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ALGEBRAS OF INFINITE DOMINANT DIMENSION

By

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A well know conjecture posed by Nakayama says that a finite dimensional K-algebra Λ over a field K and of infinite dominant dimension must be selfinjective. In this paper we will continue our investigations on algebras of infinite dominant dimension started in [11]. We will study for such algebras the category $\underline{\mathrm{mod}}(\mathrm{mod}_\Lambda)$ of all finitely presented contravariant functors vanishing on projectives. This category is an abelian category with enough projectives and injectives, so we can extend to it the notion of dominant dimension. Let $\mathcal{M}=\mathrm{mod}_\Lambda$ be the category of finitely generated modules and $\underline{\mathcal{M}}=\underline{\mathrm{mod}}_\Lambda$ the stable category, $\mathrm{mod}(\underline{\mathcal{M}})$ is the category of finitely presented contravariant functors from $\underline{\mathcal{M}}$ to the category of abelian groups. $\underline{\mathrm{mod}}(\mathrm{mod}_\Lambda)$ can be interpreted in a different way: $\mathrm{mod}(\underline{\mathcal{M}})$ and $\underline{\mathrm{mod}}(\mathrm{mod}_\Lambda)$ are equivalent categories.

Let Λ be a K-algebra of infinite dominant dimension, $\mathcal{D}=\mathcal{D}om_\Lambda$ the subcategory mod_Λ of all modules of infinite dominant dimension and $\underline{\mathcal{D}}$ the stable category, the inclusion of $\underline{\mathcal{D}}$ in $\underline{\mathcal{M}}$ induces functors:

$$\underline{\mathcal{M}}\underset{\underline{\mathcal{D}}}{\otimes}- : \mathrm{mod}(\underline{\mathcal{D}}) \longrightarrow \mathrm{mod}(\underline{\mathcal{M}}) \quad \text{and} \quad \mathrm{res}: \mathrm{mod}(\underline{\mathcal{M}}) \longrightarrow \mathrm{Mod}(\underline{\mathcal{D}}).$$

We call to the image of $\underline{\mathcal{M}}\underset{\underline{\mathcal{D}}}{\otimes}-$ the category of induced functors.

It is proved in this paper that if one of the following two conditions hold, then Λ is selfinjective:

a) The image of res lies in $\mathrm{mod}(\underline{\mathcal{D}})$.

b) The category of induced functors is contravariantly finite in $\mathrm{mod}(\underline{\mathcal{M}})$.

Another related result that is proved here is the following:

With the same notation and hypothesis as above, let $\mathcal{D}om(\underline{\mathcal{M}})$ denote the full subcategory of $\mathrm{mod}(\underline{\mathcal{M}})$ of all functors of infinite dominant dimension then:

i) The induced functors form a subcategory of $\mathcal{D}om(\underline{\mathcal{M}})$.

ii) If the induced functors form a contravariantly finite subcategory of $\mathcal{D}om(\underline{\mathcal{M}})$ then Λ is selfinjective.

We will use freely the notion of contravariantly finiteness developed in [6],

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[8].

We recall the following definitions from [5], [6]:

A full subcategory $\mathcal{C}$ of mod_Λ is called resolving if it satisfies the following conditions:

i) The subcategory $\mathcal{P}$ of all finitely generated projectives is contained in $\mathcal{C}$.

ii) If $0 \to A \to B \to C \to 0$ is an exact sequence and $A, C \in \mathcal{C}$ then $B \in \mathcal{C}$.

iii) With the same exact sequence as in ii), if $B, C \in \mathcal{C}$ then $A \in \mathcal{C}$.

We say that an additive category $\mathcal{C}$ has pseudokernels if given any map $f: X \to Y$ in $\mathcal{C}$ there exists a map $g: Z \to X$ such that the induced sequence of functors and natural transformations:

$$\text{Hom}_{\mathcal{C}}(-, Z) \xrightarrow{\text{Hom}(-g)} \text{Hom}_{\mathcal{C}}(-, X) \xrightarrow{\text{Hom}(-f)} \text{Hom}_{\mathcal{C}}(-, Y)$$

is exact.

Given any additive category $\mathcal{C}$ we denote by $\text{Mod}(\mathcal{C})$ the category of contravariant functors from $\mathcal{C}$ to the category of abelian groups and by $\text{mod}(\mathcal{C})$ the full subcategory of $\text{Mod}(\mathcal{C})$ of all finitely presented functors.

The following proposition is proved in [1]:

PROPOSITION 1. *Let $\mathcal{C}$ be an additive category, the following conditions are equivolent*:

i) *$\mathcal{C}$ has pseudokernels.*

ii) $\text{mod}(\mathcal{C})$ *is abelian.*

We need the following:

LEMMA 2. *Let $\mathcal{C}$ be a resolving subcategory of* mod_Λ *then the stable category $\underline{\mathcal{C}}$ has pseudokernels.*

PROOF. Let $\underline{f}: B \to C$ be a map in $\underline{\mathcal{C}}$ and f a representative of $\underline{f}$. Let P be the projective cover of C and $\pi: P \to C$ an epimorphism. Consider the exact sequence:

$$*) \quad 0 \to A \xrightarrow{\binom{g}{s}} B \oplus P \xrightarrow{(f\ \pi)} C \to 0, \text{ where the kernel of } (f\ \pi)\ A,\ A \in \mathcal{C}.$$

We have $\underline{g} = \underline{\binom{g}{s}}$ and $\underline{f} = \underline{(f\ \pi)}$. We claim that $\underline{g}: A \to C$ is the pseudokernel of $\underline{f}$.

