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A BOOLEAN POWER AND A DIRECT PRODUCT OF ABELIAN GROUPS

By

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A group means an abelian group in this paper. A Boolean power and a direct product of groups consist of all global sections of groups in some Boolean extensions $V^{(B)}$. We shall study about a homomorphism h whose domain is a group consisting of all the global sections of a group in $V^{(B)}$. We investigate two cases: one of them is that the range of h is a slender group, which is related to a torsion-free group, and the other is that the range of h is an infinite direct sum, which is related to a torsion group. We extend a few theorems which have been obtained in [4] and [5]. As in [5], we not only extend theorems, but improve them and give a good standing point of view.

We refer the reader to [9] or [1], for a Boolean extension $V^{(B)}$. We shall use notations and terminologies in [5], [6] and [7]. Throughout this paper, $\boldsymbol{B}$ is a complete Boolean algebra and $\mathfrak{F}$ is the set of all countably complete maximal filters on $\boldsymbol{B}$. We do not mention these any more. $\check{x}$ is the element of $V^{(B)}$ such that dom $\check{x}=\{\check{y}\,;\,y\in x\}$ and range $x\subseteq\{1\}$. As noted in [5], "$\hat{x}$" in [1] means our "$\check{x}$". $\hat{x}=\{y\,;\,[\![y\in x]\!]=\mathbf{1}$ and $y\in V^{(B)}\}$ for $x\in V^{(B)}$, where $V^{(B)}$ is separated. For $b\in\boldsymbol{B}$ and a group A in $V^{(B)}$, i.e. $[\![A$ is a group$]\!]=\mathbf{1}$, $\hat{A}^b$ is the subgroup of $\hat{A}$ such that $x\in\hat{A}^b$ iff $x\in\hat{A}$ and $-b\leqq[\![x=0]\!]$, where 0 is the unit of A. By this notation, $\hat{A}=\hat{A}^1$. For $x\in\hat{A}$, x^b is the element of $\hat{A}^b$ such that $b\leqq[\![x=x^b]\!]$.

1. A general setting about a complete Boolean algebra

Let $\Phi(b)$ be a property of $b\in\boldsymbol{B}$ which satisfies the following conditions:

(1) if $\{b_n\,;\,n\in N\}$ is a pairwise disjoint subset of $\boldsymbol{B}$, there exists k such that $\Phi(\bigvee_{n\geqq k} b_n)$ and $\Phi(b_n)$ hold for each $n\geqq k$;

(2) if $b\wedge c=\mathbf{0}$, $\Phi(b)$ and $\Phi(c)$ hold, then $\Phi(b\vee c)$ holds.

Let S be the subset of $\boldsymbol{B}$ such that $b\in S$ iff $\Phi(b)$ does not hold and $c\wedge c'=\mathbf{0}$ implies $\Phi(c)$ or $\Phi(c')$ for any $c,\,c'\leqq b$.

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LEMMA 1. *Let F^b be the subset of $\boldsymbol{B}$ defined by: $c \in F^b$ iff $\Phi(b \wedge c)$ does not hold. Then, $F^b \in \mathcal{F}$ for every $b \in S$.*

PROOF. We prove only the countable completeness. Let $b_n \in F^b$ for $n \in N$. Let $c_1 = \mathbf{0}$ and $c_{n+1} = \bigwedge_{k=1}^{n} b_k - b_{n+1}$. Then, $b_1 = \bigvee_{n \in N} c_n \vee \bigwedge_{n \in N} b_n$. By the condition (1) and (2) of Φ and the property of S, $\Phi(b \wedge \bigvee_{n \in N} c_n)$ and so $\Phi(b \wedge \bigwedge_{n \in N} b_n)$ does not hold.

