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A THEOREM ON LENGTHS OF PROOF OF PRESBURGER FORMULAS

By

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Introduction.

In this paper we consider two subsystems $\mathfrak{N}_0$ and $\mathfrak{N}_1^{\dagger}$ of Peano arithmetic $\mathfrak{N}_1$ and compare lengths of proofs of $\mathfrak{N}_0$ with lengths of proofs of $\mathfrak{N}_1^{\dagger}$.

The language $\mathfrak{L}_1$ of the system $\mathfrak{N}_1$ is a first order language which consists of three function symbols: a constant symbol 0 (zero), a unary function symbol $'$ (successor) and a binary function symbol $+$ (addision); and two predicate symbols: the equality symbol $=$ and a ternary predicate symbol P. The formula $P(a, b, c)$ represents the statement that $a\times b$ is equal to c. The axioms of $\mathfrak{N}_1$ are the following axioms and all instances of the following schemata:

(A-1) $\forall x(x=x)$, (A-2) $\forall x\neg(x'=0)$, (A-3) $\forall x\forall y(x'=y'\supset x=y)$,
(A-4) $\forall x(x+0=x)$, (A-5) $\forall x\forall y(x+y'=(x+y)')$,
(A-6) $\mathfrak{A}(0)\wedge\forall x(\mathfrak{A}(x)\supset\mathfrak{A}(x'))\supset\forall x\mathfrak{A}(x)$,
(A-7) $\forall x\forall y(x=y\supset(\mathfrak{A}(x)\supset\mathfrak{A}(y)))$, (A-8) $\forall xP(x, 0, 0)$,
(A-9) $\forall x\forall y\forall z(P(x, y, z)\supset P(x, y', z+x))$,
(A-10) $\forall x\forall y\forall z\forall w(P(x, y, z)\wedge P(x, y, w)\supset z=w)$,

where $\mathfrak{A}(x)$ in (A-6) or (A-7) is any formula of $\mathfrak{N}_1$.

The language $\mathfrak{L}_0$ of the system $\mathfrak{N}_0$ is the language obtained from $\mathfrak{L}_1$ by deleting the predicate symbol P. The axioms of $\mathfrak{N}_0$ are the axioms (A-1)-(A-5) and all instances of the schemata (A-6), (A-7), where $\mathfrak{A}(x)$ in (A-6) or (A-7) is any formula of $\mathfrak{N}_0$.

Presburger proved in [P] that $\mathfrak{N}_0$ is complete. $\mathfrak{L}_0$-formulas are called *Presburger formulas.*

An $\mathfrak{L}_1$-formula $\mathfrak{A}$ is *P-eliminable* if, for each part of the form $P(r, s, t)$ in $\mathfrak{A}$, s does not contain bound variables. $\mathfrak{N}_1^{\dagger}$ is the formal system obtained from $\mathfrak{N}_1$ by restricting induction axioms (A-6) to P-eliminable formulas.

We define the *length of proof* $\mathfrak{P}$, denoted by $\mathrm{lh}(\mathfrak{P})$, as the maximal length

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of *threads* of $\mathfrak{P}$. (For the term 'thread', see [T, p. 14].)

The purpose of this paper is to prove the following theorem.

THEOREM. *There is a function* $f:\omega\to\omega$ *such that, for each natural number* n *and each* $\mathfrak{L}_0$*-formula* $\mathfrak{A}$, *if* $\mathfrak{A}$ *is provable in* $\mathfrak{N}_1^+$ *with length* $\leqq n$, *then* $\mathfrak{A}$ *is provable in* $\mathfrak{N}_0$ *with length* $\leqq f(n)$.

Let S_1 and S_2 be systems. Let F be a set of formulas which are provable in both S_1 and S_2. We say that a function $f:\omega\to\omega$ is an upper bound for speed-up by S_2 over S_1 with respect to F if the following condition is satisfied. (*Condition*) *For each natural number* n *and each formula* $\mathfrak{A}$ *in* F, *if* $\mathfrak{A}$ *is provable in* S_2 *with length* $\leqq n$ *then* $\mathfrak{A}$ *is provable in* S_1 *with length* $\leqq f(n)$. This definition is due to [*S*]. Using this notion, we can paraphrase the above theorem in the following form.