The sequence $*)$ induces an exact sequence of functors:

$$0 \longrightarrow \mathrm{Hom}_\Lambda(-, A) \xrightarrow{\mathrm{Hom}\left(-\binom{g}{s}\right)} \mathrm{Hom}_\Lambda(-, B\oplus P) \xrightarrow{\mathrm{Hom}(-(f\ \pi))} \mathrm{Hom}_\Lambda(-, C).$$

It was proved in [10] that passing to the stable category the above sequence induces an exact sequence:

$$\underline{\mathrm{Hom}}_\Lambda(-, A) \xrightarrow{\underline{\mathrm{Hom}}(-g)} \underline{\mathrm{Hom}}_\Lambda(-, B) \xrightarrow{\underline{\mathrm{Hom}}(-f)} \underline{\mathrm{Hom}}_\Lambda(-, C),$$

as claimed.

PROPOSITION 3. *Let Λ be a finite dimensional associative K-algebra. Denote by $\mathcal{M}=\mathrm{mod}_\Lambda$ and let C be a resolving subcategory of $\mathcal{M}$. The inclusion of $\underline{C}$ in $\underline{\mathcal{M}}$ induces functors:*

$$\underline{\mathcal{M}} \underset{\underline{C}}{\otimes} - : \mathrm{Mod}(\underline{C}) \longrightarrow \mathrm{Mod}(\underline{\mathcal{M}}) \quad \text{and} \quad \mathrm{res}: \mathrm{Mod}(\underline{\mathcal{M}}) \longrightarrow \mathrm{Mod}(\underline{C}),$$

with res the restriction functor and $\underline{\mathcal{M}}\underset{\underline{C}}{\otimes}-$ its right adjoint. res is exact and $\underline{\mathcal{M}}\underset{\underline{C}}{\otimes}-$ is right exact preserving projective functors.

$\mathrm{mod}(\underline{C})$ *and* $\mathrm{mod}(\underline{\mathcal{M}})$ *are abelian categories and we have induced functors:*

$$\underline{\mathcal{M}} \underset{\underline{C}}{\otimes} - : \mathrm{mod}(\underline{C}) \longrightarrow \mathrm{mod}(\underline{\mathcal{M}}) \quad \text{and} \quad res: \mathrm{mod}(\underline{\mathcal{M}}) \longrightarrow \mathrm{Mod}(\underline{C}).$$

PROOF. This is a particular case of the functors considered in [2].

The following stable category was considered in [5]: $\underline{\mathrm{mod}}(\mathrm{mod}_\Lambda)$ denotes the full subcategory of $\mathrm{mod}(\mathrm{mod}_\Lambda)$ of all functors F vanishing on projectives, i.e. $F(\Lambda)=0$. We have the following proposition:

PROPOSITION 4 [5]. *There exists an isomorphism of categories:* $\underline{\mathrm{mod}}(\mathrm{mod}_\Lambda) \cong \mathrm{mod}(\underline{\mathrm{mod}}_\Lambda)$.

PROOF. We sketch the proof:

The canonical functor: $\pi: \mathrm{mod}_\Lambda \to \underline{\mathrm{mod}}_\Lambda$ induces a functor $\Phi: \mathrm{Mod}(\underline{\mathrm{mod}}) \to \underline{\mathrm{Mod}}(\mathrm{mod}_\Lambda)$ given by: $\Phi(F)=F\pi$, which in turn induces a functor: $\Phi': \mathrm{mod}(\underline{\mathrm{mod}}_\Lambda) \to \underline{\mathrm{mod}}(\mathrm{mod}_\Lambda)$. It is easily proved that Φ' is an isomorphism.

It was proved in [4] that the category $\underline{\mathrm{mod}}(\mathrm{mod}_\Lambda)$ has projective objects the functors $\underline{\mathrm{Hom}}_\Lambda(-, X)$ and injective objects the functors $\mathrm{Ext}^1_\Lambda(-, Y)$. We will consider the isomorphism of categories given in the above proposition as an identification. In this way we obtain a description of the projective and injective objects in $\mathrm{mod}(\underline{\mathrm{mod}}_\Lambda)$.

Returning to resolving categories we have the following proposition:

PROPOSITION 5. *Let Λ be any K-algebra and $\mathcal{C}$ a resolving subcategory af* $\mathrm{mod}_{\Lambda}=\mathcal{M}$, *then the following conditions are equivalent*:

i) *$\mathcal{C}$ is contravariantly finite.*

ii) *The functor res*: $\mathrm{mod}(\underline{\mathcal{M}})\to\mathrm{Mod}(\underline{\mathcal{C}})$ *has image contained in* $\mathrm{mod}(\underline{\mathcal{C}})$.

