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A STUDY ON THE UNICELLULARITY OF SOME LOWER TRIANGULAR OPERATORS

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1. Introduction

The investigation of invariant subspaces is the first step in the attempt to understand the structure of operators. We will investigate bounded linear operators on Hilbert spaces which have the simplest possible invariant subspace structure. In this paper, we are going to study some strictly lower triangular operators which are shown to be unicellular under certain conditions.

We introduce a simple but key result which transforms the problem of establishing whether a vector is cyclic for an operator to that of determining whether a related operator is one-to-one. We first introduce some definitions. Let $\mathcal{H}$ be a Hilbert space and A an operator on $\mathcal{H}$. Let M denote a subspace of $\mathcal{H}$. M is invariant under A means that $Ax \in M$ for all $x \in M$. The collection of all subspaces of $\mathcal{H}$ invariant under A is denoted by Lat A. The operator A is unicellular if the collection Lat A is totally ordered by inclusion. If $\mathcal{K}$ is a subset of $\mathcal{H}$, the span of $\mathcal{K}$ is the smallest subspace containing $\mathcal{K}$ and denoted by span $\mathcal{K}$. If $x \in \mathcal{H}$ then span $\{x, Ax, A^2x, \cdots\}$ is easily seen to be invariant under A. The vector x is cyclic for A if span $\{x, Ax, A^2x, \cdots\} = \mathcal{H}$ and M is a cyclic subspace for A if span $\{x, Ax, A^2x, \cdots\} = M$.

Let A be a bounded operator with $\|A\| < 1$ on l^2, and let $\{e_0, e_1, e_2, \cdots\}$ denote the standard basis for l^2. Let x be a column vector in l^2. Then $A^n x$ is a column vector in l^2 for each $n = 1, 2, \cdots$. Then we have an infinite matrix $[x, Ax, A^2x, \cdots]^t$ which will be denoted by $S_x(A)$. The matrix $S_x(A)$ is a bounded linear transformation on l^2.

Let A be a bounded operator with $\|A\| < 1$ on l^2 represented by a strictly lower triangular matrix. Let M_n be the subspace span $\{e_n, e_{n+1}, e_{n+2}, \cdots\}$ for each $n = 0, 1, 2, \cdots$. Then every M_n is invariant under A, and $\{M_n \mid n = 0, 1, 2, \cdots\}$ is totally ordered by inclusion;

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$$l^2=M_0\supset M_1\supset M_2\supset\cdots.$$

Hence A is unicellular if its only invariant subspaces are $\{0\}$ and M_n, $n=0, 1, 2, \cdots$, i. e. the collection Lat A of all subspaces of l^2 which are invariant under A is $\{\{0\}, M_n \mid i=0, 1, 2, \cdots\}$. Let M be a subspace of l^2 and $M^*=\{\sum_{n=0}^{\infty}\bar{c}_n e_n : \sum_{n=0}^{\infty} c_n e_n \in M\}$. If we let $\underline{x}_N=(\overbrace{0, \cdots, 0}^{N\text{-terms}}, 1, x_{N+1}, \cdots)^t \in l^2$ and $M_{\underline{x}_N}=\operatorname{span}\{\underline{x}_N, A\underline{x}_N, A^2\underline{x}_N, \cdots\}$ then $M_{\underline{x}_N}^{\perp}=(\operatorname{Ker}(S_{\underline{x}_N}(A)))^*$ and always $M_N^{\perp}\subset(\operatorname{Ker} S_{\underline{x}_N}(A))^*$.

