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AN INTERPRETATION OF INTUITIONISTIC ANALYSIS WITH RESTRICTED TRANSFINITE INDUCTIVE DEFINITIONS

By

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Introduction.

In some exact sciences such as the foundations of some systems of arithmetic, the characterization of the *methods* to verify certain properties are often essential. A claim of existence of certain objects is thus coupled with the presentation of a *concrete method* which produces these objects. That is, a statement of the form $\exists xA(x)$ be read: there exists a concrete method which produces an object x satisfying $A(x)$. $A(x)$ itself may in turn contain existential quantifiers, and hence such a statement be regarded as a nest of claims for desirable methods. Statements of this kind cannot be formalized in usual languages, but can only be characterized through cooperation of a formal system (in a usual language) in which certain reasonings are constrained. Due to the restrictions imposed upon the system in consideration, the existential quantifiers can be interpreted as claiming the existence of concrete methods.

The logic which underlies such a system in *intuitionistic,* or constructive. The central factor in the study of concrete mathematics is the interpretation of *implication;* that is, if $\exists X$and $\exists Y$ respectively claim existences of methods X and Y, then

$$\exists XA(X)\vdash\exists YB(Y)$$

should assert

$$\exists Z\forall X(A(X)\vdash B(Z(X))),$$

which should be read: there exists a method Z such that, for each method X verifying A, $Z(X)$ is a method to verify B. The universe of these methods varies according to the systems under consideration.

Here we consider a system of intuitionistic analysis, $\mathcal{S}(\boldsymbol{I})$, which is characterized by the bar induction as well as successive inductive definitions along some *accessible* orderings. $\mathcal{S}(\boldsymbol{I})$ is at the same time an abstraction of and a modest extension of the system ASOD in [3]. In ASOD, the formulas for in-

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ductive definitions were strictly regulated, so that existential quantifiers were admitted only over natural numbers and were *isolated* from the inductive predicates. In $\mathcal{S}(\boldsymbol{I})$, they are still quite restricted, but are allowed to envelop inductive predicates in certain circumstances. The realization of the claimed methods (by existential quantifiers) will then be rendered by the *hyper-principle for the two-storied universe of transfinite mechanisms* (symbolized by TM [2]), which was proposed in our preluding work [5]. See [3] for the original version of the hyper-principle, HP.

As is seen in the introduction of [5], we call the objects in the universe of HP (or of TM [2]) *mechanisms* rather than methods. We wish to make clear the distinction between these terms (in our usage). That is, a statement $\exists XA(X)$ claims the existence of a *method* which is yet to be specified, while an object which *realizes* such a statement must be a clearly defined *mechanism*.

§ 1. A system of intuitionistic analysis with restricted transfinite inductive definitions.

Let $\boldsymbol{I}$ be a primitive recursive scheme such that, for each $l=1, 2, \cdots$, $\boldsymbol{I}(l)$ represents the pair of a set and its order, say $(I_l, <^l)$, which admits (a concrete) accessibility proof.

DEFINITION 1.1. 1) Language $\mathcal{L}(\boldsymbol{I})$. The language $\mathcal{L}(\boldsymbol{I})$ is the language of HA, Heyting arithmetic, augmented by the following.

1.1) Function variables and the corresponding quantifiers ($\forall$ and $\exists$).

1.2) Predicate symbols for inductive definitions, H_l, $l=1, 2, \cdots$.

1.3) Some function constants may be assumed.

2) Terms and formulas of $\mathcal{L}(\boldsymbol{I})$ are defined as usual.

3) An expression e is said to be *strictly positive* in an $\mathcal{L}(\boldsymbol{I})$-formula A if no subformula of A containing e occurs in the antecedent of $\vdash$.

4) *Admissible formulas* of $\mathcal{L}(\boldsymbol{I})$ are defined as follows.

4.1) For any two occurrences of symbols q_1 and q_2 in a formula A, q_2 is *negative* to q_1 in A if there is a subformula of A in which q_1 is strictly positive and q_2 is negative (in the usual sense).

4.2) Let q_1 and q_2 be respectively an existential quantifier or an inductive predicate (one of H_l's) occurring in a formula A. q_1 is said to *depend on* q_2 in A if q_2 is negative to q_1 in A.