THEOREM. *There is an upper bound for speed-up by* $\mathfrak{N}_1^+$ *over* $\mathfrak{N}_0$ *with respect to the set of* $\mathfrak{L}_0$*-formulas which are provable in* $\mathfrak{N}_0$.

We make the following two remarks on Theorem.

1. We do not know whether or not we can replace $\mathfrak{N}_1^+$ by $\mathfrak{N}_1$.
2. Let $\mathfrak{P}$ be a proof of $\mathfrak{N}_1$ of an $\mathfrak{L}_0$-formula $\mathfrak{A}$ with length m. Assume that every axiom in $\mathfrak{P}$ is not $\forall x\forall y\forall z\forall w\{P(x, y, z)\wedge P(x, y, w)\supset z=w\}$ (respectively, $\forall xP(x, 0, 0)$). Replace every part in $\mathfrak{P}$ of the form $P(r, s, t)$ by $\forall x(x=x)$ (respectively, $\neg\forall x(x=x)$). Then axioms $\forall xP(x, 0, 0)$ and $\forall x\forall y\forall z\{P(x, y, z)\supset P(x, y', z+x)\}$ (respectively, $\forall x\forall y\forall z\{P(x, y, z)\supset P(x, y', z+x)\}$ and $\forall x\forall y\forall z\forall w\{P(x, y, z)\wedge P(x, y, w)\supset z=w\}$) in $\mathfrak{P}$ become formulas which are provable in $\mathfrak{N}_0$ with length at most 5 (respectively, 9). Hence we can get a proof of $\mathfrak{N}_0$ of $\mathfrak{A}$ with length $\leqq m+4$ (respectively, $m+8$).

The reader can find in [*Y*] some other results on speed-up for subsystems of Peano arithmetic.

We prove in §1 two lemmas which are used in the proof of Theorem. We give in §2 the proof of Theorem.

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§ 1. Preliminaries.

1. $\mathfrak{N}_0'$ is the formal system obtained from $\mathfrak{N}_0$ by replacing all axioms by the following rules of inference.

(I-1)
$$\frac{\forall x(x=x),\ \Gamma\to\Delta}{\Gamma\to\Delta}$$

(I-2)
$$\frac{\forall x\neg(x'=0),\ \Gamma\to\Delta}{\Gamma\to\Delta}$$

(I-3)
$$\frac{\forall x\forall y(x'=y'\supset x=y),\ \Gamma\to\Delta}{\Gamma\to\Delta}$$

(I-4)
$$\frac{\forall x(x+0=x),\ \Gamma\to\Delta}{\Gamma\to\Delta}$$

(I-5)
$$\frac{\forall x\forall y(x+y'=(x+y)'),\ \Gamma\to\Delta}{\Gamma\to\Delta}$$

(I-6) (induction inference)
$$\frac{\mathfrak{A}(0)\wedge\forall x(\mathfrak{A}(x)\supset\mathfrak{A}(x'))\supset\forall x\mathfrak{A}(x),\ \Gamma\to\Delta}{\Gamma\to\Delta},$$ where the formula

$\mathfrak{A}(0)\wedge\forall x(\mathfrak{A}(x)\supset\mathfrak{A}(x'))\supset\forall x\mathfrak{A}(x)$ is called an *induction axiom*.

(I-7) (equality inference)
$$\frac{\Gamma\to\Delta,\ s=t\quad \mathfrak{A}(t),\ \Pi\to\Sigma}{\mathfrak{A}(s),\ \Gamma,\ \Pi\to\Delta,\ \Sigma}$$

$\mathfrak{N}_1'$ is the formal system obtained from $\mathfrak{N}_1$ by replacing all axioms by the above rules of inference (I-1)-(I-7) and the following rules of inference.

(I-8)
$$\frac{P(r,\ 0,\ 0),\ \Gamma\to\Delta}{\Gamma\to\Delta}$$

(I-9)
$$\frac{P(r,\ s,\ t)\supset P(r,\ s',\ t+r),\ \Gamma\to\Delta}{\Gamma\to\Delta}$$

(I-10)
$$\frac{P(r,\ s,\ t)\wedge P(r,\ s,\ u)\supset t=u,\ \Gamma\to\Delta}{\Gamma\to\Delta}$$

It is easy to prove the following two facts.