LEMMA 1.1. *Let A be a strictly lower triangular operator on l^2. Then $(U^{*N}AU^N)^n U^{*N}=U^{*N}A^n P_N$ for every n, $N=0, 1, 2, \cdots$, where U is the unilateral shift on l^2 and P_N the orthogonal projection on M_N.*

PROOF. Let N be a non-negative integer. For $n=0$, $U^{*N}=U^{*N}P_N$. We assume that $(U^{*N}AU^N)^n U^{*N}=U^{*N}A^n P_N$. Then

$$\begin{aligned}(U^{*N}AU^N)^{n+1}U^{*N}&=(U^{*N}AU^N)(U^{*N}AU^N)^n U^{*N}\\&=(U^{*N}AU^N)U^{*N}A^n P_N, \text{ by induction hypothesis}\\&=U^{*N}AP_N A^n P_N\\&=U^{*N}AA^n P_N, \text{ since } A \text{ is strictly lower triangular}\\&=U^{*N}A^{n+1}P_N. \quad\square\end{aligned}$$

LEMMA 1.2. *Let A be a strictly lower triangular operator with $\|A\|<1$ on l^2, N a non-negative integer and let $\underline{x}_N=(\overbrace{0, \cdots, 0}^{N\text{-terms}}, 1, x_{N+1}, \cdots)^t \in M_N$. M_N is a cyclic subspace for A, i. e. $M_N=M_{\underline{x}_N}$, if and only if $S_{U^{*N}\underline{x}_N}(U^{*N}AU^N)$ is one-to-one.*

PROOF. $M_N=M_{\underline{x}_N}$, i. e. M_N is a cyclic subspace for A, if and only if $(\operatorname{Ker} S_{\underline{x}_N}(A))^*=M_N^{\perp}$ if and only if $(\operatorname{Ker} S_{\underline{x}_N}(A))^*\subset M_N^{\perp}$. Let $y\in l^2$.

$$S_{\underline{x}_N}(A)\bar{y}=\begin{pmatrix}\underline{x}_N{}^t\\ A\underline{x}_N{}^t\\ A^2\underline{x}_N{}^t\\ \vdots\\ \end{pmatrix}\begin{pmatrix}\bar{y}_0\\ \bar{y}_1\\ \bar{y}_2\\ \vdots\end{pmatrix}$$

$$=\begin{pmatrix}0\cdots 0 & 1 & x_{N+1} & & \cdots\\ 0\cdots 0 & 0 & * & & \cdots\\ 0\cdots 0 & 0 & 0 & * & \cdots\\ & 0 & & & \ddots\end{pmatrix}\begin{pmatrix}\bar{y}_0\\ \bar{y}_1\\ \bar{y}_2\\ \\ \end{pmatrix}$$

$S_{\underline{x}_N}(A)\bar{y}=0$ if and only if

$$0=\begin{pmatrix} 1 & x_{N+1} & & \cdots \\ 0 & * & * & \cdots \\ 0 & 0 & * & \cdots \\ & 0 & & \ddots \end{pmatrix}\begin{pmatrix} \bar{y}_N \\ \bar{y}_{N+1} \\ \bar{y}_{N+2} \\ \vdots \end{pmatrix}$$

$$=\begin{pmatrix} U^{*N}\underline{x}_N{}^t \\ U^{*N}A\underline{x}_N{}^t \\ U^{*N}A^2\underline{x}_N{}^t \\ \vdots \end{pmatrix}\begin{pmatrix} \bar{y}_N \\ \bar{y}_{N+1} \\ \bar{y}_{N+2} \\ \vdots \end{pmatrix}.$$

By Lemma 1.1, $U^{*N}A^n\underline{x}_N=U^{*N}A^nP_N\underline{x}_N=(U^{*N}AU^N)^nU^{*N}\underline{x}_N$ for each $n=0, 1, 2, \cdots$. Hence $(\operatorname{Ker} S_{\underline{x}_N}(A))^*\subset M_N^\perp$ if and only if $S_{U^{*N}\underline{x}_N}(U^{*N}AU^N)$ is one-to-one. □

THEOREM 1.3. *Let A be a strictly lower triangular operator with $\|A\|<1$ and U the unilateral shift on l^2. Then A is unicellular if and only if for any $x=(1, x_1, \cdots)^t\in l^2$, $S_x(U^{*N}AU^N)$ is one-to-one for every $N=0, 1, 2, \cdots$.*

PROOF. If A is unicellular, then $\operatorname{Lat} A=\{0\}\cup\{M_n\}_{n=0}^\infty$. Let $x=(1, x_1, \cdots)^t\in l^2$ and N be a fixed non-negative integer. Then $U^Nx=(\overbrace{0, \cdots, 0}^{N\text{-terms}}, 1, x_1, \cdots)^t\in l^2$ and $M_{U^Nx}=\operatorname{span}\{U^Nx, AU^Nx, \cdots\}$ is an invariant subspace of l^2 for A and $M_{U^Nx}=M_n$ for some n. Clearly, $M_{U^Nx}=M_N$. Hence M_N is a cyclic subspace for A and $U^{*N}U^Nx=x$. Therefore $S_x(U^{*N}AU^N)$ is one-to-one.