LEMMA 2. *Let M be a maximal pairwise disjoint subfamily of S. Then, M is finite and $\Phi(c)$ holds for any c such that $c \wedge \bigvee M = \mathbf{0}$.*

PROOF. By the condition of Φ, M is finite. Suppose that there exists c such that $\Phi(c)$ does not hold and $c \wedge \bigvee M = \mathbf{0}$. By the maximality of M, there is no element of S below c. So, there are $b_0, c_0 \leqq c$ such that $b_0 \wedge c_0 = \mathbf{0}$ and $\Phi(b_0)$ nor $\Phi(c_0)$ does not hold. Then, take $b_1, c_1 \leqq c_0$ with the same property of b_0 and c_0. In such a way, we obtain a pairwise disjoint family $\{b_n;\ n \in N\}$ such that $\Phi(b_n)$ does not hold for any $n \in N$, which is a contradiction.

2. Hom($\hat{A}$, G)

Let F be a maximal filter on $\boldsymbol{B}$. For a group A in $V^{(\boldsymbol{B})}$, $\hat{A}/F$ is the quotient of $\hat{A}$ by the equivalence relation $\sim_F$ such that $x \sim_F y$ iff $[\![x=y]\!] \in F$. In the case $A = \check{X}$, $\hat{A}$ is known as a Boolean power $X^{(\boldsymbol{B})}$ and $\hat{A}/F$ is a Boolean ultrapower $X^{(\boldsymbol{B})}/F$. (Ref. [8]) In the case that $\boldsymbol{B} = \boldsymbol{P}(I)$ and $\hat{A} = \prod_{i \in I} A_i$, where A is defined by a natural way, $\hat{A}/F$ is known as an ultraproduct $\prod_{i \in I} A_i/F$. (Ref. [2]) However, the following fact is enough to read the main part of this paper. Let K be the subgroup of $\hat{A}$ defined by: $x \in K \leftrightarrow [\![x=0]\!] \in F$. Then, $\hat{A}/F \cong \hat{A}/K$, where the right part is the quotient group.

THEOREM 1. *Let A be a group in $V^{(\boldsymbol{B})}$ and G a slender group. Then, $\mathrm{Hom}(\hat{A}, G) \cong \bigoplus_{F \in \mathcal{F}} \mathrm{Hom}(\hat{A}/F, G)$ holds.*

PROOF. Let h be a homomorphism from $\hat{A}$ to G and $\Phi(b)$ the property "$h''\hat{A}^b = 0$". Let $\{b_n;\ n \in N\}$ be a pairwise disjoint subset of $\boldsymbol{B}$ and $x_n \in \hat{A}^{b_n}$ for each $n \in N$. Think of the homomorphism $g: \boldsymbol{Z}^N \to \hat{A}$ such that $g(\sum_{n \in N} a_n e_n) = \sum_{n \in N} a_n x_n$, where $x = \sum_{n \in N} a_n x_n$ is the element of $\hat{A}^b$ such that $b = \bigvee_{n \in N} b_n$ and $b_n \leqq [\![x = a_n x_n]\!]$ for each $n \in N$, and apply the slenderness of G to $h \cdot g$, then $h \cdot g(e_n) = 0$ and so $h(x_n) = 0$ for almost all n. Hence, there exists k such that $\Phi(b_n)$ for any $n \geqq k$ and $h(\sum_{n \geqq k} x_n) = 0$, by Specker's theorem. (Ref. Prop. 1 of [5] or

Lem. 94.1 of [7])