PROOF. i) imply ii):

Let $\underline{\mathrm{Hom}}(-, C)\in\mathrm{mod}(\underline{\mathcal{M}})$. $\mathcal{C}$ contravariantly finite implies there exists $X\in\mathcal{C}$ and an epimorphism of functors: $H: \mathrm{Hom}_{\mathcal{C}}(-, X)\to\mathrm{Hom}(-, C)|_{\mathcal{C}}\to 0$ which is given by a map: $f: X\to C$. Taking the projective cover of C we complete f to an exact sequence: $0\to A\to X\oplus P\to C\to 0$ which induces exact sequence of functors:

$$0\longrightarrow \mathrm{Hom}_{\Lambda}(-, A)\longrightarrow \mathrm{Hom}_{\Lambda}(-, X\oplus P)\longrightarrow \mathrm{Hom}_{\Lambda}(-, C)\quad\text{and}$$

$$\underline{\mathrm{Hom}}_{\Lambda}(-, A)\longrightarrow \underline{\mathrm{Hom}}_{\Lambda}(-, X)\longrightarrow \underline{\mathrm{Hom}}_{\Lambda}(-, C)\,.$$

Applying the functor res to the last sequence we obtain: $\underline{\mathrm{Hom}}_{\Lambda}(-, A)|_{\mathcal{C}}$ $\to\underline{\mathrm{Hom}}_{\mathcal{C}}(-, X)\to\underline{\mathrm{Hom}}_{\Lambda}(-, C)|_{\mathcal{C}}\to 0$.

Using again that $\mathcal{C}$ is contravariantly finite we obtain $Y\in\mathcal{C}$ and an epimorphism:

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(-, Y)\longrightarrow \underline{\mathrm{Hom}}(-, A)|_{\mathcal{C}}\longrightarrow 0\,.$$

Hence; we have an exact sequence of functors:

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(-, Y)\longrightarrow \underline{\mathrm{Hom}}_{\mathcal{C}}(-, X)\longrightarrow \underline{\mathrm{Hom}}_{\Lambda}(-, C)|_{\mathcal{C}}\longrightarrow 0\,.$$

Therefore res $\underline{\mathrm{Hom}}(-, C)\in\mathrm{mod}(\mathcal{C})$.

Now let $F\in\mathrm{mod}(\underline{\mathcal{M}})$. There exists a presentation:

$$\underline{\mathrm{Hom}}(-, B)\longrightarrow \underline{\mathrm{Hom}}(-, C)\longrightarrow F\longrightarrow 0$$

with $B, C\in\mathrm{mod}_{\Lambda}$.

Applying res we obtain an exact sequence:

$$\mathrm{res}\,\underline{\mathrm{Hom}}(-, B)\longrightarrow \mathrm{res}\,\underline{\mathrm{Hom}}(-, C)\longrightarrow \mathrm{res}\,F\longrightarrow 0\,.$$

res $\underline{\mathrm{Hom}}(-, B)$, res $\underline{\mathrm{Hom}}(-, C)\in\mathrm{mod}(\underline{\mathcal{C}})$ and $\mathrm{mod}(\underline{\mathcal{C}})$ abelian imply that res $F\in$ $\mathrm{mod}(\underline{\mathcal{C}})$.

ii) imply i):

Let $B\in\mathrm{mod}_{\Lambda}$. res $\underline{\mathrm{Hom}}(-, B)$ finitely presented implies there exists $X\in\mathcal{C}$ and an epimorphism: $\underline{\mathrm{Hom}}_{\mathcal{C}}(-, X)\to\underline{\mathrm{Hom}}(-, B)|_{\mathcal{C}}\to 0$ induced by a map: $\underline{f}: X\to B$. Let P be the projective cover of B and f a representative of $\underline{f}$, f can be completed to an epimorphism: $X\oplus P\to B\to 0$ which induces an epimorphism of functors: $\mathrm{Hom}_{\mathcal{C}}(-, X\oplus P)\to\mathrm{Hom}(-, B)|_{\mathcal{C}}\to 0$. Since $X\oplus P$ is an object of $\mathcal{C}$, $\mathcal{C}$ is contravariantly finite.

COROLLARY. *Let Λ be a finite dimensional K-algebra of infinite dominant dimension and $\mathcal{D}$ the category of modules of infinite dominant dimension. If the functor*: res: $\mathrm{mod}(\underline{\mathcal{M}})\to\mathrm{Mod}(\underline{\mathcal{D}})$ *has image contained in* $\mathrm{mod}(\underline{\mathcal{D}})$ *then Λ is selfinjective.*

PROOF. It was proved in [11] that if $\mathcal{D}$ is contravariantly finite and Λ of infinite dominant dimension then Λ is selfinjective.

PROPOSITION 6. *Let Λ be an artin algebra then*:

i) $\underline{\mathrm{mod}}(\mathrm{mod}_\Lambda)$ *has projective objects the functors* $\underline{\mathrm{Hom}}(-, B)$ *and injective objects the functors* $\mathrm{Ext}^1_\Lambda(-, B)$ *with* $B\in\mathrm{mod}_\Lambda$.

ii) *A projective functor* $\underline{\mathrm{Hom}}(-, B)$ *is injective if and only if the projective cover of B is injective. In this case* $\underline{\mathrm{Hom}}(-, B)\cong\mathrm{Ext}^1_\Lambda(-, \Omega(B))$.

iii) *An injective functor* $\mathrm{Ext}^1_\Lambda(-, B)$ *is projective if and only if the injective envelope of B is projective. In this case* $\mathrm{Ext}^1_\Lambda(-, B)\cong\underline{\mathrm{Hom}}(-, \Omega^{-1}(B))$.

We will say that $\underline{\mathrm{mod}}(\mathrm{mod}_\Lambda)$ is Frobenius when every projective functor is injective and every injective functor is projective.