Conversely, we assume that for any $x=(1, x_1, \cdots)^t\in l^2$ $S_x(U^{*N}AU^N)$ is one-to-one for every $N=0, 1, 2, \cdots$. Let M be an invariant subspace of l^2. We need to show that M is $\{0\}$ or M_n for some non-negative integer n. Assume that $M\neq\{0\}$. Let N be the least index of non-zero entries of all elements of M. Then $0\leqq N<\infty$, $M\subset M_N$ and M contains a vector x of form $(0, \cdots, 0, 1, x_{N+1}, \cdots)^t$. From the assumption $S_x(U^{*N}AU^N)$ is one-to-one. Hence M_N is a cyclic subspace for A, i. e. $M_N=M_x\subset M$. Hence $M_N=M$. □

Now we need some properties of strictly upper triangular matrices in oder to determine whether they are one-to-one. The following Theorem leads to results on unicellularity.

THEOREM 1.4 [6]. *Let T and S be bounded operators on a Hilbert space $\mathcal{H}$ represented by upper triangular matrices with respect to a fixed orthonormal basis. Assume that all diagonal entries of T are non-vanishing, and that all* *diagonal entries of* S *are* 0. *If* T *is invertible and* S *is compact, then* $T+S$ *is one-to-one.*

COROLLARY 1.5 [6]. *Let* C *be a strictly upper triangular matrix on a Hilbert space* $\mathcal{H}$ *with an orthonormal basis* $\{e_n\}_{n=0}^{\infty}$. *Let* C_N *be an upper triangular matrix whose first* N *super-diagonals are zero, and the other super-diagonals are same as* C. *If each of the* N *super-diagonals of* C *has entries converging to zero, and* C_N *is a compact operator for some* N, *then* $I+C$ *is one-to-one.*

2. Some Triangular Operators

In this section, we investigate the unicellularity of some strictly lower triangular operator by using the results established in 1.

LEMMA 2.1 [4]. *If* A *is the unilateral weighted shift operator with the weight squence* $\{\alpha_n\}$, *then* $\|A^k\|=\sup_n|\prod_{i=0}^{k-1}\alpha_{n+i}|$, $k=1, 2, \cdots$.

LEMMA 2.2. *Let* A *be the unilateral weighted shift operator on* l^2 *with the weight sequence* $\{\alpha_n\}$ *and* m *a non-negative integer. If* $\alpha_n\downarrow 0$, *then* $U_m(A)=\sum_{n=1}^{\infty} n^m A^n$ *is a bounded operator on* l^2.

PROOF. Let $\{e_j|j=0, 1, 2, \cdots\}$ be the given orthonormal basis in l^2. Then

$$A^n e_j=\begin{cases} 0 & j<n \\ (w_j/w_{j-n})e_{j-n} & j\geqq n \text{ for each } n=1, 2, \cdots \end{cases}$$

where $w_n=\prod_{k=0}^{n-1}\alpha_k$ and $w_0=1$. So

$$\begin{aligned}(A^n)_{kj}&=\langle A^n e_j, e_k\rangle \\ &=\begin{cases} w_j/w_{j-n} & k=j-n\geqq 0 \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

It follows from Lemma 2.1 that $\|A^n\|=w_n$ for each $n=1, 2, \cdots$. For a fixed non-negative integer m,

$$\lim_{n\to\infty}\frac{(n+1)^m w_{n+1}}{n^m w_n}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^m \lim_{n\to\infty}\frac{w_{n+1}}{w_n}=0<1\,.$$

$$\|U_m(A)\|=\|\sum_{n=1}^{\infty} n^m A^n\|\leqq\sum_{n=1}^{\infty} n^m\|A^n\|=\sum_{n=1}^{\infty} n^m w_n<\infty\,.$$