4.3) An occurrence q of an existential quantifier $\exists y$ in A is *infavorable* in A if the variable y occurs in an $H_l(i, f)$ which is within the scope of q and if q depends on an occurrence of an inductive predicate.

4.4) An occurrence of an inductive predicate H is *related to* an infavorable quantifier q if there is a finite chain of dependence relations starting from H and ending with q in $\boldsymbol{A}$.

4.5) A formula A of $\mathcal{L}(\boldsymbol{I})$ is said to be *admissible* if no existential quantifier which binds a variable in an $H(i, f)$ is strictly positive and no H is related to an infavorable quantifier in $\boldsymbol{A}$.

5) An $\mathcal{L}(\boldsymbol{I})$-expression is said to be *$\mathcal{L}(\boldsymbol{I})$-recursive* if it is free of quantifiers as well as H_l, $l=1, 2, \cdots$. These are certainly admissible.

6) An $\mathcal{L}(\boldsymbol{I})$-formula is $\mathcal{L}(\boldsymbol{I})-u-a$ (*universally arithmetical*) if it is constructed from $\mathcal{L}(\boldsymbol{I})$-recursive formulas by applications of $\wedge$, $\vdash$ and and at-type $\forall$ (see Definition 1.1 in Part I of [3]).

DEFINITION 1.2. The system $\mathcal{S}(\boldsymbol{I})$ is HA with the *admissible* formulas of $\mathcal{L}(\boldsymbol{I})$ augmented by the following axioms and inferences.

(1) The well-foundedness of the orders of $\boldsymbol{I}$, $WF(\boldsymbol{I})$; that is,

$$\forall l \forall f(\text{If } f \text{ is a } <^l\text{-decreasing sequence from } I_l,$$
$$\text{then } \exists x(x \text{ is a modulus of finiteness of } f)).$$

(See Definition 1.2 in Part I of [3].)

(2) The axioms of inductive definitions, ID_l, $l=1, 2, \cdots$. Let l be a fixed number, and let

$$\boldsymbol{F}_l \equiv \boldsymbol{F}_l(i, f_l, \theta_1, \cdots, \theta_{l-1}, \theta_l)$$

be a formula-like expression in $\mathcal{L}(\boldsymbol{I})$ without predicate symbols H's and with (possibly) parameter symbols $\theta_1, \cdots, \theta_l$, where i is a variable of at (atomic)-type and $\boldsymbol{f}_l$ stands for the sequence of all the other free variables in $\boldsymbol{F}_l$ (either of at-type or of fn (function)-type). $\theta_1, \cdots, \theta_l$ can occur at various places in various circumstances. $\boldsymbol{F}_l$ is assumed to become an admissible formula under appropriate substitutions for the parameter symbols (see below).

$$ID_l: \forall i \in I_l \forall \boldsymbol{f}_l(H_l(i, \boldsymbol{f}_l) \equiv \boldsymbol{F}_l(i, \boldsymbol{f}_l, H_1, \cdots, H_{l-1}, H_l[i])),$$

where $H_l[i]$ abbreviates

$$\{j, \boldsymbol{g}\}(j <^l i \wedge H_l(j, \boldsymbol{g})),$$

(3) The axioms of bar induction, $BI(R; \boldsymbol{A})$, where R is $\mathcal{L}(\boldsymbol{I})$-recursive and $\boldsymbol{A}$ is an arbitrary (admissible) formula (see 8) in Definition 1.3, Part I of [3]).

(4) The axioms on function quantifiers, $\text{fnc}(\boldsymbol{A}, h, \forall)$ and $\text{fnc}(\boldsymbol{A}, h, \exists)$ respectively:

$$\forall f \boldsymbol{A}(f) \vdash \boldsymbol{A}(h) \quad \text{and} \quad \boldsymbol{A}(h) \vdash \exists f \boldsymbol{A}(f),$$

where h is an arbitrary $\mathcal{L}(\boldsymbol{I})$-term of fn-type.

(5) The rules of inference on function quantifiers, fncv$(A, \forall)$ and fncv$(A, \exists)$ respectively:

$$\text{to infer } B\vdash \forall f A(f) \text{ from } B\vdash A(f);$$

$$\text{to infer } \exists f A(f)\vdash B \text{ from } A(f)\vdash B,$$

where f occurs only at the indicated places in A.