We recall the following definitions from [9] and [4]:

Let Λ be an artin algebra. A node of Λ is a non projective, non injective, simple module S such that there exists an almost split sequence: $0\to\tau(S)\to P\to S\to 0$ with P projective.

A non splittable exact sequence $0\to A\to B\to C\to 0$ is minimal if it is not isomorphic to a direct sum of a non splittable exact sequence $0\to A'\to B'\to C'\to 0$ and a splittable sequence $0\to A''\to B''\to C''\to 0$ such that not all A'', B'', C'' are zero.

It was proved in [4] that given a minimal exact sequence $0\to A\to B\to C\to 0$, we have an exact sequence of functors: $0\to\mathrm{Hom}_\Lambda(, A)\to\mathrm{Hom}_\Lambda(, B)\to\mathrm{Hom}_\Lambda(, C)\to F\to 0$. The sequence is a minimal projective resolution of F in $\mathrm{mod}(\mathrm{mod}_\Lambda)$, $F\in\underline{\mathrm{mod}}(\mathrm{mod}_\Lambda)$ and we have a minimal projective presentation of F in $\underline{\mathrm{mod}}(\mathrm{mod}_\Lambda)$: $\underline{\mathrm{Hom}}(, B)\to\underline{\mathrm{Hom}}(, C)\to F\to 0$.

We have the following:

PROPOSITION 7. *Let Λ be an indecomposable artin algebra with no nodes. The the following conditions are equivalent*:

i) *Λ is selfinjective or Morita equivalent to* $A_1=\begin{pmatrix} D & 0 \\ D & D \end{pmatrix}$ *with D a finite dimensional division algebra.*

ii) $\underline{\mathrm{mod}}(\mathrm{mod}_\Lambda)$ *is Frobenius.*

iii) *Every projective functor in* $\underline{\mathrm{mod}}(\mathrm{mod}_\Lambda)$ *is injective.*

iv) *Every injective functor in* $\underline{\mathrm{mod}}(\mathrm{mod}_\Lambda)$ *is projective.*

PROOF. We may assume Λ is basic.

It is clear that ii) implies iii) and iv).

i) implies ii):

By Proposition 6 if Λ is selfinjective then $\underline{\mathrm{mod}}(\mathrm{mod}_\Lambda)$ is Frobenius.

If $\Lambda=A_1$ then Λ has only three indecomposable modules: a simple projective S_1, a projective injective P and a simple injective S. They are related by the exact sequence: $0\to S_1\to P\to S\to 0$.

Therefore $\underline{\mathrm{Hom}}(-, S)\cong\mathrm{Ext}^1_\Lambda(-, S_1)$ is a projective injective functor which is the only indecomposable projective functor and the only indecomposable injective functor.

We prove that iv) implies i).

Let S be a non injective simple module and I its injective envelope. The exact sequence: $0\to S\to I\to I/S\to 0$ is minimal and induces an exact sequence of functors:

$$*)\quad \underline{\mathrm{Hom}}(-, I)\longrightarrow \underline{\mathrm{Hom}}(-, I/S)\longrightarrow F\longrightarrow 0\,,\quad \text{where } F\cong\mathrm{Ext}^1_\Lambda(-, S).$$

The sequence $*$) is a minimal projective presentation of F, if we assume that F is projective then $\underline{\mathrm{Hom}}(-, I)=0$ and I is projective. Hence; every non simple indecomposable injective is projective.

Let S be a simple injective which is non projective. We know by [3] that the almost split sequence of S is: $0\to\tau(S)\to E\to S\to 0$ with E injective. It follows that E is projective and by [3], E is projective and $\tau(S)$ is simple. If $\tau(S)=S_1$ is non projective then it is a node. Contradicting the hypothesis.

Assume S_1 is projective. Let P' be an indecomposable projective and assume that there is a non zero map: $f: S_1\to P'$. $P=E$ is an indecomposable projective injective of length two. The inclusion $i: S_1\to P$ extends to P'. i.e. there exists a map $g: P'\to P$ such that $gf=i$. If $\mathrm{im}\, g=S_1$ then f is an isomorphism. If $\mathrm{im}\, g=P$ then P is isomorphic to P'. It follows that if P' is an indecomposable projective which is isomorphic neither to S_1 nor to P then $\mathrm{Hom}_\Lambda(S_1\oplus P, P')=0$ $=\mathrm{Hom}_\Lambda(P', S_1\oplus P)$.

Since Λ is indecomposable $\Lambda\cong\mathrm{End}_\Lambda(S_1\oplus P)^{op}\cong\begin{pmatrix} D & 0 \\ D & D \end{pmatrix}$.

$$D \cong \mathrm{End}_\Lambda(S_1)^{op} \cong \mathrm{End}_\Lambda(P)^{op} \cong \mathrm{Hom}_\Lambda(S_1, P)\,.$$

iii) implies i) is proved by dual arguments.

We say that a functor $F \in \mathrm{mod}(\underline{\mathcal{M}})$ has dominant dimension n if there exists a minimal coresolution:

$$0 \longrightarrow F \longrightarrow \mathrm{Ext}^1_\Lambda(-, A_1) \longrightarrow \mathrm{Ext}^1_\Lambda(-, A_2) \longrightarrow$$
$$\cdots \mathrm{Ext}^1_\Lambda(-, A_n) \longrightarrow \mathrm{Ext}^1_\Lambda(-, A_{n+1}) \longrightarrow \cdots$$

with $A_i \in \mathcal{D}om_i$ for $1 \leqq i \leqq n$ and $A_{n+1} \notin \mathcal{D}om_1$.