So $U_m(A)$ is bounded operator on l^2. □

LEMMA 2.3. *Let* A *be a strictly lower triangular matrix on* l^2 *with respect* *to the orthonormal basis* $\{e_n\}$ *where the first lower diagonal entries are nonzero. That is,*

$$A=\begin{pmatrix} 0 & & & & \\ a_{1,0} & 0 & & & \\ a_{2,0} & a_{2,1} & 0 & & 0 \\ a_{3,0} & a_{3,1} & a_{3,2} & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $a_{i+1,i}\neq 0$ *for each* $i=0, 1, 2, \cdots$. *Then for* $i\leqq j+n-1$ $(A^n)_{ij}=0$ *and for* $i\geqq j+n$

$$(A^n)_{ij}=\sum_{k_{n-1}=j+(n-1)}^{i-1} a_{i,k_{n-1}}\Bigg(\sum_{k_{n-2}=j+(n-2)}^{k_{n-1}-1} a_{k_{n-1},k_{n-2}}$$

$$\Bigg(\cdots\Bigg(\sum_{k_2=j+2}^{k_3-1} a_{k_3,k_2}\Bigg(\sum_{k_1=j+1}^{k_2-1} a_{k_2,k_1}a_{k_1,j}\Bigg)\Bigg)\cdots\Bigg)\Bigg)$$

where $(A^n)_{ij}$ *is the* (i, j)*-component of* A^n *for* $n=1, 2, \cdots$.

PROOF. Since A is a strictly lower triangular matrix acting on l^2, A^n is a strictly lower triangular matrix acting on l^2 for each n. So $(A^n)_{ij}=0$ for $i\leqq j$. For $i\geqq j+1$, using the induction,

$$A_{ij}=\langle Ae_j,\ e_i\rangle=\Big\langle\sum_{k_1=0}^{\infty} a_{k_1,j}e_{k_1}, e_i\Big\rangle$$

$$=\sum_{k_1=0}^{\infty} a_{k_1,j}\langle e_{k_1},\ e_i\rangle=a_{ij}$$

$$(A^2)_{ij}=\langle A^2e_j,\ e_i\rangle=\langle A(Ae_j),\ e_i\rangle$$

$$=\Big\langle A\Big(\sum_{k_1=0}^{\infty} a_{k_1,j}e_{k_1}\Big),\ e_i\Big\rangle=\sum_{k_1=0}^{\infty} a_{k_1,j}\langle Ae_{k_1},\ e_i\rangle$$

$$=\sum_{k_1=0}^{\infty} a_{i,k_1}a_{k_1,j}\,.$$

Since $a_{k_1,j}=0$ for $k_1\leqq j$, and $a_{i,k_1}=0$ for $i\leqq k_1$,

$$(A^2)_{ij}=\sum_{k_1=j+1}^{i-1} a_{i,k_1}a_{k_1,j}\,.$$

Assume that

$$(A^{n-1})_{ij}=\sum_{k_{n-2}=j+(n-2)}^{i-1} a_{i,k_{n-2}}\Bigg(\sum_{k_{n-3}=j+(n-3)}^{k_{n-2}-1} a_{k_{n-2},k_{n-3}}$$

$$\Bigg(\cdots\Bigg(\sum_{k_2=j+2}^{k_3-1} a_{k_3,k_2}\Bigg(\sum_{k_1=j+1}^{k_2-1} a_{k_2,k_1}a_{k_1,j}\Bigg)\Bigg)\cdots\Bigg)\Bigg).$$