(1) For each natural number m there exists a natural number n such that, for each $\mathfrak{L}_0$-formula $\mathfrak{A}$, if $\mathfrak{A}$ is provable in $\mathfrak{N}_0'$ with length $\leqq m$ then $\mathfrak{A}$ is provable in $\mathfrak{N}_0$ with length $\leqq n$.

(2) Let $\mathfrak{N}_1^{+\prime}$ be the formal system obtained from $\mathfrak{N}_1'$ by restricting induction axioms to P-eliminable formulas. For each natural number m there exists a natural number n such that, for each $\mathfrak{L}_1$-formula $\mathfrak{A}$, if $\mathfrak{A}$ is provable in $\mathfrak{N}_1^+$ with length $\leqq m$ then $\mathfrak{A}$ is provable in $\mathfrak{N}_1^{+\prime}$ with length $\leqq n$.

Hence, to prove Theorem, it is sufficient to prove the following theorem.

THEOREM′. *There exists an upper bound for speed-up by* $\mathfrak{N}_1^{+\prime}$ *over* $\mathfrak{N}_0'$ *with* *respect to the set of $\mathfrak{L}_0$-formulas which are provable in $\mathfrak{N}_0$.*

In the remainder of this paper we consider only $\mathfrak{N}_0'$, $\mathfrak{N}_1'$ and $\mathfrak{N}_1^{\dagger\prime}$. Hence there is no confusion if we remove primes from notations $\mathfrak{N}_0'$, $\mathfrak{N}_1'$ and $\mathfrak{N}_1^{\dagger\prime}$.

LEMMA 1. *There is a function $k: \omega \rightarrow \omega$ such that, for each natural number n and each proof $\mathfrak{P}$ of $\mathfrak{N}_1$, if $\mathfrak{P}$ is a proof of a sequent $\Gamma \rightarrow \Delta$ with length $\leqq n$ and every induction axiom of $\mathfrak{P}$ is P-eliminable, then there is a cut-free proof of $\mathfrak{N}_1$ of $\Gamma \rightarrow \Delta$ with length $\leqq k(n)$ whose induction axioms are P-eliminable.*

PROOF. For formulas $\mathfrak{A}$, $\mathfrak{B}$ and an equation $s=t$, we write $\mathfrak{A} \Rightarrow \mathfrak{B}$ (mod. $s=t$) if, for some formula $\mathfrak{C}(x)$, $\mathfrak{C}(s)$ is $\mathfrak{A}$ and $\mathfrak{C}(t)$ is $\mathfrak{B}$. Further we write

$$\mathfrak{A} \Rightarrow \mathfrak{B} \text{ (mod. } s_1=t_1, \cdots, s_\mu=t_\mu)$$

if, for some sequence of formulas $\mathfrak{A}_1, \cdots, \mathfrak{A}_{\mu-1}$, $\mathfrak{A} \Rightarrow \mathfrak{A}_1$ (mod. $s_1=t_1$) $\cdots$ and $\mathfrak{A}_{\mu-1} \Rightarrow \mathfrak{B}$ (mod. $s_\mu=t_\mu$). When μ is 0, $\mathfrak{A} \Rightarrow \mathfrak{B}$ (mod.) means that $\mathfrak{A}$ is $\mathfrak{B}$.

Mix is the following inference figure with the stipulations below:

$$\frac{\Gamma_1 \rightarrow \Delta_1,\ s_1=t_1 \ \cdots\ \Gamma_\mu \rightarrow \Delta_\mu,\ s_\mu=t_\mu \quad \Gamma \rightarrow \Delta \quad \Pi \rightarrow \Sigma}{\Gamma_1, \cdots, \Gamma_\mu, \Gamma, \Pi^* \rightarrow \Delta_1, \cdots, \Delta_\mu, \Delta^*, \Sigma} (\mathfrak{D})$$

Stipulations: a) Δ^* (respectively, Π^*) is a proper subsequence of Δ (respectively, Π). b) For each formula $\mathfrak{A}$ in Δ (respectively, Π), if $\mathfrak{A}$ does not occur in Δ^* (respectively, Π^*) then $\mathfrak{D} \Rightarrow \mathfrak{A}$ (mod. $s_1=t_1, \cdots, s_\mu=t_\mu$).