We can extend the notions of [11] as follows:

$$\mathcal{D}om_n(\underline{\mathcal{M}}) = \{F \in \mathrm{mod}(\underline{\mathcal{M}}) \mid \mathrm{domdim}\, F \geqq n\}, \quad \mathcal{D}om(\underline{\mathcal{M}}) = \bigcap_{n \geqq 1} \mathcal{D}om_n(\underline{\mathcal{M}})\,.$$

We say that $\underline{\mathcal{M}}$ has dominant dimension greater or equal to n if $\underline{\mathrm{Hom}}(-, B) \in \mathcal{D}om_n(\underline{\mathcal{M}})$ for all $B \in \mathrm{mod}_\Lambda$. $\underline{\mathcal{M}}$ has infinite dominant dimension if $\underline{\mathrm{Hom}}(-, B) \in \mathcal{D}om_n(\underline{\mathcal{M}})$ for all $B \in \mathrm{mod}_\Lambda$ and all n.

PROPOSITION 8. *Let Λ be a 1-Gorenstein artin algebra, $\mathcal{M} = \mathrm{mod}_\Lambda$ then $\underline{\mathcal{M}}$ has dominant dimension greater or equal to two.*

PROOF. Let $B \in \mathrm{mod}_\Lambda$, and $0 \to \Omega(B) \to \Omega \to B \to 0$ exact with Q the projective cover of B. The injective envelope of Q is projective, hence the injective envelope of $\Omega(B)$ is also projective. This means that $\mathrm{Ext}^1_\Lambda(-, Q)$, $\mathrm{Ext}^1_\Lambda(-, \Omega(B))$ are projective injective functors.

By the long homology sequence of functors we have an exact sequence: $0 \to \underline{\mathrm{Hom}}(-, B) \to \mathrm{Ext}^1_\Lambda(-, \Omega(B)) \to \mathrm{Ext}^1_\Lambda(-, Q)$.

It follows that $\underline{\mathrm{Hom}}(-, B) \in \mathcal{D}om_2(\underline{\mathcal{M}})$.

Compare Propositions 6 and 8 with the following:

PROPOSITION 9. *Let Λ be an algebra of infinite dominant dimension, $\mathcal{D}$ is the category of modules of infinite dominant dimension then:*

i) $\mathrm{mod}(\underline{\mathcal{D}})$ *is an abelian category with enough projective and injective objects.*

ii) *Every injective functor in* $\mathrm{mod}(\underline{\mathcal{D}})$ *is projective.*

iii) $\underline{\mathcal{D}}$ *has infinite dominant dimension. Moreover, every element of* $\mathrm{mod}(\underline{\mathcal{D}})$ *has infinite dominant dimension.*

PROOF. i) Consider the tensor product functor: $\underline{\mathcal{M}} \otimes_{\underline{\mathcal{D}}} - : \mathrm{mod}(\underline{\mathcal{D}}) \to \mathrm{mod}(\underline{\mathcal{M}})$, denote by $\mathcal{A}$ its image. $\mathcal{A}$ is called the category of induced functors. It is known by [2] that the tensor product induces an equivalence of categories:

$\mathrm{mod}(\underline{\mathscr{D}})\cong\mathscr{A}$.

If $F\in\mathrm{mod}(\underline{\mathscr{D}})$ and $\underline{\mathrm{Hom}}_{\mathscr{D}}(-, B)\to\underline{\mathrm{Hom}}_{\mathscr{D}}(-, C)\to F\to 0$ is a minimal presentation then applying $\underline{\mathscr{M}}\underset{\underline{\mathscr{D}}}{\otimes}-$ to the presentation of F we obtain:

$$\underline{\mathscr{M}}\underset{\underline{\mathscr{D}}}{\otimes}\underline{\mathrm{Hom}}_{\mathscr{D}}(-, B) \longrightarrow \underline{\mathscr{M}}\underset{\underline{\mathscr{D}}}{\otimes}\underline{\mathrm{Hom}}_{\mathscr{D}}(-, C) \longrightarrow \underline{\mathscr{M}}\underset{\underline{\mathscr{D}}}{\otimes}F \longrightarrow 0\,.$$

But $\underline{\mathscr{M}}\underset{\underline{\mathscr{D}}}{\otimes}\underline{\mathrm{Hom}}_{\mathscr{D}}(-, B)\cong\underline{\mathrm{Hom}}(-, B)$ and $\underline{\mathscr{M}}\underset{\underline{\mathscr{D}}}{\otimes}\underline{\mathrm{Hom}}_{\mathscr{D}}(-, C)\cong\underline{\mathrm{Hom}}(-, C)$.