Therefore, Φ satisfies the conditions (1) and (2) of § 1. Hence, Lem. 1 and Lem. 2 hold for this Φ. Now, let $M=\{b_1 \cdots b_n\}$ and $b_0=\mathbf{1}-\bigvee M$. Let $h_i: \hat{A}/F^{b_i}\to G$ be defined by: $h_i([x]_i)=h(x^{b_i})$, where $[x]_i$ is the equivalence class containing x with respect to F^{b_i}, for each $1\leqq i\leqq n$. Since $[\![x=0]\!]\in F^{b_i}$ implies $h(x^{b_i})=0$ for $x\in\hat{A}^{-[x=0]}$, h_i is well-defined for $1\leqq i\leqq n$. For $x\in\hat{A}$, $h(x)=h(\sum_{i=0}^{m} x^{b_i})=\sum_{i=0}^{m} h(x^{b_i})=\sum_{i=1}^{m} h(x^{b_i})=\sum_{i=1}^{m} h_i([x]_i)$. The linear independence of $\{\mathrm{Hom}(\hat{A}/F, G);\ F\in\mathcal{F}\}$ is clear. Now, the proof is completed.

In view of the paragraph preceding Th. 1, Th. 1 includes Th. 2 of [5] and Th. 94.4 of [7]. We express these as corollaries.

COROLLARY 1. *Let A be a group and G a slender group. Then,* $\mathrm{Hom}(A^{(B)}, G)\cong\bigoplus_{F\in\mathcal{F}}\mathrm{Hom}(A^{(B)}/F, G)$.

COROLLARY 2. *Let A_i be a group for each $i\in I$ and G a slender group. Then,* $\mathrm{Hom}(\prod_{i\in I} A_i, G)\cong\bigoplus_{F\in\mathcal{F}}\mathrm{Hom}(\prod_{i\in I} A_i/F, G)$.

If the cardinality of A is less than the least measurable cardinal M_c or $\boldsymbol{B}$ satisfies M_c—*c. c.*, $A^{(B)}/F\cong A$ holds, so Cor. 1 is an extended form of Th. 2 of [5]. If the cardinality of I is less than M_c, then every $F\in\mathcal{F}$ is principal. Therefore, $\mathrm{Hom}(\prod_{i\in I} A_i, G)\cong\bigoplus_{i\in I}\mathrm{Hom}(A_i, G)$, which is a famous theorem. If the cardinalities of the A_i are bounded below M_c, then $\prod_{i\in I} A_i/F\cong A_i$ for some i, which was used in the proof of Cor. 2 of [5].

By Cor. 2, we can calculate a dual group of $\prod_{\lambda_1}\bigoplus_{\lambda_2}\cdots\prod_{\lambda_{2n-1}}\boldsymbol{Z}$. Now, we shall do it in a simple case. Let $j_F: V\to M_F$ be the elementary embedding, where F is a countably complete maximal filter on $\boldsymbol{P}(\lambda)$ and M_F is the transitive model which is isomorphic to V^{λ}/F. (Ref. [10]) Let $\boldsymbol{B}=\boldsymbol{P}(\lambda_1)$, then

$$\begin{aligned}\mathrm{Hom}(\prod_{\lambda_1}\bigoplus_{\lambda_2}\boldsymbol{Z}, \boldsymbol{Z}) &\cong\bigoplus_{F\in\mathcal{F}}\mathrm{Hom}(\prod_{\lambda_1}(\bigoplus_{\lambda_2}\boldsymbol{Z})/F, \boldsymbol{Z})\\ &\cong\bigoplus_{F\in\mathcal{F}}\mathrm{Hom}(\bigoplus_{j_F(\lambda_2)}\boldsymbol{Z}, \boldsymbol{Z})\\ &\cong\bigoplus_{F\in\mathcal{F}}\prod_{j_F(\lambda_2)}\boldsymbol{Z}.\end{aligned}$$

In the calculation, we have used the absoluteness of direct sums. Unfortunately, direct products are not absolute among transitive models. So, for the calculation of $\mathrm{Hom}(\prod_{\lambda_1}\bigoplus_{\lambda_2}\prod_{\lambda_3}\boldsymbol{Z}, \boldsymbol{Z})$, we must prepare a proposition which is obtained by modifying Cor. 2. That can be done, if we notice the fact that only the countably completeness of B, not the full completeness, has been used in the proof of Th. 1.

