymbol{T}$ is not simple nor isogenous to the direct sum of two elliptic curves.

If $\boldsymbol{T}$ has a subtorus of dimension 1, we may assume the period matrix of $\boldsymbol{T}$ is

$$\begin{pmatrix}1 & z_1 & 0 & w\\ 0 & 0 & 1 & z_2\end{pmatrix}$$

for some complex numbers z_1, z_2, w.

LEMMA 4-3. *The complex torus* $\boldsymbol{T}=\boldsymbol{C}^2/\begin{pmatrix}1 & z_1 & 0 & w\\ 0 & 0 & 1 & z_2\end{pmatrix}$ *is isogenous to the direct sum of two elliptic curves if and only if* $w=q_0+q_1z_1+q_2z_2+q_3z_1z_2$ *for some rational* *numbers* q_0, q_1, q_2, q_3.

PROOF. If $w=q_0+q_1z_1+q_2z_2+q_3z_1z_2$, it is easy to transform $\begin{pmatrix} 1 & z_1 & 0 & w \\ 0 & 0 & 1 & z_2 \end{pmatrix}$ by some isogeny into $\begin{pmatrix} 1 & z_1 & 0 & 0 \\ 0 & 0 & 1 & z_2 \end{pmatrix}$. Conversely if $\boldsymbol{T}$ is isogenous to the direct sum of elliptic curves, there exist an element $\omega=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $GL(2, \boldsymbol{C})$ and an element $\Omega=(a_{ij})$ of $GL(4, \boldsymbol{Q})$ and complex numbers x, y such that

$$\omega\begin{pmatrix} 1 & z_1 & 0 & w \\ 0 & 0 & 1 & z_2 \end{pmatrix}=\begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 0 & 1 & y \end{pmatrix}\Omega\,,$$

that is,

$$\begin{pmatrix} a & az_1 & b & aw+bz_2 \\ c & cz_1 & d & cw+dz_2 \end{pmatrix}=\begin{pmatrix} a_{11}+a_{21}x & a_{12}+a_{22}x & a_{13}+a_{23}x & a_{14}+a_{24}x \\ a_{31}+a_{41}y & a_{32}+a_{42}y & a_{33}+a_{43}y & a_{34}+a_{44}y \end{pmatrix}.$$

Eliminating x from the equation of the first line, we have

$$(a_{11}a_{22}-a_{21}a_{12})w=(a_{22}a_{14}-a_{24}a_{11})+(a_{24}a_{11}-a_{12}a_{21})z_1+(a_{12}a_{23}-a_{22}a_{13})z_2$$
$$+(a_{21}a_{13}-a_{23}a_{11})z_1z_2\,.$$

Considering the second line, if necessary, we may assume $a=a_{11}+a_{21}x\neq 0$. Since z_1 is not a rational number, $a=a_{11}+a_{21}x$ and $az_1=a_{12}+a_{22}x$ are linearly independent over $\boldsymbol{Q}$, hence $a_{11}a_{22}-a_{21}a_{12}\neq 0$. Therefore w is a linear combination of 1, z_1, z_2, z_1z_2 with coefficients in $\boldsymbol{Q}$. (q. e. d.)

LEMMA 4-4. *Let $\boldsymbol{T}$ be a complex torus which is not simple nor isogenous to the direct sum of two elliptic curves. Then $\boldsymbol{T}$ has the unique subtorus $\boldsymbol{T}'$ of dimension* 1, *which is invariant. If* $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})\neq\boldsymbol{Q}$, *$\boldsymbol{T}'$ is isogenous to the factor torus $\boldsymbol{T}/\boldsymbol{T}'$. Therefore $\boldsymbol{T}$ is isogenous to a complex torus of the following type*;

$$\boldsymbol{T}_2(z\,;\,w)=\boldsymbol{C}^2/\begin{pmatrix} 1 & z & 0 & w \\ 0 & 0 & 1 & z \end{pmatrix}.$$