Set $G=\underline{\mathscr{M}}\underset{\underline{\mathscr{D}}}{\otimes}F$. G has a projective presentation:

$\underline{\mathrm{Hom}}(-, B)\to\underline{\mathrm{Hom}}(-, C)\to G\to 0$. Hence: $\mathscr{A}$ is the full subcategory of $\mathrm{mod}(\underline{\mathscr{M}})$ consisting of all functors G with a minimal projective presentation:

$$\underline{\mathrm{Hom}}(-, B) \longrightarrow \underline{\mathrm{Hom}}(-, C) \longrightarrow G \longrightarrow 0\,, \quad \text{with } B, C\in\mathscr{D}.$$

By [1] $\mathscr{A}$ is abelian, by [2] has enough projectives and they are of the form: $\underline{\mathrm{Hom}}(\ , B)$ with $B\in\mathscr{D}$. By [4] $\mathscr{A}$ has enough injectives and they are of the form $\mathrm{Ext}^1_\Lambda(\ , B)$ with $B\in\mathscr{D}$.

ii) is clear.

iii) Let $F\in\mathrm{mod}(\underline{\mathscr{D}})$ and $\underline{\mathrm{Hom}}(-, B)\to\underline{\mathrm{Hom}}(-, C)\to F\to 0$ with $B, C\in\mathscr{D}$ (B may be projective). We know by [10], that there exists a finitely generated projective P and a map $\pi: P\to C$ such that: $*$) $0\to A\to B\oplus P\xrightarrow{(f\ \pi)} C\to 0$ is a minimal exact sequence.

Since $\mathscr{D}$ is resolving, $A\in\mathscr{D}$. The sequence $*$) induces exact sequences of functors:

$$0 \longrightarrow \mathrm{Hom}_\Lambda(-, A) \longrightarrow \mathrm{Hom}_\Lambda(-, B\oplus P) \longrightarrow \mathrm{Hom}_\Lambda(-, C) \longrightarrow F \longrightarrow 0$$

$$\begin{aligned}0 \longrightarrow F \longrightarrow \mathrm{Ext}^1_\Lambda(-, A) &\longrightarrow \mathrm{Ext}^1_\Lambda(-, B\oplus P) \longrightarrow \mathrm{Ext}^1_\Lambda(-, C) \longrightarrow\\ \longrightarrow \mathrm{Ext}^2_\Lambda(-, A) &\longrightarrow \mathrm{Ext}^2_\Lambda(-, B\oplus P) \longrightarrow \mathrm{Ext}^2_\Lambda(-, C) \longrightarrow\\ \cdots \qquad &\qquad\quad \cdots \qquad\qquad\qquad \cdots\\ \longrightarrow \mathrm{Ext}^k_\Lambda(-, A) &\longrightarrow \mathrm{Ext}^k_\Lambda(-, B\oplus P) \longrightarrow \mathrm{Ext}^k_\Lambda(-, C) \longrightarrow \cdots.\end{aligned}$$

But $\mathrm{Ext}^k_\Lambda(-, X)\cong\mathrm{Ext}^1_\Lambda(-, \Omega^{-k+1}(X))$ for all $k\geqq 1$ and $X\in\mathscr{D}$ implies $\Omega^{-k}X\in\mathscr{D}$ for all $k\geqq 0$. Therefore the above long sequence is an injective coresolution of F consisting of projective injective functors.

We want to compare the categories $\mathscr{D}_{om}(\underline{\mathscr{M}})$ and $\mathrm{mod}(\underline{\mathscr{D}})$. We do so in the following:

THEOREM 10. *Let Λ be a finite dimensional associative algebra of infinite dominant dimension. $\mathscr{M}$, $\mathscr{D}$, $\mathrm{mod}(\underline{\mathscr{D}})$, $\mathrm{mod}(\underline{\mathscr{M}})$ the categories of functors defined above. Then the following two conditions hold:*

i) *The tensor product functor*: $\underline{\mathcal{M}}\underset{\underline{\mathcal{D}}}{\otimes}-:\mathrm{mod}(\underline{\mathcal{D}})$ *has image contained in* $\mathcal{D}om(\underline{\mathcal{M}})$.

ii) *Let* $F\in\mathcal{D}om(\underline{\mathcal{M}})$ *be a functor with no projective injective summands, then there exists a functor* $F'\in\mathrm{mod}(\underline{\mathcal{D}})$ *such that*: $F\in\underline{\mathcal{M}}\underset{\underline{\mathcal{D}}}{\otimes}F'$.

PROOF. i) Was proved in the previous proposition.

We prove ii) now:

Let $F\in\mathcal{D}om(\underline{\mathcal{M}})$ be a functor with no projective injective summands and $\underline{\mathrm{Hom}}(-, B')\xrightarrow{\underline{\mathrm{Hom}}(-f')}\underline{\mathrm{Hom}}(-, C)\to F\to 0$ a minimal projective presentation of F. We want to prove that $B', C\in\mathcal{D}om_n$ for every integer n. We will prove it by induction on n.

Let:

$$0\longrightarrow F\longrightarrow \mathrm{Ext}^1_\Lambda(-, A_1)\longrightarrow \mathrm{Ext}^1_\Lambda(-, A_2)\longrightarrow\cdots \mathrm{Ext}^1_\Lambda(-, A_n)\longrightarrow$$

be a minimal injective coresolution of F consisting of projective injectives. i.e. $A_i\in\mathcal{D}om_1$ for each i.