(6) The rule of choice, choice(A):

$$\text{to infer } B\vdash \exists f\forall x A(x, f(x)) \text{ from } B\vdash \forall x\exists y A(x, y).$$

Note. 1) $\mathcal{S}(I)$ can be regarded as a system of intuitionistic analysis "of strength I."

2) In subsequent sections, we deal with the cases where there are just two accessible sets I_1 and I_2, and hence just two inductive predicate symbols H_1 and H_2. Furthermore, we assume that $f_1\equiv f_2\equiv f$ (a single function variable) and F_1 conforms to the regulation on the formulas in Definition 1.1, Part I of [3]; that is, existential quantifiers are restricted to at-type and their scopes are $\mathcal{L}(I)-u-a$. (F_2 is arbitrary if admissible.) We do not lose anything essential by these simplifications.

§ 2. Concrete translation.

We transform the formulas of $\mathcal{L}(I)$ into the *generalized formulas*, which are generalizations of the formula forms of $\mathcal{L}_2$ in § 3 of [5]. It is an extension of the mr-translation defined in § 2, Part III of [3]. The mr-translation of an admissible $\mathcal{L}(I)$-formula is of the form $\exists\mathfrak{X} A(\mathfrak{X})$, where $\exists\mathfrak{X}$ abbreviates a sequence of existential quantifiers and $A(\mathfrak{X})$ is a formula-form of $\mathcal{L}_2$. $A(\mathfrak{X})$ will be called the matrix of $\exists\mathfrak{X} A(\mathfrak{X})$. We would rather call such a translation the *concrete translation* in as much as the translation represents the intended meaning of $\exists$ and $\vdash$, which is a concrete construction, and hence we denote the translation by crt.

We assume [5] throughout this section.

DEFINITION 2.1. The crt-*translation* of (admissible) $\mathcal{L}(I)$-formulas are defined as follows.

(i) For C, $\mathcal{L}(I)$-recursive,

$$\mathrm{crt}(C)\equiv C',$$

where C' is obtained from C by replacing V by a combination of $\wedge$ and $\vdash$.

Suppose in the subsequent cases

$$\mathrm{crt}(\boldsymbol{C})\equiv\exists\mathcal{X}C(\mathcal{X}) \quad\text{and}\quad \mathrm{crt}(\boldsymbol{D})\equiv\exists\mathcal{Y}D(\mathcal{Y}).$$

(ii) $\mathrm{crt}(\boldsymbol{C}\wedge\boldsymbol{D})\equiv\exists\mathcal{X}\exists\mathcal{Y}(C(\mathcal{X})\wedge D(\mathcal{Y}))$.

(iii) $\mathrm{crt}(\boldsymbol{C}\vee\boldsymbol{D})\equiv\exists z\exists\mathcal{X}\exists\mathcal{Y}((z=0\vdash C(\mathcal{X}))\wedge(z>0\vdash D(\mathcal{Y})))$.

(iv) $\mathrm{crt}(\boldsymbol{C}\vdash\boldsymbol{D})\equiv\exists\mathcal{Z}\forall\mathcal{X}(C(\mathcal{X})\vdash D(\Pi(\mathcal{Z};\, x)))$.

(v) $\mathrm{crt}(\forall f\boldsymbol{C}(f))\equiv\exists\mathcal{Z}\forall fC(f,\, \Pi(\mathcal{Z};\, f))$,

where $\mathrm{crt}(\boldsymbol{C}(f))\equiv\exists\mathcal{X}C(f,\, \mathcal{X})$.

(vi) $\mathrm{crt}(\exists f\boldsymbol{C}(f))\equiv\exists f\exists\mathcal{X}C(f,\, \mathcal{X})$.

(vii) $\mathrm{crt}(H_l(i,\, f))\equiv\exists X_l\Delta_l(i,\, f,\, X_l)$, $l=1, 2$,

where the type-form of X_l has been specified in Definition 3.1 of [5].