Let $\mathfrak{A}_1, \cdots, \mathfrak{A}_\nu$ be *a bundle* of a proof $\mathfrak{P}$. (For the term "bundle", see [T, p. 72].) We define the degree of the bundle as the number of such $\mathfrak{A}_i$'s that $\mathfrak{A}_i$ is the chief formula of a logical inference in $\mathfrak{P}$. Let $\mathfrak{A}$ be a formula in the end-sequent of $\mathfrak{P}$. *The degree of* $\mathfrak{A}$ *with respect to* $\mathfrak{P}$ is defined as the greatest degree of bundles which end with $\mathfrak{A}$. By $d(\mathfrak{A}, \mathfrak{P})$ we denote the degree of $\mathfrak{A}$ with respect to $\mathfrak{P}$.

We can define by double recursion a function $h: \omega^3 \rightarrow \omega$ which satisfies the following sublemma. We omit the definition of h.

SUBLEMMA 1. *Let*

$$\frac{\Gamma_1 \rightarrow \Delta_1,\ s_1=t_1 \ \cdots\ \Gamma_\mu \rightarrow \Delta_\mu,\ s_\mu=t_\mu \quad \Gamma \rightarrow \Delta \quad \Pi \rightarrow \Sigma}{\Gamma_1, \cdots, \Gamma_\mu, \Gamma, \Pi^* \rightarrow \Delta_1, \cdots, \Delta_\mu, \Delta^*, \Sigma} (\mathfrak{D})$$

be a mix. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_\mu$, $\mathfrak{P}$ and $\mathfrak{Q}$ be cut-free proofs of $\mathfrak{N}_1$ of $\Gamma_1 \rightarrow \Delta_1, s_1=t_1$; $\cdots$; $\Gamma_\mu \rightarrow \Delta_\mu, s_\mu=t_\mu$; $\Gamma \rightarrow \Delta$ and $\Pi \rightarrow \Sigma$, respectively. Assume that μ, $\mathrm{lh}(\mathfrak{P}_1), \cdots, \mathrm{lh}(\mathfrak{P}_\mu)$, $\mathrm{lh}(\mathfrak{P})$, $\mathrm{lh}(\mathfrak{Q}) \leqq m$ and the rank of the mix is less than or equal to ρ. Assume that every induction axiom of $\mathfrak{P}_1, \cdots, \mathfrak{P}_\mu$, $\mathfrak{P}$ and $\mathfrak{Q}$ is P-eliminable. Further assume that, for each formula $\mathfrak{A}$ in Δ (respectively, Π), if $\mathfrak{A}$ does not *occur in* Δ^* *(respectively,* Π^**), then* $d(\mathfrak{A}, \mathfrak{P}) \leqq d$ *(respectively,* $d(\mathfrak{A}, \mathfrak{Q}) \leqq d$*). Then there exists a cut-free proof* $\mathfrak{P}^*$ *of* $\mathfrak{N}_1$ *of* $\Gamma_1, \cdots, \Gamma_\mu, \Gamma, \Pi^* \rightarrow \Delta_1, \cdots, \Delta_\mu, \Delta^*, \Sigma$ *such that* $\mathrm{lh}(\mathfrak{P}^*) \leqq h(d, \rho, m)$, $d(\mathfrak{A}, \mathfrak{P}^*) \leqq d(\mathfrak{A}, \mathfrak{P})$ *for each formula* $\mathfrak{A}$ *in* Γ, Δ^*, $d(\mathfrak{B}, \mathfrak{P}^*) \leqq d(\mathfrak{B}, \mathfrak{Q})$ *for each formula* $\mathfrak{B}$ *in* Π^*, Σ *and all induction axioms of* $\mathfrak{P}^*$ *are P-eliminable.*

PROOF OF SUBLEMMA 1. We get $\mathfrak{P}^*$ by similar reductions to those in [G].

Note that $d(\mathfrak{A}, \mathfrak{P}) \leqq \mathrm{lh}(\mathfrak{P})$. Hence we can derive Lemma 1 from Sublemma 1.1 as we can derive Hauptsatz from Hilfssatz in [G].