In this paper, we deal with the case that B is a complete Boolean algebra. Therefore, unless B is very large, every element of $\mathcal{F}$ is principal. Concerning a Boolean power, a countably complete Boolean algebra can give us interesting groups, for there can be a non-principal c.c. max-filter on a non-complete but countably complete and small Boolean algebra.

3. A homomorphism into an infinite sum

In this section, we shall extend some results of [4]. We do not prove the next lemma, because the proof is in [3] and [4], and the essential idea of it will be developed in the proof of Lem. 5. For $X \subseteq I$, we identify $\prod_{i\in X} A_i$ with the subgroup of $\prod_{i\in I} A_i$ such that $x \in \prod_{i\in X} A_i$ iff $x \in \prod_{i\in I} A_i$ and $x(i)=0$ for each $i \notin X$. Similarly, we do $\bigoplus_{i\in X} A_i$ with the subgroup of $\bigoplus_{i\in I} A_i$.

LEMMA 3. (Chase [3]) *Let $h: \prod_{i\in N} A_i \to \bigoplus_{j\in J} G_j$ $(=G)$ be a homomorphism. Then, there exist an integer $n>0$ and finite subsets $F \subseteq N$ and $J' \subseteq J$ such that*

$$h''n \prod_{i\in N-F} A_i \subseteq \bigoplus_{j\in J'} G_j + \bigcap_{n\in N} nG.$$

THEOREM 2. *Let A be a group in $V^{(B)}$ and $h: \hat{A} \to \bigoplus_{j\in J} G_j$ $(=G)$ a homomorphism. Then, there exist $F_1, \cdots, F_m \in \mathcal{F}$, an integer $n^*>0$ and a finite subset J^* of J that satisfy the following condition: Let K be the subgroup of $\hat{A}$ such that $x \in K$ iff $[\![x=0]\!] \in F_i$ for each $1 \leqq i \leqq m$, then $h''n^*K \subseteq \bigoplus_{j\in J^*} G_j + \bigcap_{n\in N} nG$.*(*)

Let $\Phi(b)$ be the property "There exist an integer $n>0$ and a finite subset J' of J such that $h''n\hat{A}^b \subseteq \bigoplus_{j\in J'} G_j + \bigcap_{n\in N} nG$."

LEMMA 4. *This Φ satisfies the conditions* (1) *and* (2) *in* § 1.

PROOF. Let $b = \bigvee_{n\in N} b_n$, for a pairwise disjoint family $\{b_n; n \in N\}$. Then, $\hat{A}^b \cong \prod_{n\in N} \hat{A}^{b_n}$. $b \leqq c$ and $\Phi(c)$ imply $\Phi(b)$. Hence, Φ satisfies the condition (1), by virtue of Lem. 3. Φ satisfies the condition (2) clearly.

LEMMA 5. *There exist an integer $n^*>0$ and a finite subset J^* of J such that, for any b which satisfies $\Phi(b)$, $h''n^*\hat{A}^b \subseteq \bigoplus_{j\in J^*} G_j + \bigcap_{n\in N} nG$.*

(*) Here we admit $m=0$ and in such a case $K=\hat{A}$.

PROOF. Suppose the negation of the conclusion. Let $\pi_j: \bigoplus_{j\in J} G_j \to G_j$ be the projection for $j\in J$. We construct $b_k\in \boldsymbol{B}$, $a_k\in \hat{A}$, $n_k\in N$, $j_k\in J$ and a finite subset J_k of J satisfying the following conditions:

(1) $\langle b_k; k\in N\rangle$ are pairwise disjoint and $\Phi(b_k)$ for $k\in N$;

(2) $a_k\in n_{k-1}!\,\hat{A}^{b_k}$ and $\pi_{j_k}h(a_k)\notin n_k!\,G_{j_k}$ and $\pi_{j_i}h(a_k)=0$ for each $i<k$;

(3) $h''n_{k-1}!\,\hat{A}^b\subseteq \bigoplus_{j\in J_{k-1}} G_j+\bigcap_{n\in N} nG$, where $b=\bigvee_{i=1}^{k-1} b_i$;

(4) $j_k\in J_k$ and $j_k\notin J_i$ for $i<k$;

(5) $\langle n_k; k\in N\rangle$ and $\langle J_k; k\in N\rangle$ are increasing.