PROOF. Of course $\boldsymbol{T}$ has a subtorus $\boldsymbol{T}'$ of dimension 1. If there exists another subtorus $\boldsymbol{T}''$ of dimension 1, $\boldsymbol{T}$ is isogenous to $\boldsymbol{T}'\oplus\boldsymbol{T}''$. Hence $\boldsymbol{T}'$ is the unique subtorus of dimension 1. Now assume that $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})\neq\boldsymbol{Q}$. If there exists an endomorphism ϕ such that $\phi(\boldsymbol{T})=\boldsymbol{T}'$, $\boldsymbol{T}'$ is contained in the kernel of ϕ, so ϕ induces an isogeny of $\boldsymbol{T}/\boldsymbol{T}'$ to $\boldsymbol{T}'$. If there does not exist such a ϕ, $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})$ is division algebra. We have seen in § 3 that $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})$ is considered to be a subalgebra of $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T}')$ and of $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T}/\boldsymbol{T}')$. Since $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})\neq\boldsymbol{Q}$, we have $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T}')\cong\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})\cong\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T}/\boldsymbol{T}')$. So $\boldsymbol{T}'$ is isogenous to $\boldsymbol{T}/\boldsymbol{T}'$. (q. e. d.)

Now to study the endomorphism ring of $\boldsymbol{T}_2(z\,;\,w)$ we prepare a lemma.

LEMMA 4-5. *Let $\boldsymbol{T}=\boldsymbol{E}/\boldsymbol{G}$ be a complex torus of dimension n and $\boldsymbol{T}'$ an invariant subtorus of dimension r. If $(1_r\ T')$ and $(1_s\ T'')$ are the period matrices of $\boldsymbol{T}'$ and $\boldsymbol{T}/\boldsymbol{T}'$ respectively where $r+s=n$, then we can choose a $\boldsymbol{C}$-base of $\boldsymbol{E}$ and a $\boldsymbol{Z}$-base of $\boldsymbol{G}$ such that the period matrix is of the following type;*

$$\begin{pmatrix} 1_r & 0 & T' & * \\ 0 & 1_s & 0 & T'' \end{pmatrix}.$$

Then the analytic representation ω and the rational representation Ω of any element of $\mathrm{End}^{Q}(\boldsymbol{T})$ are matrices of the following types;

$$\omega=\begin{array}{cc} \begin{matrix} \leftarrow r \rightarrow & \leftarrow s \rightarrow \end{matrix} & \\ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} & \begin{matrix} \updownarrow r \\ \updownarrow s \end{matrix} \end{array} \qquad \Omega=\begin{array}{cc} \begin{matrix} \leftarrow r \rightarrow & \leftarrow s \rightarrow & \leftarrow r \rightarrow & \leftarrow s \rightarrow \end{matrix} & \\ \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ * & * & * & * \\ 0 & * & 0 & * \end{pmatrix} & \begin{matrix} \updownarrow r \\ \updownarrow s \\ \updownarrow r \\ \updownarrow s \end{matrix} \end{array}.$$

PROOF. Putting $\boldsymbol{T}=\boldsymbol{E}/\boldsymbol{G}$, $\boldsymbol{T}'=\boldsymbol{E}'/\boldsymbol{G}'$ ($\boldsymbol{E}\subset\boldsymbol{E}'$), $\boldsymbol{E}'$ is invariant by the linear extension of any endomorphism. The lemma follows immediately.

We now pass on to the consideration on a complex torus

$$\boldsymbol{T}_2=\boldsymbol{T}_2(z\,;\,w)=\boldsymbol{C}^2/\begin{pmatrix} 1 & 0 & z & w \\ 0 & 1 & 0 & z \end{pmatrix}$$

and $\mathrm{End}^{Q}(\boldsymbol{T}_2)$. Let

$$\omega=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad\text{and}\quad \Omega=\begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{pmatrix}$$

be the analytic representation and the rational representation of an endomorphism of $\boldsymbol{T}_2$. $\gamma=a_{21}=b_{21}=c_{21}=d_{21}=0$ by lemma 4-5. Since

$$\omega\begin{pmatrix} 1 & 0 & z & w \\ 0 & 1 & 0 & z \end{pmatrix}=\begin{pmatrix} 1 & 0 & z & w \\ 0 & 1 & 0 & z \end{pmatrix}\Omega,$$

we have

i) $c_{11}z^2+(a_{11}-d_{11})z-b_{11}=0$

ii) $c_{22}z^2+(a_{22}-d_{22})z-b_{22}=0$

iii) $\{(a_{11}-d_{22})+(c_{11}+c_{22})z\}\,w=b_{12}+(d_{12}-a_{12})z-c_{12}z^2.$

a) The case of $[\boldsymbol{Q}(z):\boldsymbol{Q}]\geqq 3$.