3. We define an equivalence relation $\sim$ between proofs inductively as follows: 1. $\mathfrak{A} \rightarrow \mathfrak{A} \sim \mathfrak{B} \rightarrow \mathfrak{B}$. 2. If $\mathfrak{P}$ is $\dfrac{\mathfrak{P}_0}{\Gamma \rightarrow \Delta}$, $\mathfrak{Q}$ is $\dfrac{\mathfrak{Q}_0}{\Pi \rightarrow \Sigma}$, $\mathfrak{P}_0 \sim \mathfrak{Q}_0$ and the last inference rules of $\mathfrak{P}$ and $\mathfrak{Q}$ are the same type, then $\mathfrak{P} \sim \mathfrak{Q}$. 3. If $\mathfrak{P}$ is $\dfrac{\mathfrak{P}_0 \mathfrak{P}_1}{\Gamma \rightarrow \Delta}$, $\mathfrak{Q}$ is $\dfrac{\mathfrak{Q}_0 \mathfrak{Q}_1}{\Pi \rightarrow \Sigma}$, $\mathfrak{P}_0 \sim \mathfrak{Q}_0$, $\mathfrak{P}_1 \sim \mathfrak{Q}_1$ and the last inference rules of $\mathfrak{P}$ and $\mathfrak{Q}$ are the same type, then $\mathfrak{P} \sim \mathfrak{Q}$.

LEMMA 2. *For each natural number m there is a natural number n such that, for each proof* $\mathfrak{P}$ *of* $\mathfrak{N}_1$ *and each* $\mathfrak{L}_0$*-formula* $\mathfrak{A}$*, if* $\mathfrak{P}$ *is a proof of* $\mathfrak{A}$ *with length* $\leqq m$ *and every induction axiom of* $\mathfrak{P}$ *is P-eliminable, then we can find a proof* $\mathfrak{Q}$ *of* $\mathfrak{A}$ *with the properties that* $\mathfrak{P} \sim \mathfrak{Q}$*, the number of occurrences of P in* $\mathfrak{Q}$ *is less than or equal to n and every inductihn axiom of* $\mathfrak{Q}$ *is P-eliminable.*

PROOF. First we define inductively $\mathfrak{F}(\mathfrak{A}, \mathfrak{B})$ in the following manner:
1.1. If $\mathfrak{A}$ is $s=t$ and $\mathfrak{B}$ is an $\mathfrak{L}_0$-formula, then $\mathfrak{F}(\mathfrak{A}, \mathfrak{B})$ is $\mathfrak{A}$. 1.2. If $\mathfrak{A}$ is $s=t$ and $\mathfrak{B}$ is not $\mathfrak{L}_0$-formulas, then $\mathfrak{F}(\mathfrak{A}, \mathfrak{B})$ is $0=0$. 2.1. If $\mathfrak{A}$ is $P(r, s, t)$ and $\mathfrak{B}$ is $P(u, v, w)$, then $\mathfrak{F}(\mathfrak{A}, \mathfrak{B})$ is $\mathfrak{A}$. 2.2. If $\mathfrak{A}$ is $P(r, s, t)$ and $\mathfrak{B}$ is not of the form $P(u, v, w)$, then $\mathfrak{F}(\mathfrak{A}, \mathfrak{B})$ is $0=0$. 3.1. If $\mathfrak{A}$ is $\mathfrak{A}^1 \wedge \mathfrak{A}^2$ and $\mathfrak{B}$ is $\mathfrak{B}^1 \wedge \mathfrak{B}^2$, then $\mathfrak{F}(\mathfrak{A}, \mathfrak{B})$ is $\mathfrak{F}(\mathfrak{A}^1, \mathfrak{B}^1) \wedge \mathfrak{F}(\mathfrak{A}^2, \mathfrak{B}^2)$. 3.2. If $\mathfrak{A}$ is $\mathfrak{A}^1 \wedge \mathfrak{A}^2$, $\mathfrak{B}$ is not of the form $\mathfrak{B}^1 \wedge \mathfrak{B}^2$ and $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathfrak{L}_0$-formulas, then $\mathfrak{F}(\mathfrak{A}, \mathfrak{B})$ is $\mathfrak{A}$. 3.3. If $\mathfrak{A}$ is $\mathfrak{A}^1 \wedge \mathfrak{A}^2$, $\mathfrak{B}$ is not of the form $\mathfrak{B}^1 \wedge \mathfrak{B}^2$ and $\mathfrak{A}$ or $\mathfrak{B}$ is not $\mathfrak{L}_0$-formula, then $\mathfrak{F}(\mathfrak{A}, \mathfrak{B})$ is $0=0$. 4. Similar to 3 for the case where the outermost logical symbol is $\neg$, $\vee$ or $\supset$. 5.1. If $\mathfrak{A}$ is $\forall x \mathfrak{A}_0$ and $\mathfrak{B}$ is $\forall y \mathfrak{B}_0$, then $\mathfrak{F}(\mathfrak{A}, \mathfrak{B})$ is $\forall x \mathfrak{F}(\mathfrak{A}_0, \mathfrak{B}_0)$. 5.2. If $\mathfrak{A}$ is $\forall x \mathfrak{A}_0$, $\mathfrak{B}$ is not of the form $\forall y \mathfrak{B}_0$ and $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathfrak{L}_0$-formulas, then $\mathfrak{F}(\mathfrak{A}, \mathfrak{B})$ is $\mathfrak{A}$. 5.3. If $\mathfrak{A}$ is $\forall x \mathfrak{A}_0$, $\mathfrak{B}$ is not of the form $\forall y \mathfrak{B}_0$ and $\mathfrak{A}$ or $\mathfrak{B}$ is not $\mathfrak{L}_0$-formula, then $\mathfrak{F}(\mathfrak{A}, \mathfrak{B})$ is $0=0$. 6. Similar to 5 for the case where the outermost logical symbol is $\exists$.