Since the presentation of F give above is minimal, the map $f': B'\to C$, with f' a representative of $\underline{f}'$, can be completed to a minimal exact sequence: (see [10]) $0\to A\xrightarrow{\binom{g'}{j'}} B'\oplus P'\xrightarrow{(f'\,p')} C\to 0$, where P' is finitely generated projective. Assume $P'\cong Q\oplus P$ where Q has no injective summands and P is projective injective. Set $B=B'\oplus Q$. B has no projective injective summands. We write the above sequence as: *) $0\to A\xrightarrow{\binom{g}{j}} B\oplus P\xrightarrow{(f\ p)} C\to 0$. By [4], *) induces a minimal injective copresentation of F: $0\to F\to \mathrm{Ext}^1_\Lambda(-, A)\xrightarrow{\mathrm{Ext}(-g)}\mathrm{Ext}^1_\Lambda(-, B)$.

Decompose B as $B=B_1\oplus I$, I injective and B_1 with no injective summands. The map g decomposes as: $g: A\to B_1\oplus I$ $g=\binom{g_1}{g_2}$. We have the following commutative square with the vertical maps isomorphisms:

$$\begin{array}{ccc}\mathrm{Ext}^1_\Lambda(-, A) & \xrightarrow{\mathrm{Ext}(-g_1)} & \mathrm{Ext}^1_\Lambda(-, B_1)\\ \downarrow & & \downarrow\\ \mathrm{Ext}^1_\Lambda(-, A) & \xrightarrow{\mathrm{Ext}(-g)} & \mathrm{Ext}^1_\Lambda(-, B)\end{array}$$

Therefore: $F\cong\mathrm{Ker}\,\mathrm{Ext}^1_\Lambda(-g_1)$.

A has no injective summands, otherwise *) would not be minimal. It follows $A\cong A_1$ and $B_1\cong A_2$. So, A, $B_1\in\mathcal{D}om_1$.

Let P_1 be the injective envelope of A. P_1 is projective, the map $g_1: A\to B_1$

can be completed to an exact sequence: $0 \leftarrow A \xrightarrow{\binom{g_1}{f_1}} B_1 \oplus P_1 \rightarrow Y \rightarrow 0$, which induces exact sequences of functors:

$$0 \longrightarrow \mathrm{Hom}_\Lambda(-, A) \longrightarrow \mathrm{Hom}_\Lambda(-, B_1 \oplus P_1) \longrightarrow \mathrm{Hom}_\Lambda(-, Y) \longrightarrow H \longrightarrow 0$$

$$0 \longrightarrow H \longrightarrow \mathrm{Ext}^1_\Lambda(-, A) \longrightarrow \mathrm{Ext}^1_\Lambda(-, B_1 \oplus P_1).$$

It follows as above that $F \cong H$.

F has a minimal projective resolution in mod(mod$_\Lambda$):

$$0 \longrightarrow \mathrm{Hom}_\Lambda(-, A) \longrightarrow \mathrm{Hom}_\Lambda(-, B \oplus P) \longrightarrow \mathrm{Hom}_\Lambda(-, C) \longrightarrow F \longrightarrow 0.$$

It follows that $\mathrm{Hom}_\Lambda(-, B \oplus P)$ is a summand of $\mathrm{Hom}_\Lambda(-, B_1 \oplus P_1)$. Therefore there exists a module Z such that: $B \oplus P \oplus Z \cong B_1 \oplus P_1 \cong B_1 \oplus I \oplus P \oplus Z$. Hence; I is projective.

Since B has no projective injective injective summands $I=0$ and $B \cong A_2$.

Consider now the exact sequence of functors:

$$\underline{\mathrm{Hom}}(-, C) \longrightarrow \underline{\mathrm{Hom}}(-, \Omega^{-1}A) \longrightarrow \Omega^{-1}F \longrightarrow 0.$$

Since F has no projective injective summands the above sequence is a minimal projective presentation of $\Omega^{-1}F$.

Repeating the arguments used, we have a minimal exact sequence: $0 \rightarrow X \rightarrow C \oplus P_2 \rightarrow \Omega^{-1}A \rightarrow 0$, with P_2 projective.

This sequence induces a minimal injective copresentation of $\Omega^{-1}F$: $0 \rightarrow \Omega^{-1}F \rightarrow \mathrm{Ext}^1_\Lambda(-, X) \rightarrow \mathrm{Ext}^1_\Lambda(-, C \oplus P_2)$, but $\Omega^{-1}F$ has a copresentation: $0 \rightarrow \Omega^{-1}F \rightarrow \mathrm{Ext}^1_\Lambda(-, B) \rightarrow \mathrm{Ext}^1_\Lambda(-, C)$, minimal in the first term and X, B with no injective summands, it follows that $X \cong B$ and P_2 is injective.

It follows as in the first part of the proof, that C has no injective summands and $C \in \mathcal{D}om_1$.

The sequence: $\underline{\mathrm{Hom}}(-, \Omega^{-1}B) \rightarrow \underline{\mathrm{Hom}}(-, \Omega^{-1}C) \rightarrow \Omega^{-3}F \rightarrow 0$ is a minimal projective presentation of the functor $\Omega^{-3}F$ with no projective injective summands and of infinite dominant dimension. So we have the same hypothesis on F and $\Omega^{-3}F$, if we assume $\Omega^{-1}B$, $\Omega^{-1}C \in \mathcal{D}om_n$ then B, $C \in \mathcal{D}om_{n+1}$. It follows by induction that B, C, $A \in \mathcal{D}om$.

Therefore $F \cong \underline{\mathcal{M}} \otimes_{\underline{\mathcal{D}}} \mathrm{res}\, F$, as claimed.

Denote by $\mathcal{C}odom_1$ the full subcategory of mod$_\Lambda$ of all modules with a projective cover which is also injective.