PROPOSITION 2.1. 1) *The* crt-*translation of admissible $\mathcal{L}(\boldsymbol{I})$-formulas are of the form $\exists\mathcal{X}C(\mathcal{X})$ where $C(\mathcal{X})$ is a formula-form of $\mathcal{L}_2$ (see Definition* 3.1 *in* [5] *for $\mathcal{L}_2$-formula-forms).*

2) *If $\boldsymbol{C}$ does not have an infavorable quantifier, then* $\mathrm{crt}(\boldsymbol{C})$ *is of first-floor.*

3) *The $\exists\mathcal{X}$ in $\exists\mathcal{X}C(\mathcal{X})\equiv\mathrm{crt}(\boldsymbol{C})$ originates in the strictly positive existential quantifiers in $\boldsymbol{C}$, and hence of first floor.*

PROOF OF 1) and 2). By induction on the complexity of $\boldsymbol{C}$. We shall work on a crucial case for 1) with an example. Suppose there is a subformula of $\boldsymbol{C}$ of the form

$$\boldsymbol{D}\equiv H_1(h_1(i),\, h_2(f))\vdash\exists j(j<^2 i\wedge H_2(j,\, f)),$$

where h_1 and h_2 are some $\mathcal{L}_0(\boldsymbol{I})$-functions. The $\exists j$ is an infavorable quantifier. Due to the condition that $\boldsymbol{C}$ be admissible, $\boldsymbol{D}$ should occur in a subformula of the form $\boldsymbol{D}\vdash\boldsymbol{B}$, where $\boldsymbol{B}$ is free of H. So, if $\mathrm{crt}(\boldsymbol{B})\equiv\exists ZB(Z)$,

$$\begin{aligned}\mathrm{crt}(\boldsymbol{C})\equiv\exists\mathcal{Z}\forall J\forall V\{&\forall U[\Delta_1(h_1(i),\, h_2(f),\, U)\\ &\vdash\Pi(J;\, U)<^2 i\wedge\Delta_2(\Pi(J;\, U),\, f,\, \Pi(V;\, U))]\\ &\vdash B(\Pi(\mathcal{Z};\, j,\, V))\}.\end{aligned}$$

The type-form of $\Pi(V;U)$ (and hence that of V also) depends on the variable-forms J and U, and hence is of second floor, but since $\boldsymbol{B}$ is free of H, J and V do not occur in the type-form of $\mathcal{Z}$.

Note. A counter-example for non-admissible formulas will be given later.

THEOREM. *Let $\boldsymbol{C}$ be any closed theorem of $\mathcal{S}(\boldsymbol{I})$ with*

$$\mathrm{crt}(\boldsymbol{C})\equiv\exists\mathscr{X}C(\mathscr{X}).$$

Then, there is a sequence of first-floor-hyper-functionals Φ such that $C(\Phi)$ is TM[2]-valid. (*See* Definition 3.3 *in* [5] *for TM[2]-validity.*)

We shall call such Φ a solution for $\mathscr{X}$ with respect to $\exists\mathscr{X}C(\mathscr{X})$. We shall also say that $\boldsymbol{C}$ *can be interpreted in TM[2].*

PROOF. We can follow the proof of the Theorem in §4, Part III of [3]. We prove a generalized statement:

(*) If $\boldsymbol{C}$ is any theorem of $\mathcal{S}(\boldsymbol{I})$ with

$$\mathrm{crt}(\boldsymbol{C})\equiv\exists\mathscr{X}C(\mathscr{X}, Q),$$

where Q stands for the free variables in $\boldsymbol{C}$, then there is a sequence of first-floor-term-forms Ψ with the parameters Q such that

$$C(\Psi(Q), Q)$$

is TM[2]-valid.

It suffices to deal with some cases which are characteristic of $\mathcal{S}(\boldsymbol{I})$, and we shall see simultaneously what type-forms are involved in the translations and solutions.

fnc$(\boldsymbol{C}, t, \exists)$ Suppose $\mathrm{crt}(\boldsymbol{C}(f))\equiv\exists XC(f, X)$. (For the sake of simplicity, we deal with the case where $\exists\mathscr{X}$ consists of a single quantifier.)

$$\begin{aligned}\mathrm{crt}(\boldsymbol{C}(t)\vdash\exists f\boldsymbol{C}(f))&\equiv\mathrm{crt}(\exists Y C(t, Y)\vdash\exists f\exists CX(f, X))\\&\equiv\exists F\exists Z\forall Y(C(t, Y)\vdash C(\Pi(F; Y), \Pi(Z; Y))).\end{aligned}$$