The following sublemma is easily proved by induction on m.

SUBLEMMA 2.1. *For each m, the number of equivalence classes by* $\sim$ *which* *contain proofs with length* $\leqq m$ *is finite.*

SUBLEMMA 2.2. 1. $\mathfrak{F}\left(\mathfrak{A}\left(\begin{matrix}x\\t\end{matrix}\right), \mathfrak{B}\right)$ *is* $(\mathfrak{F}(\mathfrak{A}, \mathfrak{B}))\left(\begin{matrix}x\\t\end{matrix}\right)$. 2. $\mathfrak{F}(\mathfrak{A}, \mathfrak{B})$ *is* $\mathfrak{F}\left(\mathfrak{A}, \mathfrak{B}\left(\begin{matrix}x\\t\end{matrix}\right)\right)$. 3. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* $\mathfrak{L}_0$*-formulas, then* $\mathfrak{F}(\mathfrak{A}, \mathfrak{B})$ *is* $\mathfrak{A}$. 4. $\mathfrak{F}(\mathfrak{A}(0)\wedge\forall x(\mathfrak{A}(x)\supset\mathfrak{A}(x'))\supset\forall x\mathfrak{A}(x), \mathfrak{B}(0)\wedge\forall y(\mathfrak{B}(y)\supset\mathfrak{B}(y'))\supset\forall y\mathfrak{B}(y))$ *is* $\mathfrak{C}(0)\wedge\forall x(\mathfrak{C}(x)\supset\mathfrak{C}(x'))\supset\forall x\mathfrak{C}(x)$, *where* $\mathfrak{C}(x)$ *is* $\mathfrak{F}(\mathfrak{A}(x), \mathfrak{B}(y))$. 5. *The number of occurrences of* P *in* $\mathfrak{F}(\mathfrak{A}, \mathfrak{B})$ *is less than or equal to* $\min(n_1, n_2)$, *where* n_1 *and* n_2 *are the numbers of occurrences of* P *in* $\mathfrak{A}$ *and* $\mathfrak{B}$, *respectively.* 6. *If* $\mathfrak{A}$ *is P-eliminable then* $\mathfrak{F}(\mathfrak{A}, \mathfrak{B})$ *is P-eliminable.*

PROOF OF SUBLMMA 2.2.

1, 2, 3, 5 and 6. Easily proved by induction corresponding to the inductive definition of $\mathfrak{F}$.

4. By 1 and 2.

When $\mathfrak{P}\sim\mathfrak{Q}$, $\mathfrak{F}(\mathfrak{P}, \mathfrak{Q})$ denotes the proof figure obtained from $\mathfrak{P}$ by replacing each formula $\mathfrak{A}$ in $\mathfrak{P}$ by $\mathfrak{F}(\mathfrak{A}, \mathfrak{B})$, where $\mathfrak{B}$ is the formula in $\mathfrak{Q}$ corresponding to $\mathfrak{A}$.

We can prove the following sublemma by induction on $\mathrm{lh}(\mathfrak{P})$.