Then i) and ii) imply that $a_{11}=d_{11}$, $a_{22}=d_{22}$, $c_{11}=b_{11}=c_{22}=b_{22}=0$, and hence iii) implies

$$(a_{11}-d_{22})w=b_{12}+(d_{12}-a_{12})z-c_{12}z^{2}.$$

If $a_{11}\neq d_{22}$, $\boldsymbol{T}_2$ is isogenous to the direct sum of two elliptic curves. Therefore $a_{11}=d_{22}$ and $b_{12}=c_{12}=0$, $d_{12}=a_{12}$. Hence the rational representation of $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T}_2)$ is

$$\left\{\left.\begin{pmatrix} a & b & 0 & 0\\ 0 & a & 0 & 0\\ 0 & 0 & a & b\\ 0 & 0 & 0 & a\end{pmatrix}\right| a,\ b\in\boldsymbol{Q}\right\}.$$

The dimension of $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T}_2)$ over $\boldsymbol{Q}$ is 2, and the analytic representation of a base is

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

$\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T}_2)$ is isomorphic to $\boldsymbol{Q}[X]/(X^2)$.

b) The case of $[\boldsymbol{Q}(z):\boldsymbol{Q}]=2$.

Then we may assume that $z=\sqrt{m}$ where m is a square-free integer. i) and ii) imply $a_{11}=d_{11}$, $mc_{11}=b_{11}$, $a_{22}=d_{22}$, $mc_{22}=b_{22}$. If $(a_{11}-d_{22})+(c_{11}+c_{22})z\neq 0$, w is an element of $\boldsymbol{Q}(z)$ and hence $\boldsymbol{T}_2$ is isogenous to the direct sum of two elliptic curves. Therefore $(a_{11}-d_{22})+(c_{11}+c_{22})z=0$. This equation implies $a_{11}=d_{22}$, $c_{11}+c_{22}=0$ and $b_{12}=mc_{12}$, $d_{12}=a_{12}$. It follows that the rational representation of $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T}_2)$ is

$$\left\{\left.\begin{pmatrix} a & b & mc & d\\ 0 & a & 0 & -mc\\ c & d & a & b\\ 0 & -c & 0 & a\end{pmatrix}\right| a,\ b,\ c,\ d\in\boldsymbol{Q}\right\}.$$

The dimension of $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})$ over $\boldsymbol{Q}$ is 4 and the analytic representation of a base is

$$1_2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad e_1=\begin{pmatrix}\sqrt{m} & -w\\ 0 & -\sqrt{m}\end{pmatrix},\quad e_2=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\quad e_3=\begin{pmatrix}0 & \sqrt{m}\\ 0 & 0\end{pmatrix}.$$

There are the following equation among those four elements;

$$e_1 1_2=e_1,\quad e_2 1_2=e_2,\quad e_1^2=m1_2,\quad e_2^2=0,\quad e_1e_2=-e_2e_1=e_3.$$

Hence $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})$ is isomorphic to $(m, 0)_{\boldsymbol{Q}}$.

(3) The case that $\boldsymbol{T}$ is isogenous to the direct sum of two elliptic curves.

There is no difficulty in this case. We may assume that $\boldsymbol{T}=\boldsymbol{T}'\oplus\boldsymbol{T}''$ for some elliptic curves $\boldsymbol{T}'$ and $\boldsymbol{T}''$. If $\boldsymbol{T}'$ is isogenous to $\boldsymbol{T}''$, $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})\cong M(2, \mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T}'))$. And if $\boldsymbol{T}'$ is not isogenous to $\boldsymbol{T}''$, $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})\cong\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T}')\oplus\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T}'')$.

Now we will summarize the facts we have seen in this section. Let m, m' be integers which are square-free and z, z' complex numbers which are not contained in $\boldsymbol{R}$ nor any quadratic field over $\boldsymbol{Q}$. Consider complex tori of the following types.