Suppose that we have already defined b_i, a_i, n_i, j_i and J_i for $i\leqq k$ satisfying the above conditions. By the hypothesis, there exists b_{k+1} such that $b_{k+1}\wedge \bigvee_{i=1}^{k} b_i=0$, $\Phi(b_{k+1})$ and $h''n_k!\,\hat{A}^{b_{k+1}}\not\subseteq \bigoplus_{j\in J_k} G_j+\bigcap_{n\in N} nG$. So, there exists $a_{k+1}\in n_k!\,\hat{A}^{b_{k+1}}$ such that $h(a_{k+1})\notin \bigoplus_{j\in J_k} G_j+\bigcap_{n\in N} nG$. Hence, there are $j_{k+1}\notin J_k$ and $n>n_k$ such that $\pi_{j_{k+1}}h(a_{k+1})\notin n!\,G_{j_{k+1}}$. Let $J'=J_k\cup\{j;\, \pi_j h(a_{k+1})\neq 0\}$. By the property of b_{k+1}, there exist n_{k+1} and a finite subset J_{k+1} such that $n<n_{k+1}$ and $J'\subseteq J_{k+1}$ and $h''n_{k+1}!\,\hat{A}^{b_{k+1}}\subseteq \bigoplus_{j\in J_{k+1}} G_j+\bigcap_{n\in N} nG$. $\sum_{k\in N} a_k$ exists in $\hat{A}$ and so let it be a. Then, $a-\sum_{i=1}^{k} a_i\in n_k!\,A$ and $\pi_{j_k}h(a_k)\notin n_k!\,G_{j_k}$ and $\pi_{j_k}h(a_i)=0$ for each $i<k$. Hence, $\pi_{j_k}h(a)=\pi_{j_k}h(a-\sum_{i=1}^{k} a_i)+\pi_{j_k}h(a_k)\neq 0$ for each k. Since $k\neq k'$ implies $j_k\neq j_{k'}$, it is a contradiction.

PROOF OF TH. 2. By Lem. 1, Lem. 2 and Lem. 4, M is finite and so let $M=\{b_1, \cdots, b_m\}$ and $b_0=\mathbf{1}-\bigvee M$. Let $F_i=F^{b_i}$ for $1\leqq i\leqq m$. Now, the theorem is clear by Lem. 5 and the fact that $x\in K$ implies $x\in\hat{A}^b$ for some b which satisfies $\Phi(b)$.

For a Group A, $\bar{A}$ denotes the corresponding Hausdorff group $A/\bigcap_{n\in N} nA$.

LEMMA 6. *For a group A in $V^{(B)}$, $\hat{\bar{A}}\cong\bar{\hat{A}}$.*

PROOF. By the absoluteness of N, $\bigcap_{n\in N} n\hat{A}\cong\bigcap_{n\in N}\widehat{nA}$. Hence, $\bar{\hat{A}}\cong\hat{A}/\bigcap_{n\in N} n\hat{A}\cong \hat{A}/\bigcap_{n\in N}\widehat{nA}\cong\hat{\bar{A}}$.

Let F be a maximal filter on $\boldsymbol{B}$ and $K_F^{\hat{A}}$ the subgroup of $\hat{A}$ such that $x\in K_F^{\hat{A}}$ iff $[\![x=0]\!]\in F$.