SUBLEMMA 2.3. *Let* $\mathfrak{P}$ *and* $\mathfrak{Q}$ *be proofs of sequents* $\mathfrak{A}_1, \cdots, \mathfrak{A}_i\rightarrow\mathfrak{B}_1, \cdots, \mathfrak{B}_j$ *and* $\mathfrak{C}_1, \cdots, \mathfrak{C}_i\rightarrow\mathfrak{D}_1, \cdots, \mathfrak{D}_j$ *respectively. Assume* $\mathfrak{P}\sim\mathfrak{Q}$. *Then* $\mathfrak{F}(\mathfrak{P}, \mathfrak{Q})$ *is a proof of*

$$\mathfrak{F}(\mathfrak{A}_1, \mathfrak{C}_1), \cdots, \mathfrak{F}(\mathfrak{A}_i, \mathfrak{C}_i)\rightarrow\mathfrak{F}(\mathfrak{B}_1, \mathfrak{D}_1), \cdots, \mathfrak{F}(\mathfrak{B}_j, \mathfrak{D}_j)$$

and $\mathfrak{F}(\mathfrak{P}, \mathfrak{Q})\sim\mathfrak{P}$.

Now, to prove Lemma 2, let m be given. By Sublemma 2.1 there is a natural number ν such that $\Omega_1, \cdots, \Omega_\nu$ are all the equivalence classes by $\sim$ which contain proofs with length $\leqq m$. For each i $(1\leqq i\leqq\nu)$, let $\{\mathfrak{P}_\lambda\}_{\lambda\in\Lambda_i}$ be the set of proofs which are elements of Ω_i and proofs of $\mathfrak{L}_0$-formulas. Define: 1) $k_\lambda=$the number of occurrences of P in $\mathfrak{P}_\lambda$. 2) $n_i=\inf_{\lambda\in\Lambda_i} k_\lambda$. 3) $n=\max(n_1, \cdots, n_\nu)$.

To verify that this n has the property in the lemma, let $\mathfrak{P}$ be a proof of an $\mathfrak{L}_0$-formula $\mathfrak{A}$ with length $\leqq m$. Further assume that every induction axiom of $\mathfrak{P}$ is P-eliminable. Then $\mathfrak{P}$ is an element of Ω_i for some i $(1\leqq i\leqq\nu)$. Take out from $\{\mathfrak{P}_\lambda\}_{\lambda\in\Lambda_i}$ such a proof $\mathfrak{Q}$ that the number of occurrences of P in $\mathfrak{Q}$ is just n_i. By Sublemma 2.3 and 3.6 in Sublemma 2.2, $\mathfrak{F}(\mathfrak{P}, \mathfrak{Q})\sim\mathfrak{P}$, $\mathfrak{F}(\mathfrak{P}, \mathfrak{Q})$ is a proof of $\mathfrak{A}$ and every induction axiom of $\mathfrak{F}(\mathfrak{P}, \mathfrak{Q})$ is P-eliminable. Furthermore, by 5 in Sublemma 2.2, the number of occurrences of P in $\mathfrak{F}(\mathfrak{P}, \mathfrak{Q})$ is $n_i\leqq n$.

§ 2. Proof of Theorem.

By Lemmas 1 and 2, to prove Theorem′, it is sufficient to prove

LEMMA 3. *For each natural number m, there exists a natural number n such that if $\mathfrak{P}$ is a cut-free proof of an $\mathfrak{L}_0$-formula $\mathfrak{A}_0$ of $\mathfrak{N}_1^+$ with length $\leqq m$ and the number of occurrences of P in $\mathfrak{P}$ is at most m then $\mathfrak{A}_0$ is provable in $\mathfrak{N}_0$ with length $\leqq n$.*

PROOF. In this proof we can assume, without loss of generality, that eigen-variables of a proof $\mathfrak{P}$ are distinct each other and also that the eigen-variable of an inference in $\mathfrak{P}$ does not occur below the inference. Hence we consider only proofs satisfying these conditions on eigen-variables.

In the remainder of this paper we consider only cut-free proofs of $\mathfrak{N}_1^+$ of $\mathfrak{L}_0$-formulas. Hence every formula in a proof is P-eliminable.

Step. 1. In this step we define, for each sequent $\mathfrak{S}$ in a proof $\mathfrak{P}$, a finite sequence $\Theta(\mathfrak{P}, \mathfrak{S})$ of equations or negations of equations and a finite sequence $\alpha(\mathfrak{P}, \mathfrak{S})$ of eigen-variables of $\mathfrak{P}$. These definitions are done by induction from the end-sequent up to begining sequents.