I)

$$\boldsymbol{T}'(\zeta, \xi)=\boldsymbol{C}^2/\begin{pmatrix}1 & \zeta & \zeta^2 & \zeta^3\\ 1 & \xi & \xi^2 & \xi^3\end{pmatrix}$$

where ζ, ξ are algebraic numbers of degree 4 over $\boldsymbol{Q}$ such that $\{\zeta, \xi, \bar{\zeta}, \bar{\xi}\}$ is the set of all conjugates of ζ over $\boldsymbol{Q}$. Moreover if we consider the Galois group G^* of $F=\boldsymbol{Q}(\zeta, \xi, \bar{\zeta}, \bar{\xi})$ to be a subgroup of S_4 by the correspondence $1\longleftrightarrow\zeta$, $2\longleftrightarrow\xi$, $3\longleftrightarrow\bar{\zeta}$, $4\longleftrightarrow\bar{\xi}$, G is not V_4 nor $V_4\cup(12)V_4$.

II) (complex tori of quatenion types)

$$\boldsymbol{T}_1(m\,;\,b,\,d)=\boldsymbol{C}^2/\begin{pmatrix}1 & \sqrt{m} & b & b\sqrt{m}\\ 1 & -\sqrt{m} & d & -d\sqrt{m}\end{pmatrix}$$

where b, d are complex numbers which are not contained in $\boldsymbol{Q}(\sqrt{m})$, and $bd=q$ is a rational number which is not contained in $N(\boldsymbol{Q}(\sqrt{m}))$. And there is no element α of $\boldsymbol{Q}(\sqrt{m})$ but zero such that $\alpha b+\alpha^\sigma d$ is a rational number. Moreover if $m>0$, b, d are not real number, and if $m<0$, $b\neq\bar{d}$.

III) Simple complex tori of the following type

$$\boldsymbol{T}_1(m\,;\,b,\,d)=\boldsymbol{C}^2/\begin{pmatrix}1 & \sqrt{m} & b & b\sqrt{m}\\ 1 & -\sqrt{m} & d & -d\sqrt{m}\end{pmatrix}$$

which are not isogenous to any complex torus of the type (I) nor the type (II). If $m>0$, b, d are not contained in $\boldsymbol{R}$, and if $m<0$, $b\neq\bar{d}$.

IV)

$$\boldsymbol{T}_2(\sqrt{m}\,;\,w)=\boldsymbol{C}^2/\begin{pmatrix}1 & \sqrt{m} & 0 & w\\ 0 & 0 & 1 & \sqrt{m}\end{pmatrix}$$

where $m<0$, and w is not contained in $\boldsymbol{Q}(\sqrt{m})$.

V)

$$\boldsymbol{T}_2(z\,;\,w)=\boldsymbol{C}^2/\begin{pmatrix}1 & z & 0 & w\\ 0 & 0 & 1 & z\end{pmatrix}$$

where w is not contained in $\boldsymbol{Q}+\boldsymbol{Q}z+\boldsymbol{Q}z^2$.

VI) $$\boldsymbol{T}_3(\sqrt{m},\,\sqrt{m})=\boldsymbol{C}/(1\ \ \sqrt{m})\oplus\boldsymbol{C}/(1\ \ \sqrt{m})$$

where $m<0$.

VII) $$T_3(\sqrt{m}, \sqrt{m'})=\boldsymbol{C}/(1\ \sqrt{m})\oplus\boldsymbol{C}/(1\ \sqrt{m'})$$

where $m, m'<0$ and $m\neq m'$.

VIII) $$T_3(\sqrt{m}, z)=\boldsymbol{C}/(1\ \sqrt{m})\oplus\boldsymbol{C}/(1\ z)$$

where $m<0$.

IX) $$T_3(z, z)=\boldsymbol{C}/(1\ z)\oplus\boldsymbol{C}/(1\ z).$$

X) $$T_3(z, z')=\boldsymbol{C}/(1\ z)\oplus\boldsymbol{C}/(1\ z')$$

where $z'\notin\boldsymbol{Q}(z)$.

Then a complex torus $\boldsymbol{T}$ of dimension 2 is isogenous to a complex torus of one of the above types if and only if $\mathrm{End}^{Q}(\boldsymbol{T})$ is isomorphic to a $\boldsymbol{Q}$-algebra of the following corresponding type.