LEMMA 7. *$nx\in K_F^{\hat{A}}$ implies $nx\in nK_F^{\hat{A}}$, where n is an integer.*

PROOF. Let $b=[\![nx=0]\!]$. Let x' be the element of $\hat{A}$ such that $-b\leqq[\![x'=x]\!]$

and $b \leqq [\![x'=0]\!]$. Then, $x' \in K_F^{\hat{A}}$ and $nx'=nx$.

LEMMA 8. *Let* $\pi: \hat{A} \to \hat{\hat{A}}$ $(\cong \bar{A})$ *be the canonical homomorphism. Then,* $\pi'' K_F^{\hat{A}} = K_F^{\hat{\hat{A}}}$.

PROOF. $\pi'' K_F^{\hat{A}} \subseteq K_F^{\hat{\hat{A}}}$ is obvious. Let $x \in K_F^{\hat{\hat{A}}}$. Then, there exists y in $\hat{A}$ such that $\pi(y)=x$. So, there exists b such that $b \in F$ and $b \leqq [\![x=0]\!]$. Let y' be the element of $\hat{A}$ such that $-b \leqq [\![y'=y]\!]$ and $b \leqq [\![y'=0]\!]$. Then, $\pi(y')=\pi(y)$ and $y' \in K_F^{\hat{A}}$.

LEMMA 9. *Let A be a torsion group in $V^{(B)}$, then $\hat{A}/F$ is also a torsion group for $F \in \mathcal{F}$.*

PROOF. Let $a \in \hat{A}$, then $\bigvee_{n \in N} [\![na=0]\!] = [\![\exists n \in N(na=0)]\!] = \mathbf{1}$. By the countable completeness of F, $[\![na=0]\!] \in F$ for some $n \in N$. So, $\hat{A}/F$ is a torsion group.

THEOREM 3. *Let A be a torsion group in $V^{(B)}$. Then, for each direct sum decomposition $\bigoplus_{j \in J} G_j$ of $\hat{A}$, $\bar{G}_j$ is a torsion group for almost all $j \in J$.*

PROOF. Applying Th. 2 directly, we have $F_1, \cdots, F_m \in \mathcal{F}$, an integer n and a finite subset J' of J such that $nK \subseteq \bigoplus_{j \in J'} G_j + \bigcap_{n \in N} nG$, where K and G are the same as Th. 2. Let $\pi: G \to \bar{G}$ be the canonical homomorphism. Then, $\pi'' G_j \cong \bar{G}_j$ for each $j \in J$ and $n\pi'' K \subseteq \bigoplus_{j \in J'} \pi'' G_j$.

Let $\psi: \bar{G}$ $(=\bar{\hat{A}}) \to \bar{G}/\pi'' K$ be the canonical homomorphism. Then, the restriction ψ to $n \bigoplus_{j \in J-J'} \pi'' G_j$ is a monomorphism, by Lem. 6, 7 and 8. On the other hand, $\bar{G}/\pi'' K \cong \hat{\hat{A}}^{b_1}/F_1 \oplus \cdots \oplus \hat{\hat{A}}^{b_m}/F_m \cong \hat{\hat{A}}/F_1 \oplus \cdots \oplus \hat{\hat{A}}/F_m$, by virtue of Lem. 6, 7 and 8 and the fact: $K = \hat{A}^{b_0} \oplus K_{F_1}^{\hat{A}^{b_1}} \oplus \cdots \oplus K_{F_m}^{\hat{A}^{b_m}}$. Therefore, it is a torsion group by Lem. 9 and hence $\bigoplus_{j \in J-J'} \bar{G}_j$ is a torsion group.

Let A_i be a torsion group for each $i \in I$. In view of the first paragraph of §2, we can take a torsion group A in $V^{(P(I))}$ such that $\hat{A} \cong \prod_{i \in I} A_i$. So, Th. 3 is an improvement of Lem. 8 of [4], even in the case of a direct product, i. e. dropping the cardinality hypothesis for I. Hence, we have Th. 9 of [4] without the cardinality hypothesis for I.

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