Then

$$(A^n)_{ij}=\langle A^{n-1}Ae_j,\ e_i\rangle=\left\langle A^{n-1}\Big(\sum_{k_1=0}^{\infty}a_{k_1,j}e_{k_1}\Big),\ e_i\right\rangle$$

$$=\sum_{k_1=0}^{\infty}a_{k_1,j}\langle A^{n-1}e_{k_1},\ e_i\rangle\,.$$

By the induction hypothesis,

$$(A^n)_{ij}=\sum_{k_1=0}^{\infty}a_{k_1,j}\Big(\sum_{k_{n-1}=k_1+(n-2)}^{i-1}a_{i,k_{n-1}}\Big(\cdots\Big(\sum_{k_3=k_1+2}^{k_4-1}a_{k_4,k_3}\Big(\sum_{k_2=k_1+1}^{k_3-1}a_{k_3,k_2}a_{k_2,k_1}\Big)\Big)\cdots\Big)\Big)$$

$$=\sum_{k_{n-1}=k_1+(n-2)}^{i-1}a_{i,k_{n-1}}\Big(\cdots\Big(\sum_{k_3=k_1+2}^{k_4-1}a_{k_4,k_3}\Big(\sum_{k_2=k_1+1}^{k_3-1}a_{k_3,k_2}\Big(\sum_{k_1=0}^{\infty}a_{k_2,k_1}a_{k_1,j}\Big)\Big)\Big)\cdots\Big).$$

Since $a_{k_2,k_1}=0$ for $k_2\leqq k_1$, and $a_{k_1,j}=0$ for $k_1\leqq j$, we have

$$(A^n)_{ij}=\sum_{k_{n-1}=k_1+n-2}^{i-1}a_{i,k_{n-1}}\Big(\cdots\Big(\sum_{k_3=k_1+2}^{k_4-1}a_{k_4,k_3}\Big(\sum_{k_2=k_1+1}^{k_3-1}a_{k_3,k_2}\Big(\sum_{k_1=j+1}^{k_2-1}a_{k_2,k_1}a_{k_1,j}\Big)\Big)\Big)\cdots\Big)$$

$$=\sum_{k_{n-1}=j+(n-1)}^{i-1}a_{i,k_{n-1}}\Big(\cdots\Big(\sum_{k_3=j+3}^{k_4-1}a_{k_4,k_3}\Big(\sum_{k_2=j+2}^{k_3-1}a_{k_3,k_2}\Big(\sum_{k_1=j+1}^{k_2-1}a_{k_2,k_1}a_{k_1,j}\Big)\Big)\Big)\cdots\Big).$$

Thus this proof is complete. □

Consider a strictly lower triangular matrix A ($\|A\|<1$) acting on l^2 with respect to the orthonormal basis $\{e_n\}_{n=0}^{\infty}$ which has the first non-zero lower diagonal entries. That is,

$$A_{ij}=\begin{cases}0 & \text{if } i\leqq j\\ a_{ij} & \text{if } i>j+1\end{cases}$$

where $a_{i+1,i}\neq 0$ for each $i=0,1,2,\cdots$.

Let $w_0=1$ and $w_n=\prod_{i=0}^{n-1}a_{i+1,i}$. By Lemma 2.3, $(A^n)_{n+k,k}=w_{n+k}/w_k$ for $n=1,2,\cdots$ and $k=0,1,2,\cdots$. For $x=(1,x_1,x_2,\cdots)^t\in l^2$,

$$(A^nx)_{n+s}=\sum_{p=0}^{s}(A^n)_{n+s,p}x_p$$

$$=\sum_{p=0}^{s-1}(A^n)_{n+s,p}x_p+\frac{w_{n+s}}{w_s}x_s \qquad \text{for all } s\geqq 0\,.$$

So

$$(S_x(A))_{ij}=\begin{cases}\sum_{k=0}^{j-(i+1)}(A^i)_{jk}x_k+\dfrac{w_j}{w_{j-1}}x_{j-i} & i\leqq j\\ 0 & \text{otherwise.}\end{cases}$$

Let D_A be the diagonal operator with the diagonal sequence $\{w_n\}$,

$$(C_A)_{ij}=\begin{cases} \dfrac{w_j}{w_{j-1}w_i}x_{j-i} & i=0,\ 1,\ 2,\ \cdots \text{ and } j>i \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(F_A)_{ij}=\begin{cases} \sum_{k=0}^{j-(i+1)}(A^i)_{jk}\dfrac{x_k}{w_i} & i<j \\ 0 & \text{otherwise.} \end{cases}$$

Then $S_x(A)=D_A(I+C_A+F_A)$.