I) Algebraic fields $\boldsymbol{Q}(\zeta)$ of degree 4 over $\boldsymbol{Q}$.
II) Quatenions $(m, q)_{\boldsymbol{Q}}$ such that q is not contained in $N(\boldsymbol{Q}(\sqrt{m}))$.
III) Quadratic fields $\boldsymbol{Q}(\sqrt{m})$.
IV) Quatenions $(m, 0)_{\boldsymbol{Q}}$.
V) $\boldsymbol{Q}[X]/(X^2)$.
VI) $M(2, \boldsymbol{Q}(\sqrt{m}))$ where $m<0$.
VII) $\boldsymbol{Q}(\sqrt{m})\oplus\boldsymbol{Q}(\sqrt{m'})$ where $m, m'<0$, $m\neq m'$.
VIII) $\boldsymbol{Q}(\sqrt{m})\oplus\boldsymbol{Q}$ where $m<0$.
IX) $M(2, \boldsymbol{Q})$.
X) $\boldsymbol{Q}\oplus\boldsymbol{Q}$.

§ 5. Abelian varietis of dimension 2.

A complex torus $\boldsymbol{T}$ is called an abelian variety if $\boldsymbol{T}$ can be embedded in some projective space, in other words, if there exists an ample Riemann form on $\boldsymbol{T}$. A complex torus of dimension 2 of the type VI), VII), VIII), IX) or X) is an abelian variety. And a complex torus of the type IV) or V) is not an abelian variety. Then we will study complex tori of types I), II) and III), that is, simple tori.

Let $\boldsymbol{T}=\boldsymbol{E}/\boldsymbol{G}$ be a complex torus of dimension n where $\boldsymbol{E}$ is $\boldsymbol{C}$-vector space and $\boldsymbol{G}$ is its lattice subgroup. Fix bases of $\boldsymbol{E}$ and $\boldsymbol{G}$, and let G be the period matrix of $\boldsymbol{T}$ with respect to those bases. Put $(C\ \bar{C})=\begin{pmatrix} G \\ \bar{G}\end{pmatrix}^{-1}$ where $C\in M(2n\times n, \boldsymbol{C})$. There exists a one-to-one correspondence between the set of hermitian forms on $\boldsymbol{T}$ (namely the set of hermitian forms H on $\boldsymbol{E}\times\boldsymbol{E}$ such that $H(g, g')$ is integral for any $g, g'\in \boldsymbol{G}$) and the set of skew-symmetric matrices M of degree $2n$ with coefficients in $\boldsymbol{Z}$ which satisfy

(1) ${}^tCMC=0$.

In this correspondence an ample Riemann form on $\boldsymbol{T}$ corresponds to an M which satisfies (1) and

(2) $\sqrt{-1}\,{}^t\bar{C}MC>0$ (namely $\sqrt{-1}\,{}^t\bar{C}MC$ is positive definite.)

$\boldsymbol{T}$ is an abelian variety if and only if there exists a skew-symmetric matrix M which satisfies (1) and (2). If $G=(1_n\ T)$, $C=\begin{pmatrix}-\bar{T}\\ 1_n\end{pmatrix}(T-\bar{T})^{-1}$. Put $M=\begin{pmatrix}A & B\\ {}^tB & D\end{pmatrix}$ where $A, B, D\in M(n, \boldsymbol{Z})$ and ${}^tA=-A$, ${}^tD=-D$. Then (1), (2) imply respectively

(1′) ${}^tTAT-{}^tTB+{}^tBT+D=0$,

(2′) $\sqrt{-1}({}^tTA\bar{T}-{}^tTB+{}^tB\bar{T}+D)>0$.

When (1′) is satisfied, (2′) is equivalent to the following condition;

(2″) $\sqrt{-1}({}^tTA+{}^tB)(\bar{T}-T)>0$.

When $n=2$, put $T=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, $A=\begin{pmatrix}0 & x\\ -x & 0\end{pmatrix}$, $B=\begin{pmatrix}p & q\\ r & s\end{pmatrix}$ and $D=\begin{pmatrix}0 & y\\ -y & 0\end{pmatrix}$, and (1′) implies

i) $x(\alpha\delta-\gamma\beta)-(q\alpha+s\gamma)+(p\beta+r\delta)+y=0$

and (2″) implies

$$\sqrt{-1}\begin{pmatrix}p-x\gamma & r+x\alpha\\ q-x\delta & s+x\beta\end{pmatrix}\begin{pmatrix}\bar{\alpha}-\alpha & \bar{\beta}-\beta\\ \bar{\gamma}-\gamma & \bar{\delta}-\delta\end{pmatrix}>0,$$

which is equivalent to the following two conditions;

a) $\sqrt{-1}\{p(\bar{\alpha}-\alpha)+q(\bar{\gamma}-\gamma)+x(\alpha\bar{\gamma}-\bar{\alpha}\gamma)\}>0$,

b) $(-1)\{(p-x\gamma)(s+x\beta)-(r+x\alpha)(q-x\delta)\}\{(\bar{\alpha}-\alpha)(\bar{\delta}-\delta)-(\bar{\gamma}-\gamma)(\bar{\beta}-\beta)\}>0$.

