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3-DIMENSIONAL SPACE-LIKE SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR OF AN INDEFINITE SPACE FORM II

By

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Introduction.

For an n-dimensional complete space-like submanifold M with parallel mean curvature vector of $S_p^{n+p}(c)$, it is seen by Cheng [3] that M is totally umbilic if $n=2$ and $H^2 \leqq c$ or if $n>2$ and $n^2H^2<4(n-1)c$, where H denotes the mean curvature, i.e., the norm of the mean curvature vector $\boldsymbol{h}$. On the other hand, Aiyama and Cheng [1] prove the following

THEOREM A. *Let M be a complete space-like hypersurface in a Lorentz space form $M_1^4(c)$ with parallel mean curvature vector. If the Ricci curvature is bounded from above by some number less than $3(c-H^2)$, then c must be positive and M is congruent to a Riemannian 3-sphere $S^3(c')$.*

Recentely we verified the following which is Theorem 1 in [2] as a high codimensional version of Theorem A.

THEOREM B. *Let M be a 3-dimensional complete space-like submanifold with non-zero parallel mean curvature vector $\boldsymbol{h}$ of an indefinite space form $S_p^{3+p}(c)$, $p \geqq 2$. If it satisfies*

$$\frac{8}{9}c \leqq H^2 \leqq c \quad and \quad Ric(M) \leqq \delta_1 < 3(c-H^2), \tag{1.1}$$

then M is totally umbilic.

However, we get a more natural extending theorem. In this paper, we verify the following theorem.

THEOREM. *Let M be a complete 3-dimensional space-like submanifold in an*

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indefinite space form $M_p^{3+p}(c)$ *with parallel mean curvature vector. If the Ricci curvature is bounded from above by some number less than* $3(c-H^2)$, *then* c *must be positive and* M *is congruent to a Riemannian* 3-*sphere* $S^3(c')$.

Theorem B is included in the above theorem.

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2. Preliminaries.

Throughout this paper all manifolds are assumed to be smooth, connected without boundary. We discuss in smooth category. Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional indefinite Riemannian manifold of index p and with constant curvature c, which is called *an indefinite space form of index p*. According to $c>0$, $c=0$ or $c<0$ it is denoted by $S_p^{n+p}(c)$, R_p^{n+p} or $H_p^{n+p}(c)$. A submanifold M of an indefinite space form $M_p^{n+p}(c)$ is said to be *space-like* if the induced metric on M from that of the ambient space is positive definite.

Let M be an n-dimensional space-like submanifold of $M_p^{n+p}(c)$. We choose a local field of orthonormal frames $e_1, \cdots, e_{n+p}$ adapted to the indefinite Riemannian metric of $M_p^{n+p}(c)$ and the dual coframe $\omega_1, \cdots, \omega_{n+p}$ in such a way that, restricted to the submanifold M, $e_1, \cdots, e_n$ are tangent to M. In the sequel, the following convention on the range of indices is used, unless otherwise stated:

$$1 \leqq i, j, \cdots \leqq n ; \qquad n+1 \leqq \alpha, \beta, \cdots \leqq n+p .$$

We denote the second fundamental form α of M by

$$\alpha = -\sum_{\alpha, i, j} h_{ij}^{\alpha} \omega_i \omega_j e_{\alpha} .$$

The mean curvature vector $\boldsymbol{h}$ and the mean curvature H are defined by

$$\boldsymbol{h} = -\frac{1}{n} \sum_{\alpha} (\sum_{i} h_{ii}^{\alpha}) e_{\alpha}, \qquad H = |\boldsymbol{h}| = \frac{1}{n} \sqrt{\sum_{\alpha} (\sum_{i} h_{ii}^{\alpha})^2}. \tag{2.1}$$

Let $S = \sum (h_{ij}^{\alpha})^2$ denote the squared norm of the second fundamental form α of M. From the Gauss equation, the components of the Ricci curvature Ric is given by

$$R_{jk} = (n-1) c \delta_{jk} - \sum_{\alpha, i} h_{ii}^{\alpha} h_{jk}^{\alpha} + \sum_{\alpha, i} h_{ik}^{\alpha} h_{ij}^{\alpha} . \tag{2.2}$$

3. Proof of Theorem.

In order to prove Theorem, the following facts are needed.

PROPOSITION 1. *Let M be a 3-dimensional complete space-like submanifold with non-zero parallel mean curvature vector of $M_p^{3+p}(c)$. If the Ricci curvature of M is bounded from above by some number less than $3(c-H^2)$, then M is pseudo-umbilic.*

This is proved as Proposition 4.1 in [2].

PROPOSITION 2. *Let M be an n-dimensional complete space-like submanifold with non-zero parallel mean curvature vector of $M_p^{n+p}(c)$. If M is a pseudo-umbilical submanifold, then M is a maximal submanifold of a totally umbilical hypersurface $M_{p-1}^{n+p-1}(\tilde{c})$ of $M_p^{n+p}(c)$.*

The generalized case of Proposition 2 was proved by Chen in [4].

LEMMA 3. *Let M be a 3-dimensional complete maximal space-like submanifold of $M_q^{3+q}(\tilde{c})$. If the Ricci curvature of M is bounded from above by $3\tilde{c}$, then $\tilde{c}>0$ and M is congruent to $S^3(\tilde{c})$.*

PROOF. By using (2.2) and the assumpution that M is maximal, the diagonal components of the Ricci curvature are given by

$$R_{ii}=2\tilde{c}-\sum_{\alpha,j} h_{jj}^{\alpha}h_{ii}^{\alpha}+\sum_{\alpha,j} h_{ij}^{\alpha}h_{ij}^{\alpha}=2\tilde{c}+S\geqq 2\tilde{c}\ .$$

This combines with the condition $R_{ii}<3\tilde{c}$ to be $\tilde{c}>0$. By means of Ishihara's theorem [5], M is concluded to be a totally geodesic submanifold of $S_q^{3+q}(\tilde{c})$. Thus M is congruent to $S^3(\tilde{c})$. ■

Now, let us prove Theorem.

PROOF OF THEOREM. First of all, we consider the case which the mean curvature vector $\boldsymbol{h}$ is zero. In this case, by virtue of Lemma 3, we can prove that $c>0$ and M is a totally geodesic submanifold $S^3(c)$ of $S_p^{3+p}(c)$. Next we suppose that $\boldsymbol{h}\neq 0$. Combining Proposition 1 and Proposition 2, we have concluded that M is a maximal submanifold of a totally umbilic hypersurface $M_{p-1}^{3+p-1}(c_1)$ of $M_p^{3+p}(c)$, where $c_1=c-H^2$ and then $R_{ii}<3c_1$. Thus it follows from Lemma 3 that $c>H^2$ and M is congruent to $S^3(c_1)$. Now we have completed the proof of Theorem. ■

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