Due to the condition of admissibility, $\exists f$ is *isolated* from H. Let $\lambda Y\cdot t$ and $\lambda Y\cdot Y$ be respectively solutions for F and Z. The matrix is then equivalent to $C(t, Y)\vdash C(t, Y)$, which is a tautology.

fncv$(\boldsymbol{C}, \exists)$ From $\boldsymbol{C}(f)\vdash\boldsymbol{D}$, infer $\exists f\boldsymbol{C}(f)\vdash\boldsymbol{D}$.

$$\begin{aligned}\mathrm{crt}(\boldsymbol{C}(f)\vdash D)&\equiv\mathrm{crt}(\exists XC(f, X)\vdash\exists Y D(Y))\\&\equiv\exists Z\forall X(C(f, X)\vdash D(\Pi(Z; X))).\\\mathrm{crt}(\exists f\boldsymbol{C}(f)\vdash\boldsymbol{D})&\equiv\mathrm{crt}(\exists f\exists XC(f, X)\vdash\exists Y D(Y))\\&\equiv\exists W\forall f\forall X(C(f, X)\vdash D(\Pi(W; f, X))).\end{aligned}$$

If $[X]=s(f)$ and $[Y]=t$, then $[Z]=s(f)\to t$ and $[W]=\Lambda f(s(f)\to t)$. Let $Z_0(f)$ be a solution for Z, so that

(1) $$\forall X(C(f, X)\vdash D(\Pi(Z_0(f); X)))$$

is valid for every f an $\mathcal{L}_0(\boldsymbol{I})$-function. Let $\lambda f\lambda X\Pi(Z_0(f); X)$ be a solution for W. Then

$$\Pi(W;f,X)=\Pi(Z_0(f);X).$$

The matrix of the conclusion becomes

$$\forall f\forall X(C(f,X)\vdash D(Z_0(f);X)),$$

which is valid by (1).

The TM [2]-interpretation of the rule of choice trivially follows.

We now come to the crucial cases; ID_l, $l=1,2$. In order to crystallize the essence of the subsequent arguments, we specify the forms of $\boldsymbol{F}_l$. See Definitions 3.1 and 3.2 in [5] for these cases.

ID_1 Let $\boldsymbol{F}_1$ be of the following form.

$$\boldsymbol{F}_1\equiv\forall j<^1 i\forall g(A(i,g)\vdash H_1(j,g))$$
$$\wedge\forall j<^1 i\forall x(H_1(j,s(i,f,j,x))\vdash\exists nB(n,i,f,j,x)),$$

where A and B are $\mathcal{L}(\boldsymbol{I})-u-a$.

(3) $$\mathrm{crt}(\boldsymbol{F}_1)\equiv\mathrm{crt}(\forall j<^1 i\forall g(A(i,g)\vdash\exists V_1\Delta_1(j,g,V_1))$$
$$\wedge\forall j<^1 i\forall x(\exists V_2\Delta_1(j,s(i,f,j,x),V_2)\vdash\exists nB(n,i,f,j,x)))$$
$$\equiv\exists V\exists N[\forall j<^1 i\forall g(A(i,g)\vdash\Delta_1(j,g,\Pi(V;j,g)))$$
$$\wedge\forall j<^1 i\forall x\forall V_2(\Delta_1(j,s(i,f,j,x),V_2)\vdash B(\Pi(N;j,x,V_2),i,f,j,x))].$$

For the time being we leave the type-forms unspecified, and let us abbreviate the matrix to $G_1(i,f,V,N,\Delta_1)$. Then,

(4) $$\mathrm{crt}(ID_1)\equiv\mathcal{D}_1\wedge\mathcal{D}_2,$$

where

(5) $$\mathcal{D}_1\equiv\mathrm{crt}(H_1(i,f)\vdash\boldsymbol{F}_1)$$
$$\equiv\exists V'\exists N'\forall X_1\{\Delta_1(i,f,X_1)$$
$$\vdash G_1(i,f,\Pi(V';X_1),\Pi(N';X_1),\Delta_1)\}$$

and

(6) $$\mathcal{D}_2\equiv\mathrm{crt}(\boldsymbol{F}_1\vdash H_1(i,f))$$
$$\equiv\exists X'\forall V\forall N\{G_1(i,f,V,N)\vdash\Delta_1(i,f,\Pi(X';V,N))\}.$$