When i) is satisfied b) is equivalent to the following;

c) $\{-xy+(ps-rq)\}\{(\bar{\alpha}-\alpha)(\bar{\delta}-\delta)-(\bar{\gamma}-\gamma)(\bar{\beta}-\beta)\}<0$.

Now let $\boldsymbol{T}$ be a simple torus of dimension 2 with non-trivial endomorphisms. First we prove that if $\boldsymbol{T}$ is an abelian variety $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})$ contains some quadratic field over $\boldsymbol{Q}$. In fact, if it does not, $\boldsymbol{T}$ is isogenous to a complex torus of the type

$$\boldsymbol{C}^2/\begin{pmatrix}1 & \zeta & \zeta^2 & \zeta^3\\ 1 & \xi & \xi^2 & \xi^3\end{pmatrix}$$

where the Galois group G^* of $\boldsymbol{Q}(\zeta, \xi, \bar{\zeta}, \bar{\xi})$ over $\boldsymbol{Q}$ is isomorphic to the alternative group A_4 or the symmetric group S_4. $\boldsymbol{T}$ is isogenous to

$$\boldsymbol{T}'=\boldsymbol{C}^2/\begin{pmatrix}1 & 0 & -\xi\zeta & -\xi\zeta(\xi+\zeta)\\ 0 & 1 & \xi+\zeta & \xi^2+\xi\zeta+\zeta^2\end{pmatrix}.$$

If $\boldsymbol{T}$ is an abelian variety, so is $\boldsymbol{T}'$, hence there exist integers x, y, p, q, r, s which are not all zero and satisfy i), that is,

$$\begin{aligned}0&=x(\zeta^2\xi^2)-\{q(-\xi\zeta)+s(\zeta+\xi)\}+\{p(-\xi\zeta(\zeta+\xi))+r(\xi^2+\xi\zeta+\zeta^2)\}+y\\&=(x\xi^2-p\xi+r)\zeta^2+(-p\xi^2+q\xi+r\xi-s)\zeta+(r\xi^2-s\xi+y).\end{aligned}$$

But if $G^*=A_4$ or S_4, this is impossible. Therefore if $\boldsymbol{T}$ is an abelian variety, $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})$ contains a quadratic field $\boldsymbol{Q}(\sqrt{m})$. Then $\boldsymbol{T}$ is isogenous to a complex torus

$$\boldsymbol{T}_1(m\,;\,b,\,d)=\boldsymbol{C}^2/\begin{pmatrix}1 & \sqrt{m} & b & b\sqrt{m}\\ 1 & -\sqrt{m} & d & -d\sqrt{m}\end{pmatrix}$$

for some complex numbers b, d. Since this is isomorphism to

$$\boldsymbol{T}_1'=\boldsymbol{C}^2/\begin{pmatrix}1 & 0 & u & mv\\ 0 & 1 & v & u\end{pmatrix}$$

where $u=(b+d)/2$ and $v=(b-d)/2\sqrt{m}$, $\boldsymbol{T}$ is an abelian variety if and only if there exist integers x, y, p, q, r, s which satisfy the following i′), a′) and c′).

i′) $bdx+zb+z^\sigma d+y=0$ (where $z=z_1+z_2/\sqrt{m}$, $z_1=(r-q)/2$ and $z=(pm-s)/2$.)

a′) $\sqrt{-1}\,\{p(u-\bar{u})+q(v-\bar{v})+x(u\bar{v}-v\bar{u})\}>0$

c′) $\{-xy+(ps-rq)\}\,F(b,\,d)<0$ $\left(\text{where } F(b,\,d)=\begin{cases}(b-\bar{b})(d-\bar{d}) & \text{if } m>0\\ (b-\bar{d})(d-\bar{b}) & \text{if } m<0.\end{cases}\right)$

