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A CHARACTERIZATION OF PSEUDO-EINSTEIN REAL HYPERSURFACES IN A QUATERNIONIC PROJECTIVE SPACE

By

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0. Introduction.

Let HP^n be a quaternionic projective space, $n \geqq 3$, with metric G of constant quaternionic sectional curvature 4, and let M be a connected real hypersurface of HP^n. Let ξ be a unit local normal vector field on M and $\{I, J, K\}$ a local basis of the quaternionic structure of HP^n (cf. [4]). Then $U_1 = -I\xi, U_2 = -J\xi, U_3 = -K\xi$ are unit vector fields tangent to M. We call them *structure vectors*. Now we put $f_i(X) = g(X, U_i)$, for arbitrary $X \in TM$, i = 1, 2, 3, where TM is the tangent bundle of M and g denotes the Riemannian metric induced from the metric G. We denote D and $D^{\perp}$ the subbundles of TM generated by vectors perpendicular to structure vectors, and structure vectors, respectively. There are many theorems from the point of view of the second fundamental tensor A of M (cf. [1], [8] and [9]). It is known that if M satisfies $g(AD, D^{\perp}) = 0$ then there is a local basis of quaternionic structure such that structure vectors are principal vectors. Berndt classified the real hypersurfaces which satisfy this condition (cf. [1]). On the other hand we know some results on real hypersurfaces of HP^n in terms of the Ricci tensor S of M (cf. [3] and [8]). If the Ricci tensor satisfies that $SX = aX + b\sum_{i=1}^{3} f_i(X)U_i$ for some smooth functions a and b on M, then M is called a pseudo-Einstein real hypersurface of HP^n. This notion comes from the problem for the real hypersurfaces in complex projective space CP^n. Kon studied it under the assumption that they have constant coefficients (cf. [5]) and Cecil and Ryan gave a complete classification (cf. [2]). In [8] Martinez and Perez studied pseudo-Einstein real hypersurfaces of HP^n, $n \geq 3$ under the condition that a and b are constant. Using Berndt's classification we show that we do not need the assumption. The main purpose of this paper is to provide a characterization of pseudo-Einstein real hypersurface in HP^n by using an estimate of the length of the Ricci tensor S, which is a quaternionic version of a result of Kimura and

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Maeda (cf. [5]).

THEOREM 1. *Let M be a real hypersurface of HP^n $(n \geqq 3)$ with $f_i(SU_i) = \alpha$ for $i = 1, 2, 3$, α is a function on M. Then the following holds:*

$$\|S\|^2 \geqq 3\alpha^2 + \frac{1}{4(n-1)}(\rho - 3\alpha)^2, \tag{0.1}$$

where $\|S\|$ is the length of the Ricci tensor S of M and ρ is the scalar curvature of M. The equality of (0.1) *holds if and only if M is an open subset of one of the following:*

(a) *a geodesic hypersphere,*

(b) *a tube of radius r over a totally geodesic HP^k, $1 \leqq k \leqq n-2$, $0 < r < \pi/2$ and $\cot^2 r = (4k+2)/(4n-4k-2)$.*

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1. Preliminaries.

Let M be a connected real hypersurface of $HP^n, n \geqq 3$, and let ξ be a unit normal vector field on M. The Riemannian connection $\tilde{\nabla}$ in HP^n and ∇ in M are related by the following formulas for arbitrary vector fields X and Y on M:

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\xi, \tag{1.1}$$

$$\tilde{\nabla}_X \xi = -AX, \tag{1.2}$$

where A is the second fundamental tensor of M in HP^n. We put

$$\begin{aligned} IX &= \phi_1 X + f_1(X)U_1, \\ JX &= \phi_2 X + f_2(X)U_2, \\ KX &= \phi_3 X + f_3(X)U_3, \end{aligned} \tag{1.3}$$

for any vector field X tangent to M, where $\phi_1 X, \phi_2 X$, and $\phi_3 X$, are the tangential parts of IX, JX and KX respectively, ϕ_i are tensors of type (1, 1), f_i are 1-forms for $i = 1, 2, 3$. Then they satisfy

$$\phi_i^2 X = -X + f_i(X)U_i, \tag{1.4}$$

$$f_i(U_i) = 1, \quad f_i(U_{i+1}) = f_i(U_{i+2}) = 0, \tag{1.5}$$

(1.6) $$\phi_i U_i = 0, \quad \phi_i U_{i+1} = -\phi_{i+1} U_i = U_{i+2},$$

(1.7) $$\phi_{i+1}\phi_{i+2} X = \phi_i X + f_{i+2}(X) U_{i+1},$$

(1.8) $$\phi_{i+2}\phi_{i+1} X = -\phi_i X + f_{i+1}(X) U_{i+2},$$

for $i = 1, 2, 3$, where we take the index i modulo 3. From the expression of the curvature tensor of HP^n (cf. [4]), we have the following Gauss and Codazzi equations:

(1.9)

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y \\ & + \sum_{i=1}^{3} (g(\phi_i Y, Z)\phi_i X - g(\phi_i X, Z)\phi_i Y - 2g(\phi_i X, Y)\phi_i Z) \\ & + g(AY,Z)AX - g(AX,Z)AY, \end{aligned}$$

(1.10) $$(\nabla_X A)Y - (\nabla_X A)X = \sum_{i=1}^{3} (f_i(X)\phi_i Y - f_i(Y)\phi_i X - 2g(\phi_i X, Y)U_i).$$

We denote by S the Ricci tensor of type (1, 1) on M. Then by (1.9) we have

(1.11) $$SX = (4m+7)X - 3\sum_{i=1}^{3} f_i(X)U_i + (trace\ A)AX - A^2 X,$$

We use the following lemma.

LEMMA 2. *Let M be a real hypersurface of HP^n $n \geqq 2$. Then $g(AD, D^{\perp}) = 0$ if and only if there exists a local basis $\{I, J, K\}$ of quaternionic structure, such that the corresponding $U_1 = -I\xi, U_2 = -J\xi, U_3 = -K\xi$ are principal vectors.*

We know that Berndt classified the real hypersurface with the above condition (cf. [1]).

PROPOSITION 3. *Let M be a real hypersurface of $HP^n, n \geqq 2$, satisfying $g(AD, D^{\perp}) = 0$. Then M is congruent to an open subset of one of the following:*
(a) *a geodesic hypersphere,*
(b) *a tube of radius r over a totally geodesic $HP^k, 1 \leqq k \leqq n-2, 0 < r < \pi/2$,*
(c) *a tube of radius r over a totally geodesic $CP^n, 0 < r < \pi/4$.*

The geodesic hypersphere of HP^n has two distinct principal curvatures. Conversely Martinez and Perez proved the following proposition in [8].

PROPOSITION 4. *Let M be a real hypersurface of $HP^n, n \geqq 3$, with at most two distinct principal curvatures at each point of M. Then M is an open subset of the geodesic hypersphere of HP^n.*

We show the following result, which was proved by Martinez and Perez (cf. [8]) under the additional assumption that a and b are constant.

PROPOSITION 5. *Let M be a pseudo-Einstein real hypersurface of $HP^n, n \geqq 3$. Then M is an open subset of one of the following:*

(a) *a geodesic hypersphere,*

(b) *a tube of radius r over a totally geodesic $HP^k, 1 \leqq k \leqq n-2, 0 < r < \pi/2$ and $\cot^2 r = (4k+2)/(4n-4k-2)$.*

PROOF. Suppose that M is a pseudo-Einstein real hypersurface of $HP^n, n \geqq 3$. Let $H = A^2 - (traceA)A$. From the assumption, we have

$$HX = (4n+7-a)X, \quad HZ = (4n+4-(a+b))Z, \tag{1.12}$$

for arbitrary $X \in D, Z \in D^{\perp}$. If $b = -3$, from (1.12) we get

$$(A^2 - (traceA)A - (4n+7-a))X = 0,$$

for any $X \in TM$. Hence M has at most two distinct principal curvatures at each point of M, so that, by Proposition 4, m is an open subset of the geodesic hypersphere. If $b \neq -3$, D and $D^{\perp}$ are invariant under H. $4n+4-(a+b)$ is an eigenvalue of multiplicity three of H. On the other hand, if X is an eigenvector of A, clearly X is an eigenvector of H. If $\{X_1, \cdots, X_{4n-1}\}$ is an orthonormal basis of eigenvectors of A, then it is also a basis of eigenvectors for H. There must be three X_i for i = 1, 2, 3, by suitable choice of indices, so that $HX_i = (4n+4-(a+b))X_i$ for i = 1, 2, 3. Then X_1, X_2 and X_3 span the distribution $D^{\perp}$. Thus $g(AD, D^{\perp})$ = 0 and, by Proposition 3, we get the result.

