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A TRANSFORMATION GROUP OF THE PYTHAGOREAN NUMBERS

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Our purpose in this note is to study a transformation group of the Pythagorean numbers using the theory of Kac-Moody Lie algebras. We will essentially use the conjugacy theorem, established by Kac [1], for null roots in infinite root systems. Mariani [3] has also given a transformation group of the Pythagorean numbers in a different way. We will discuss about the relationship.

It is well-known that all the integral solutions, called the Pythagorean numbers, of the Pythagorean equation:

$$x^2+y^2=z^2$$

are given by

$$\begin{cases} x=n(a^2-b^2) \\ y=2nab \\ z=n(a^2+b^2) \end{cases} \quad \text{or} \quad \begin{cases} x=2nab \\ y=n(a^2-b^2) \\ z=n(a^2+b^2) \end{cases}$$

for all $n, a, b\in \boldsymbol{Z}$.

Put $M=\{(x, y, z)\in \boldsymbol{Z}^3 | x^2+y^2=z^2,\ gcd(x, y, z)=1\}$, the set of all the primitive Pythagorean numbers, and $M'=\{(x, y, z)\in M | y=\text{even},\ z>0\}$. We choose the following basic transformations of M:

$$\begin{aligned} r_1: &\ (x, y, z) \longmapsto (-x, y, z), \\ r_3: &\ (x, y, z) \longmapsto (x, -y, z), \\ -I: &\ (x, y, z) \longmapsto (-x, -y, -z), \\ t: &\ (x, y, z) \longmapsto (y, x, z). \end{aligned}$$

These are arising from the symmetries of the Pythagorean equation. Furthermore we can find an important transformation of M:

$$r_2: (x, y, z) \longmapsto (-x-2y+2z,\ -2x-y+2z,\ -2x-2y+3z).$$

Let W be the subgroup of $GL_3(\boldsymbol{Z})$ generated by the r_i $(1\leq i\leq 3)$, and G the subgroup of $GL_3(\boldsymbol{Z})$ generated by W, t and $-I$. Put $O_{2,1}(\boldsymbol{Z})=O(2, 1)\cap GL_3(\boldsymbol{Z})$, the orthogonal group over $\boldsymbol{Z}$ defined by the quadratic form $x^2+y^2-z^2$.

THEOREM. (a) $M'=W.(1, 0, 1)$ and $M=G.(1, 0, 1)$.

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(b) $[G:W]=4$ and $G=O_{2,1}(\boldsymbol{Z})$.

(c) The stabilizer of (1, 0, 1) in W is the infinite dihedral group generated by r_2 and r_3.

PROOF. We choose the following generalized Cartan matrix of hyperbolic type (cf. [2]):

$$A=\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}.$$

Let $\Lambda=\sum_{i=1}^{3}\boldsymbol{Z}\alpha_i$ be the root lattice and Δ the root system associated with A. We define the bilinear form on Λ by

$$((\alpha_i, \alpha_j))=\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Then $\alpha\in\Delta$ if $\alpha\in\Lambda$ has the property $(\alpha, \alpha)=0$ (cf. [1], [4]). Such an element is called a null root. Put $N=\left\{\alpha\in\Lambda \,\middle|\, \frac{1}{n}\alpha\notin\Lambda\ (n=2, 3, \cdots),\ (\alpha, \alpha)=0\right\}$, the set of primitive null roots.

Let $\beta_1=-\alpha_1$, $\beta_2=-\alpha_3$, $\beta_3=\alpha_1+\alpha_2+\alpha_3$. Then $\{\beta_1, \beta_2, \beta_3\}$ is a new basis of Λ and $(\beta_3, \beta_3)=-1$. Therefore an element $\beta=x\beta_1+y\beta_2+z\beta_3$ is a null root if and only if

$$(\beta, \beta)=x^2+y^2-z^2=0,$$

that is, (x, y, z) is a Pythagorean number. We identify M with N. Then r_i is the reflection with respect to α_i, and W is the Weyl group of Δ. In general, it has been established by Kac [1; Lemma 1.9 d)] that a null root is conjugate to a null root of an affine subdiagram under the action of the Weyl group. Therefore, in our case, we see $N=W.(\pm\alpha_1 \pm\alpha_2)\cup W.(\pm\alpha_2 \pm\alpha_3)$, which implies (a). (b): The index $[G:W]$ is 4 since $G=(W\rtimes\langle t\rangle)\times\{\pm I\}$. An element $\alpha\in A$ with $(\alpha, \alpha)=1$ is a root, so $\{g(\alpha_i)|1\leq i\leq 3\}$ is a fundamental system of Δ for all $g\in O_{2,1}(\boldsymbol{Z})$. Therefore the conjugacy theorem of Kac for fundamental systems (cf. [2]) leads to $G=O_{2,1}(\boldsymbol{Z})$. (c) follows from the fact that $\alpha_2+\alpha_3$ is in the (standard) fundamental domain for the action of W on the positive imaginary roots (cf. [2]). □

Let ϕ be the isomorphism of W into $PGL_2(\boldsymbol{Z})$ defined by $\phi(r_1)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\phi(r_2)=\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$, and $\phi(r_3)=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ mod $\begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix}$. The group which Mariani [3] has constructed is $\phi(W)$. However, his theorems 2 and 3 are misunderstanding —he claims that $\phi(W)$ is isomorphic to $GL_2(\boldsymbol{Z})$ in Theorem 2 and that the stabilizer of an element of M' is the direct product of an infinite cyclic group and a group of order 2 in Theorem 3. To be exact, $\phi(W)$ is a subgroup of $PGL_2(\boldsymbol{Z})$ with the group index $[PGL_2(\boldsymbol{Z}):\phi(W)]=3$ and the stabilizer of an element of M' is an infinite dihedral group.

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