The type-forms are given by those in ($\mathcal{A}(\boldsymbol{I})-2$) of Definition 3.2 in [5]. That is,

$$[V]=\alpha(\xi(i,f);i),$$
$$[N]=\beta(\xi(i,f);i,f),$$
$$[V_2]=v_0(\xi(i,f);j,s).$$
$$[X_1]=\gamma(i,f).$$

(Recall that $[V]$ denotes the type-form of V.) These are of first floor. The solutions for V', N' and X' are also given by the term-forms there; that is,

$$V^*=\lambda X\lambda j\lambda g\mathcal{C}[j<^1 i;\ \Pi(X;0,j,g),\ ept],$$

$$N^*=\lambda X\lambda j\lambda x\lambda V_2\mathcal{C}[j<^1 i;\ \Pi(X;1,j,x,V_2),\ ept],$$

$$X^*=\lambda V\lambda N(V,N),$$

where $[X]=\gamma(i,f)$ and (V,N) denotes the pair of V and N, With these as solutions, the matrices of (5) and (6) respectively become $(\Delta_{1,1})$ and $(\Delta_{1,2})$ in $(\mathcal{A}(\boldsymbol{I})-2)$, and hence are valid by virtue of the Theorem in § 3 of [5].

ID_2 Let $\boldsymbol{F}_2$ be of the following form.

$$\boldsymbol{F}_2\equiv((H_1(h_1(i),h_2(f))\vdash\exists j<^2 iH_2(j,f))\vdash C)$$
$$\wedge\forall j<^2 i(H_1(h_3(j),h_4(f))\vdash H_2(j,f)),$$

where C is $\mathcal{L}(\boldsymbol{I})$-recursive.

(7) $$\mathrm{crt}(\boldsymbol{F}_2)\equiv\mathrm{crt}(((\exists V_1\Delta_1(h_1(i),h_2(f),V_1)\vdash\exists j<^2 i\exists U_1\Delta_2(j,f,U_1))\vdash C)$$
$$\wedge\forall j<^2 i(\exists V_2\Delta_1(h_3(j),h_4(f),V_2)\vdash\exists U_2\Delta_2(j,f,U_2)))$$
$$\equiv\mathrm{crt}((\exists J\exists T\forall V_1(\Delta_1(h_1(i),h_2(f),V_1)$$
$$\vdash\Pi(J;V_1)<^2 i\wedge\Delta_2(\Pi(J;V_1),f,\Pi(T;V_1)))\vdash C)$$
$$\wedge\exists W\forall j<^2 i\forall V_2(\Delta_1(h_3(j),h_4(f),V_2)\vdash\Delta_2(j,f,\Pi(W;j,V_2))))$$
$$\equiv\mathrm{crt}(\forall J\forall T[\forall V_1(\Delta_1(h_1(i),h_2(f),V_1)$$
$$\vdash\Pi(J;V_1)<^2 i\wedge\Delta_2(\Pi(J;V_1),f,\Pi(T;V_1)))\vdash C]$$
$$\wedge\exists W\forall j<^2 i\forall V_2(\Delta_1(h_3(j),h_4(f),V_2)\vdash\Delta_2(j,f,\Pi(W;j,V_2))))$$
$$\equiv\exists W\{\forall J\forall T[\forall V_1(\Delta_1(h_1(i),h_2(f),V_1)$$
$$\vdash\Pi(J;V_1)<^2 i\wedge\Delta_2(\Pi(J;V_1),f,\Pi(T;V_1)))\vdash C]$$
$$\wedge\forall j<^2 i\forall V_2(\Delta_1(h_3(j),h_4(f),V_2)\vdash\Delta_2(j,f,\Pi(W;j,V_2)))\}$$
$$\equiv\exists WG_2(i,f,W,\Delta_1,\Delta_2)$$

(8) $$\mathrm{crt}(ID_2)\equiv\mathcal{D}_3\wedge\mathcal{D}_4,$$

where

(9) $$\mathcal{D}_3\equiv\mathrm{crt}(H_2(i,f)\vdash\boldsymbol{F}_2)$$
$$\equiv\exists W'\forall X_2\{\Delta_2(i,f,X_2)\vdash G_2(i,f,\Pi(W';X_2),\Delta_1,\Delta_2)\}$$

and

(10) $$\mathcal{D}_4\equiv\mathrm{crt}(\boldsymbol{F}_2\vdash H_2(i,f))$$
$$\equiv\exists X'\forall W\{G_2(i,f,W,\Delta_1,\Delta_2)\vdash\Delta_2(i,f,\Pi(X';W))\}.$$