2. Proof of Theorem 1.

We first remark that the real hypersurface M is pseudo-Einstein if and only if

$$g(SX, Y) = \lambda g(X, Y) \quad \text{for any } X, Y \in D, \tag{2.1}$$

λ is a function on M, and

(2.2) U_1, U_2 and U_3 are eigenvectors of S with the same eigenvalue α.

We can rewrite the condition (2.1) to get

(2.3) $$g(SX,Y)=\rho_0 g(X,Y) \quad \text{for any } X,Y\in D$$

and

$$\rho_0=\frac{1}{4(n-1)}(\rho-\sum_{i=1}^{3} f_i(SU_i)).$$

This equation (2.3) is equivalent to

$$g(SX-\sum_{i=1}^{3} f_i(X)SU_i, Y-\sum_{j=1}^{3} f_j(Y)U_j)$$
$$=\rho_0 g(X-\sum_{k=1}^{3} f_k(X)U_k, Y-\sum_{l=1}^{3} f_l(Y)U_l)$$

for any tangent vector fields X, Y on M. Consequently we obtain

$$\begin{aligned} &g(SX,Y)-\rho_0 g(X,Y)\\ &=\sum_{i=1}^{3}(\rho_0 f_i(X)f_i(Y)-f_i(SX)f_i(Y)-f_i(X)f_i(SY))\\ &+\sum_{j,k=1}^{3} f_j(X)f_k(Y)f_j(SU_k)=0. \end{aligned}$$

We define the tensor T by

$$\begin{aligned} T(X,Y)&=g(SX,Y)-\rho_0 g(X,Y)\\ &+\sum_{i=1}^{3}(\rho_0 f_i(X)f_i(Y)-f_i(SX)f_i(Y)-f_i(X)f_i(SY))\\ &+\sum_{j,k=1}^{3} f_j(X)f_k(Y)f_j(SU_k). \end{aligned}$$

Using (1.4), (1.5), (1.6), (1.7) and (1.8), we calculate the length of T to get

$$\|T\|^2=\|S\|^2-\frac{1}{4(n-1)}(\rho-\sum_{i=1}^{3} f_i(SU_i))^2-2\sum_{j=1}^{3}\|SU_j\|^2+\sum_{k,l=1}^{3}(f_k(SU_l))^2.$$

We know the inequality

$$\sum_{j=1}^{3}\|SU_j\|^2\geqq\sum_{k,l=1}^{3}(f_k(SU_l))^2\geqq 3\alpha^2$$

holds on any real hypersurface M of HP^n. From the assumption, the equality holds if and only if U_1, U_2 and U_3 are eigenvectors of S with the same eigenvalue α. We assert that the equality (0.1) holds if and only if M is pseudo-Einstein. By Proposition 5, we have proved our theorem.

References

[1] Berndt, J., *Real hypersurfaces in quaternionic space forms*, J. reine angew. Math. **419** (1991), 9–26.

[2] Cecil, T. E. & Ryan, P. J., *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), 481–499.

[3] Hamada, T., *On real hypersurfaces of a quaternionic projective space*, Saitama Math. J. **11** (1993), 29–39.

[4] Ishihara, S., *Quaternion Kähler manifolds*, J. Differential. Geom. **9** (1974), 483–500.

[5] Kimura, M. & Maeda, S., *Characterizations of geodesichyperspheres in a complex projective space in terms of Ricci tensors*, Yokohama Math. J. **40** (1992), 35–43.

[6] Kon, M., *Pseudo-Einstein real hypersurfaces in complex space forms*, J. Diff. Geom. **14** (1979), 339–354.

[7] Maeda, Y., *On real hypersurfaces of a complex projective space*, J. Math. Soc. Japan **28** (1976), 529–540.

[8] Martinez, A. & Perez, J. D., *Real hypersurfaces in quaternionic projective space*, Ann. Mat. Pura Appl. **145** (1986), 355–384.

[9] Pak, J. S., *Real hypersurfaces in quaternionic Kaehlerian manifolds with constant Q-sectional curvature*, Kodai Math. Sem. Rep. **29** (1977), 22–61.

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