From now on, we will express $S_x(A)=D_A(I+C_A+F_A)$ as the above way, if A is a strictly lower triangular operator such that each $A_{n+1,n}$ is non-zero for $n=0, 1, 2, \cdots$, where A_{ij} is the (i, j)-component of A.

LEMMA 2.4 [6]. *$\{\alpha_n\}\in l^p$ for $1\leqq p<\infty$ fixed, and $\alpha_n\downarrow 0$, then*

$$\sup_{k\geqq K}\left(\frac{1}{w_k^2}\right)\sum_{j=k+1}^{\infty}\frac{w_j^2}{w_{j-k}^2}<\infty \quad \textit{for some } K \textit{ where } w_n=\prod_{k=0}^{n-1}\alpha_k.$$

LEMMA 2.5 [6]. *If $\{\alpha_n\}\in l^p$ for some p with $1<p\leqq\infty$, and $\alpha_n\downarrow 0$, then the above matrix C_A is a compact operator on l^2 where $a_{n+1,n}=\alpha_n$.*

NOTATION. Let A and B be two matrices. $B\prec A$ means that $b_{ij}\leqq a_{ij}$ for all i, j.

Let A and B be strictly lower triangular operators such that $A_{n+1,n}=B_{n+1,n}\neq 0$ for all $n=0, 1, 2, \cdots$ and $B\prec A$ and let $S_x(A)=D_A(I+C_A+F_A)$ and $S_x(B)=D_B(I+C_B+F_B)$. Then for $x=(x_0, x_1, x_2, \cdots)\in l^2$ such that $x_i\geqq 0$ for all $i=0, 1, 2, \cdots$, $C_A=C_B$ and $D_A=D_B$. Since $B\prec A$, $B^n\prec A^n$. Moreover,

$$(F_A)_{ij}=\begin{cases} \sum_{k=0}^{j-(i+1)}(A^i)_{jk}\dfrac{x_k}{w_i} & i<j \\ 0 & \text{otherwise} \end{cases}$$

and

$$(F_B)_{ij}=\begin{cases} \sum_{k=0}^{j-(i+1)}(B^i)_{jk}\dfrac{x_k}{w_i} & i<j \\ 0 & \text{otherwise.} \end{cases}$$

So

$$F_B\prec F_A.$$

Thus we have the following Lemma.

LEMMA 2.6. *Let A and B be strictly lower triangular operators such that*

$B \prec A$ and $A_{n+1,n}=B_{n+1,n}\neq 0$ for all $n=0, 1, 2, \cdots$. Let $S_x(A)=D_A(I+C_A+F_A)$ and $S_x(B)=D_B(I+C_B+F_B)$. Then for $x=(1, x_1, x_2, \cdots)^t \in l^2$ such that $x_i \geqq 0$ for all $i=1, 2, \cdots$,

(a) *$F_B \prec F_A$ and*

(b) *F_B is a Hilbert-Schmidt operator if F_A is a Hilbert-Schmidt operator.*

THEOREM 2.7. *Let A be the unilateral weighted shift operator acting on l^2 with the weight sequence $\{\alpha_n\}$ and m a positive integer. If $\alpha_n \downarrow 0$ such that*

$$\sum_{n=1}^{\infty} n^2 \alpha_n^2 < \infty ,$$

then $V_m(A)=A(I+A)^{m-1}$ is unicellular.