The type-forms are given by these in $(\mathcal{A}(\boldsymbol{I})-3)$ of Definition 3.2 in [5]. That is,

$[W]=\Lambda j(\gamma(h_3(j),\ h_4(f)) \longrightarrow \eta(i,\ f,\ j))=\mathcal{R}^2(ept,\ {}_s\,;\ i)$,

$[J]=\gamma(h_1(i),\ h_2(f)) \longrightarrow N_0$,

$[T]=\Lambda V_1\mathcal{C}[\Pi(J;\ V_1)<^2 i;\ \eta(i,\ f,\ \Pi(J;\ V_1))]$,

$[V_1]=\gamma(h_1(i),\ h_2(f))$,

$[V_2]=\gamma(h_3(j),\ h_4(f))$,

$[X_2]=\delta(i,\ f)$.

T is of second floor, while W and X_2 are of first floor. The solutions for W' and X' are given by the term-forms there; that is,

$$W^*=\lambda X\lambda j\lambda V_2\mathcal{C}[j<^2 i;\ \Pi(X;\ j,\ V_2),\ ept],$$

$$X^*=\lambda W\cdot W,$$

where $[X]=\delta(i,\ f)$. With these as solutions, the matrices of (9) and (10) respectively become $(\Delta_{2,1})$ and $(\Delta_{2,2})$ in $(\mathcal{A}(\boldsymbol{I})-3)$, and hence is valid by virtue of the Theorem in § 3 of [5].

This completes the proof of the theorem, which is the main theme of this article.

REMARK. We present a very simple counter-example to our interpretation, which violates one condition of admissibility. Put

$$\boldsymbol{F}\equiv\forall j<i\exists g(A(i,\ j,\ f,\ g)\vdash H(j,\ g)),$$

where A is $\mathcal{L}(\boldsymbol{I})-u-a$, and put

$ID:\ H(i,\ f)\equiv\boldsymbol{F}$.

$\mathrm{crt}(\boldsymbol{F})\equiv\mathrm{crt}(\forall j<i\exists g(A(i,\ j,\ f,\ g)\vdash\exists U\Delta(j,\ g,\ U))$

$\equiv\exists G\exists V\forall j<i(A(i,\ j,\ g,\ \Pi(G;\ j))\vdash\Delta(j,\ \Pi(G;\ j),\ \Pi(V;\ j)))$

$\mathrm{crt}(H(i,\ f)\vdash\boldsymbol{F})\equiv\exists F\exists W\forall X\{\Delta(i,\ f,\ X)\vdash\forall j<i(A(i,\ j,\ f,\ \Pi(F;\ X,\ j))$

$\vdash\Delta(j,\ \Pi(F;\ X,\ j),\ \Pi(W;\ X,\ j)))\}$

Let us try fo determine type-forms for variable-forms.

Let θ be a parameter which is supposed to yield the type-form of U_0 in $\Delta(j,\ g,\ U_0)$ (at $j<i$ and any g); that is,

(1) $\qquad \Pi(\theta;\ j,\ g)=[U_0;\ \theta](=u_0(\theta;\ j,\ g)) \quad$ if $\ j<i$.

Define

(2) $\qquad \alpha(i)=\Lambda j\mathcal{C}[j<i;\ [g],\ ept]$,

$\beta(\theta;\ i,\ F,\ X)=\Lambda j\mathcal{C}[j<i;\ u_0(\theta;\ j,\ \Pi(F;\ X,\ j)),\ ept]$,

${}_s(\theta;\ i,\ F,\ X)=\Lambda m\mathcal{C}[m=0,\ m=1;\ \alpha(i),\ \beta(\theta;\ i,\ F,\ X),\ ept]$.

Then $[X]$ mus satisfy

$$[X]=\mathcal{T}[ept,\ {}_s(\theta\,;\, i,\, F,\, X),\, i],$$

where $\mathcal{T}$ is a constant symbol for a transfinite type-form. But this ɔauses a cycle, since the type-form of X must depend on X itself and F (and $[F]$ depends on $[X]$).