LEMMA 5-1. *If $m>0$, there exist x, y, p, q, r, s which satisfy* i′) *and* a′), c′). *Therefore $\boldsymbol{T}$ is an abelian variety.*

PROOF. Put $x=y=0$, $r=q$, $s=mp$, and i′) is of course satisfied and a′), c′) imply

a″) $\sqrt{-1}\,\{(p+q/\sqrt{m})(b-\bar{b})+(p-q/\sqrt{m})(d-\bar{d})\}>0$

c″) $(mp^2-q^2)(b-\bar{b})(d-\bar{d})<0.$

Put $X=(p+q/\sqrt{m})\sqrt{-1}(b-\bar{b})$, $Y=(p-q/\sqrt{m})\sqrt{-1}(d-\bar{d})$, and a″), c″) imply $X+Y>0$ and $XY>0$. We only have to take p, q which make X and Y positive. (q. e. d.)

LEMMA 5-2. *If $m<0$ and $\boldsymbol{T}$ is not of a quatenion type, $\boldsymbol{T}$ is not an abelian* *variety.*

PROOF. Since $\boldsymbol{T}$ is not quatenion type, x, y, z which satisfy i′) are all zero, so $x=y=0$, $mp=s$, $r=q$. Then if $m<0$, c′) implies

$$(mp^2-q^2)(b-\bar{d})(d-\bar{b})=-(mp^2-q^2)|b-\bar{d}|^2<0 .$$

But since $m<0$, this is impossible. Hence $\boldsymbol{T}$ cannot be an abelian variety. (q. e. d.)

Now we assume that $\boldsymbol{T}$ is of a quatenion type. There exist an integer q_0 which is not contained in $N(\boldsymbol{Q}(\sqrt{m}))$ such that $\boldsymbol{T}$ is isogenous to

$$\boldsymbol{T}''=\boldsymbol{C}^2/\begin{pmatrix}1 & \sqrt{m} & b & b\sqrt{m} \\ 1 & -\sqrt{m} & d & -d\sqrt{m}\end{pmatrix}$$

where $bd=q_0$. If $m>0$ or $q_0>0$, T'' is an abelian variety by Lemma 5-1. So we assume $m<0$ and $q_0<0$. If there exists an element z of $\boldsymbol{Q}(\sqrt{m})$ such that $zb+z^\sigma d$ is a rational number r_0, putting $x=0$, $y=-r_0$, the condition i*) of Lemma 4-1 is satisfied. Therefore since $bd=q_0$ is a rational number, there exists no z but zero which satisfies i′) with some x, y. Hence if T' is an abelian variety, $y=-x_0$, $r=q$, $s=pm$ and

$$-(x^2q_0+mp^2-q^2)|b-\bar{d}|^2<0 .$$

But this is impossible. Therefore we have proved the following lemma.

LEMMA 5-3. *Let $\boldsymbol{T}$ be a complex torus of a quatenion type such that* $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})\cong(m, q)_{\boldsymbol{Q}}$. *If $m>0$ or $q>0$, $\boldsymbol{T}$ is an abelian variety. If $m<0$ and $q<0$, $\boldsymbol{T}$ is not abelian variety.*

And the following theorem has been proved.

THEOREM 5-4. *Let $\boldsymbol{T}$ be a simple complex torus of dimension 2 with non-trivial endomorphisms. Then $\boldsymbol{T}$ is an abelian variety if and only if* $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})$ *contains a real quadratic field over $\boldsymbol{Q}$ as a sub-$\boldsymbol{Q}$-algebra.*

REMARK. Let $\rho(\boldsymbol{T})$ be the rank of the additive group of all hermitian forms on $\boldsymbol{T}$, which is equal to the Picard number of $\boldsymbol{T}$. When $\boldsymbol{T}$ is a simple torus of dimension 2 such that $\mathrm{End}(\boldsymbol{T})\neq\boldsymbol{Z}$, we have seen above that if $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})$ contains no quadratic field over $\boldsymbol{Q}$, $\rho(\boldsymbol{T})=0$, if $\mathrm{End}^{\boldsymbol{Q}}(\boldsymbol{T})$ contains a quadratic field but $\boldsymbol{T}$ is not of a quatenion type, $\rho(\boldsymbol{T})=2$, and if $\boldsymbol{T}$ is of a quatenion type, $\rho(\boldsymbol{T})=3$.

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