PROOF. Let m be a fixed non-negative integer and let $w_n=\prod_{i=0}^{n-1}\alpha_i$ and $w_0=1$. Since

$$\begin{aligned}[A(I+A)^m]^i &= A^i(I+A)^{mi}\\ &= A^i+{}_{mi}C_1A^{i+1}+{}_{mi}C_2A^{i+2}+\cdots+{}_{mi}C_{mi-1}A^{mi+i-1}+A^{mi+i},\end{aligned}$$

$$(A^l)_{jk}=\begin{cases} \dfrac{w_j}{w_k} & \text{when } j\geqq l,\ k=j-l,\\ 0 & \text{otherwise,}\end{cases}$$

we have

$$(V_{m+1}(A)^i)_{jk}=\begin{cases} {}_{mi}C_{j-k-1}\dfrac{w_j}{w_k} & \text{when } 1\leqq i\leqq j-k\leqq mi+1,\ k\geqq 0\\ 0 & \text{otherwise.}\end{cases}$$

Let $x=(1, x_1, x_2, \cdots)^t \in l^2$. Then $S_x(V_{m+1}(A))=D(I+C_A+F)$, where D is the diagonal operator with the diagonal sequence $\{w_n\}$, C_A is the operator described in page 8, and

$$F_{ij}=\begin{cases} \sum_{k=\max\{i,\, j-mi-i\}}^{j-i-1} {}_{mi}C_{j-k-i}\dfrac{w_j}{w_k w_i}x_k & \text{when } j>i,\\ 0 & \text{when } j\leqq i.\end{cases}$$

By replacing k by $k+i$ we get

$$F_{ij}=\begin{cases} \sum_{k=\max\{i,\, j-mi\}}^{j-1} {}_{mi}C_{j-k}\dfrac{w_j}{w_{k-i} w_i}x_{k-i} & \text{when } j>i,\\ 0 & \text{when } j\leqq i.\end{cases}$$

If F is a Hilbert-Schmidt operator, then F is compact. So, by Lemma 2.5, $F+C_A$ is compact, and hence $S_x(V_{m+1}(A))=D(I+C_A+F)$ is one-to-one by Theorem 1.4. Therefore, we only need to prove that F is a Hilbert-Schmidt operator.

We have by the Schwarz inequality that

$$\sum_{j=2}^{\infty}\sum_{i=1}^{j-1}|F_{ij}|^2\leqq\sum_{j=2}^{\infty}\sum_{i=1}^{j-1}\sum_{k=\max\{i,j-mi\}}^{j-1}{}_{mi}C_{j-k}^2\Big(\frac{w_j}{w_{k-1}w_i}\Big)^2M,$$

where $M=\sum_{k=0}^{\infty}|x_k|^2<\infty$. By interchanging the order of summation we have

$$\sum_{j=2}^{\infty}\sum_{i=1}^{j-1}|F_{ij}|^2\leqq M\sum_{j=2}^{\infty}\sum_{\max\{1,(j/m+1)\}\leqq k<j}\ \sum_{\max\{1,(j-k/m)\}\leqq i\leqq k}{}_{mi}C_{j-k}^2\Big(\frac{w_j}{w_{k-i}w_i}\Big)^2$$

$$=M\sum_{k=1}^{\infty}\sum_{k<j\leqq(m+1)k}\ \sum_{\max\{1,(j-k/m)\}\leqq i\leqq k}{}_{mi}C_{j-k}^2\Big(\frac{w_j}{w_{k-i}w_i}\Big)^2$$

$$=M\sum_{k=1}^{\infty}\sum_{j=1}^{mk}\ \sum_{\max\{1,(j/m)\}\leqq i\leqq k}{}_{mi}C_j^2\Big(\frac{w_{j+k}}{w_{k-i}w_i}\Big)^2$$

by replacing j by $j-k$. Thus, in view of the fact that

$$\sum_{\max\{1,(j/m)\}\leqq i\leqq k}{}_{mi}C_j^2\Big(\frac{w_{j+k}}{w_{k-i}w_i}\Big)^2\leqq\frac{m^{2j}}{(j\,!)^2}\sum_{i=1}^{k}i^{2j}\Big(\frac{\alpha_{k-i}\cdots\alpha_{k-1}}{\alpha_0\cdots\alpha_{i-1}}\alpha_k\cdots\alpha_{k+j-1}\Big)^2$$

$$\leqq\frac{m^{2j}k^{2j}}{(j\,!)^2\alpha_0^2}\alpha_k^{2j}\sum_{i=1}^{k}\alpha_{k-i}^2\leqq\frac{m^{2j}k^{2j}}{(j\,!)^2\alpha_0^2}\alpha_k^{2j}L\,,$$