§ 3. A progression of inductive definitions.

Here we give a brief sketch how to deal with a *progression* of the systems of intuitionistic analysis with inductive definitions applied to admissible formulas, which is an abstract treatment of the systems in [4]. We assume [4] throughout this section.

Let $\boldsymbol{I}$ be a primitive recursive scheme such that for each $r\in\omega$, $\boldsymbol{I}(r)$ represents the pair of a set and its linear order, say $(I_r, <^r)$, where $\boldsymbol{I}(r-1)$ is an initial segment of $\boldsymbol{I}(r)$.

DEFINITION 3.2. 1) The language $\mathcal{L}_\omega(\boldsymbol{I})$ is defined similarly to $\mathcal{L}(\boldsymbol{I})$ in Definition 1.1, but here $\boldsymbol{I}$ will play a different role from that of the $\boldsymbol{I}$ in the preceding sections. We assume just one predicate symbol H for inductive definitions with parameter r. That is, $H(r\,;\, i,\, \boldsymbol{f})$ is an atomic formula in which r is supposed to range over ω (natural numbers), i is supposed to range over I_r and $\boldsymbol{f}$ stands for a finite sequence of terms (of at-type or fn-type). r will be called the stage indicator (for H). For notational simplicity, we assume that $\boldsymbol{f}$ consists of a single term f.

2) A formula is $\mathcal{L}_\omega(\boldsymbol{I})$-recursive if it is free of H and quantifiers.

3) Admissible formulas are defined as in Definition 1.1, and, in the systems we are to work on, all the formulas are confined to admissible ones.

4) The language $\mathcal{L}_r(\boldsymbol{I})$ is the restriction of $\mathcal{L}_\omega(\boldsymbol{I})$ where the first argument r in $H(r,\, i,\, f)$ is a constant in ω.

DEFINITION 3.2. Formal systems $\{\mathcal{S}(r\,;\,\boldsymbol{I})\,;\ r\leqq\omega\}$ can be defined so that $\mathcal{S}(r\,;\,\boldsymbol{I})$ is a system of the language $\mathcal{L}_r(\boldsymbol{I})$, $r\leqq\omega$, similarly to the corresponding ones in Definition 1.3 of [4], the content of which will be omitted here except for some modifications.

1°. The properties on the constants concerning the elementary theory of $\boldsymbol{I}$ are formulated as axioms common to all $\mathcal{S}(r\,;\,\boldsymbol{I})$.

2°. Inductive definitions along $\boldsymbol{I}(r)$, ID_r, for each $r\in\omega$ are axioms.

$$\forall i\in I_r \forall f(H(r\,;\, i,\, f)\equiv \boldsymbol{F}(r\,;\, i,\, f,\, H[r\,;\, i])),$$

where $H[r; i]$ abbreviates

$$\{j, g\}(j <^r i \wedge H(r; j, g))$$

and $\boldsymbol{F}(r; i, f, H[r; i])$ is an admissible formula of the language $\mathcal{L}_r(\boldsymbol{I})$.

3°. Eliminations and introductions of ∀ and ∃ for fn-type are admitted.

Corresponding to Theorem 1 in §2 of [4], we assume the following basic assumption, (BA).

(BA) The accessibility of $\boldsymbol{I}(e)$ is $\mathcal{S}(\boldsymbol{\omega}; \boldsymbol{I})$-provable *uniformly* with respect to $e \in \boldsymbol{\omega}$.

Then we are led to the

CONCLUSION. The accessibility of $\boldsymbol{I}$ is $\mathcal{S}(\boldsymbol{\omega})$-provable with respect to the $\boldsymbol{\omega}$-*reasoning*.

By reading $\boldsymbol{I}$ for $Od(\boldsymbol{\omega})$ and starting from TM [2], we can define the $\boldsymbol{\omega}$-theory of two-storied transfinite mechanisms $\boldsymbol{\omega}$-TM [2], similarly to $\boldsymbol{\omega}$-HP in Definition 4.1 in [4]. See also [5] for TM [2].

We now come to the

Theorem. The accesibility of $\boldsymbol{I}$ is *valid* in $\boldsymbol{\omega}$-TM [2].

See Theorem 3 in [4] for details.

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