Given any full subcategory $\mathcal{C}$ of mod$_\Lambda$ the following notion was defined in [7]:

Rapp($\mathcal{C}$)$=\{X \in \mathrm{Mod}_\Lambda |$ there exists $C \in \mathcal{C}$ and a map $f: C \rightarrow X$ such that the following sequence of functors is exact: $\mathrm{Hom}_{\mathcal{C}}(\ ,C)\to\mathrm{Hom}_{\Lambda}(\ ,X)|_{\mathcal{C}}\to 0\}$, where $\mathrm{Hom}_{\Lambda}(\ ,X)|_{\mathcal{C}}$ denotes the restriction of $\mathrm{Hom}_{\Lambda}(\ ,X)$ to the category $\mathcal{C}$.

We have the following lemma:

LEMMA 11. *Assume Λ is a K-algebra of infinite dominant dimension. If $\mathcal{C}odom_1\subseteq \mathrm{Rapp}(\mathcal{D})$ then $\mathcal{D}$ is contravariantly finite.*

PROOF. i) Let X be any finitely generated Λ-module, I its injective envelope and $0\to X\to I\to \Omega^{-1}X\to 0$ an exact sequence. The projective cover of I is injective, ([11]) hence $\Omega^{-1}X\in\mathcal{C}odom_1$.

By [11], $\mathrm{Rapp}(\mathcal{D}om)=\{X\in \mathrm{mod}_{\Lambda}|$ there exists an integer $k\geqq 0$ such that $\Omega^{-k}(X)\in\mathcal{D}om\}$.

It follows that $X\in\mathrm{Rapp}(\mathcal{D})$.

From this lemma we obtain our main theorem:

THEOREM 12. *Let Λ be a K-algebra of infinite dominant dimension, let* $\mathrm{mod}(\underline{\mathcal{D}})$, $\mathrm{mod}(\underline{\mathcal{M}})$ *be the categories defined above and* $\underline{\mathcal{M}}\otimes_{\underline{\mathcal{D}}}-:\mathrm{mod}(\underline{\mathcal{D}})\to\mathrm{mod}(\underline{\mathcal{M}})$ *the tensor functor.*

Assume that the category of induced functors is contravariantly finite in $\mathcal{D}om(\underline{\mathcal{M}})$, then Λ is selfinjective.

PROOF. Let $B\in\mathcal{C}odom_1$ then $\underline{\mathrm{Hom}}(-,B)\cong \mathrm{Ext}^1_{\Lambda}(-,\Omega B)\in\mathcal{D}om(\underline{\mathcal{M}})$. Let F be an induced functor and $\phi: F\to\underline{\mathrm{Hom}}(-,B)$ a right approximation. The projective cover of F is of the form $\pi:\underline{\mathrm{Hom}}(-,A)\to F\to 0$, with $A\in\mathcal{D}om_{\Lambda}$. By Yoneda's lemma, the natural transformation $\phi\pi$ is $\phi\pi=\underline{\mathrm{Hom}}(-,\underline{f})$. Let $C\in\mathcal{D}om_{\Lambda}$, $\underline{\mathrm{Hom}}(-,C)$ is in $\mathcal{D}om(\underline{\mathcal{M}})$.

For any map: $\underline{\mathrm{Hom}}(-,\underline{g}):\underline{\mathrm{Hom}}(-,C)\to\underline{\mathrm{Hom}}(-,B)$ there exists a map: $\eta:\underline{\mathrm{Hom}}(-,C)\to F$ such that $\phi\eta=\underline{\mathrm{Hom}}(-,g)$. But $\underline{\mathrm{Hom}}(-,C)$ projective implies there exists a map: $\underline{\mathrm{Hom}}(-,h)$: $\underline{\mathrm{Hom}}(-,C)\to\underline{\mathrm{Hom}}(-,A)$ such that $\pi\,\underline{\mathrm{Hom}}(-,h)=\eta$.

Therefore $\underline{\mathrm{Hom}}(-,\underline{f})\underline{\mathrm{Hom}}(-,\underline{h})=\underline{\mathrm{Hom}}(-,\underline{g})$. Then the map $\underline{f}:A\to B$ has the property that any map $\underline{g}:C\to B$ lifts to A. Taking the projective cover of B, Q, we obtain a right $\mathcal{D}$-approximation to B, $A\oplus Q\to B$.

We have proved that $B\in\mathrm{Rapp}(\mathcal{D})$.

By Lemma 11, $\mathcal{D}$ is contravariantly finite and by [11], Λ is selfinjective.

COROLLARY. *Let Λ be a K-algebra of infinite dominant dimension, let* $\mathrm{mod}(\underline{\mathcal{D}})$, mod($\underline{\mathscr{M}}$) *be the categories defined above and* $\underline{\mathscr{M}}\underset{\underline{\mathscr{D}}}{\otimes}-$: mod($\underline{\mathscr{D}}$)$\rightarrow$mod($\underline{\mathscr{M}}$) *the tensor functor.*

Assume that the category of induced functors is contravariantly finite in mod($\underline{\mathscr{M}}$), *then* Λ *is selfinjective.*

PROOF. If every functor in mod($\underline{\mathscr{M}}$) can be right approximated by induced functors, in particular, every functor in $\mathscr{D}om(\underline{\mathscr{M}})$ can be right approximated by by induced functors.

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