where $L=\sum_{i=1}^{\infty}\alpha_i^2<\infty$, we get

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j-1}|F_{ij}|^2\leqq LM\sum_{k=1}^{\infty}\sum_{j=1}^{mk}\frac{m^{2j}k^{2j}}{(j\,!)^2\alpha_0^2}\alpha_j^{2j}$$

$$\leqq\frac{LMm^2}{\alpha_0^2}\sum_{k=1}^{\infty}k^2\alpha_k^2\sum_{j=0}^{mk-1}\frac{m^{2j}}{j\,!}(k\alpha_k)^{2j}<\infty\,,$$

which implies that F is a Hilbert-Schmidt operator. In fact, it follows from the assumption that there is a positive integer N such that $k\alpha_k\leqq 1$ for any $k\geqq N$, so we have for any $k\geqq N$

$$\sum_{j=0}^{mk-1}\frac{m^{2j}}{j\,!}(k\alpha_k)^{2j}\leqq\sum_{j=0}^{\infty}\frac{m^{2j}}{j\,!}=e^{m^2}<\infty\,.\quad\square$$

COROLLARY 2.8. *Let A be the unilateral weighted shift operator acting on l^2 with the weight sequence $\{\alpha_n\}$, m a non-negative integer, and n a positive integer. If $\alpha_n\downarrow 0$ such that $\sum_{n=1}^{\infty}n^2\alpha_n^2<\infty$, then*

(a) *$V(n)=\sum_{i=1}^{n}A^i$ and*

(b) *$W_m(n)=\sum_{i=1}^{n}i^mA^i$ are unicellular.*

PROOF. (a) Let $[x, V(n)x, (V(n))^2x, \cdots]^t=D_{V(n)}(I+C_{V(n)}+F_{V(n)})$. By Lemma 2.3, $D_{V(n)}=D_A$ and $C_{V(n)}=C_A$. Let $x=(1, x_1, x_2, \cdots)^t\in l^2$. Without loss of generality we may assume that $x_i\geqq 0$ for all i. Since $V(n)\prec V_n(A)$, $F_{V(n)}\prec F_{V_n(A)}$. So $F_{V(n)}$ is a Hilbert-Schmidt operator by Lemma 2.6. Thus $S_x(V(n))=D_A(I+C_A+F_{V(n)})$ is one-to-one by Theorem 1.4. Since $U^{*N}V(n)U^N$ has the same condition as $V(n)$, $S_x(U^{*N}V(n)U^N)$ is one-to-one for each N. Therefore $V(n)$ is unicellular by Theorem 1.3.

(b) Let $x=(1, x_1, x_2, \cdots)^t \in l^2$ (without loss of generality, we may assume that $x_i \geqq 0$ for all i), and $S_x(W_m(n))=D_W(I+C_W+F_W)$, and $S_x(V_{n^m+1}(A))=D_V(I+C_A+F_V)$.

$$\begin{aligned} W_m(n) &= A(I+2^m A+3^m A^2+\cdots+n^m A^{n-1}) \\ &\prec A(I+{}_{n^m+1}C_1 A+{}_{n^m+1}C_2 A^2+ \\ &\qquad \cdots+{}_{n^m+1}C_{n-1}A^{n-1}+\cdots+{}_{n^m+1}C_{n^m+1}A^{n^m+1}) \\ &= A(I+A)^{n^m+1-1}=V_{n^m+1}(A), \end{aligned}$$

we have $D_W=D_V=D_A$, $C_W=C_V=C_A$ and $F_W \prec F_V$. As in the proof of Theorem 2.7, we see that F_V is a Hilbert-Schmidt operator. Therefore F_W is a Hilbert-Schmidt operator by Lemma 2.6. Hence $S_x(W_m(n))$ is one-to-one. Since $U^{*N}W_m(n)U^N$ has the same condition as $W_m(n)$, $S_x(U^{*N}W_m(n)U^N)$ is one-to-one for each N. So $W_m(n)$ is unicellular. □

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