To simplify notations, we say sometimes that $\Theta=\Theta_0$ for $\mathfrak{S}$ (respectively, $\alpha=\alpha_0$ for $\mathfrak{S}$) if $\Theta(\mathfrak{P}, \mathfrak{S})=\Theta_0$ (respectively, $\alpha(\mathfrak{P}, \mathfrak{S})=\alpha_0$).

1.1. For the end-sequent:

Take $\Theta=\langle\,\rangle$ and $\alpha=\langle\,\rangle$.

1.2. For the case where the inference rule is $\dfrac{\Gamma\rightarrow\Delta,\ s=t\quad \mathfrak{A}(t),\ \Pi\rightarrow\Sigma}{\mathfrak{A}(s),\ \Gamma,\ \Pi\rightarrow\Delta,\ \Sigma}$: Let $\Theta=\Theta_0$ and $\alpha=\alpha_0$ for the lower sequent. Take $\Theta=\Theta_0, \neg s=t$ and $\alpha=\alpha_0$ for the left upper sequent, and take $\Theta=\Theta_0,\ s=t$ and $\alpha=\alpha_0$ for the right upper one.

1.3. For the case where the inference rule is $\dfrac{\Gamma\rightarrow\Delta,\ \mathfrak{A}(a)}{\Gamma\rightarrow\Delta,\ \forall x\mathfrak{A}(x)}$: Let $\Theta=\Theta_0$ and $\alpha=\alpha_0$ for the lower sequent. Take $\Theta=\Theta_0$ and $\alpha=\alpha_0, a$ for the upper one.

1.4. Similar to 1.3 for the case where the inference rule is $\exists$-left.

1.5. For the case where the inference rule is $\dfrac{\Gamma\rightarrow\Delta,\ \mathfrak{A}\quad \mathfrak{B},\ \Gamma\rightarrow\Delta}{\mathfrak{A}\supset\mathfrak{B},\ \Gamma\rightarrow\Delta}$: Let $\Theta=\Theta_0$ and $\alpha=\alpha_0$ for the lower sequent. Take $\Theta=\Theta_0$ and $\alpha=\alpha_0$ also for upper sequents.

1.6. Similar to 1.5 for the remaining cases.

It is easy to see

SUBLEMMA 3.1. 1) *If $\Theta=\Theta_0$ for $\mathfrak{S}$ and κ is the number of equality inferences below $\mathfrak{S}$, then the length of Θ_0 is κ. If $\alpha=\alpha_0$ for $\mathfrak{S}$ and λ is the number of $\forall$-*

-right and $\exists$-left inferences below $\mathfrak{S}$, then the length of α_0 is λ. 2) *If $\Theta=\Theta_0$ and $\alpha=\alpha_0$ for $\mathfrak{S}$, then eigen-variables occurring in Θ_0 occur in α_0.* 3) *If $\alpha=a_1, \cdots, a_\nu$ for some $\mathfrak{S}$ and i is distinct from j, then a_i is distinct from a_j.* 4) *If $\alpha=a_1, \cdots, a_\nu, a, \alpha_0$ for some $\mathfrak{S}_1$, $\alpha=a_1, \cdots, a_\nu, b, \alpha_1$ for some $\mathfrak{S}_2$ and a is distinct from b, then every variable in a, α_0 does not occur in b, α_1.*

Step 2. In this step we define, for each sequent $\mathfrak{S}$ in a proof $\mathfrak{P}$, a finite sequence $\Phi(\mathfrak{P}, \mathfrak{S})$ of $\mathfrak{L}_0$-formulas by induction from begining sequents down to the end-sequent.

To simplify notations, we say sometimes that $\Phi=\Phi_0$ for $\mathfrak{S}$ if $\Phi(\mathfrak{P}, \mathfrak{S})=\Phi_0$.

2.0. For a begining sequent $\mathfrak{A}\to\mathfrak{A}$: Let $\Theta=\Theta_0$ for the sequent $\mathfrak{A}\to\mathfrak{A}$. Take $\Phi=\wedge\Theta_0$, where $\wedge\Theta_0$ is the conjunction of all formulas in Θ_0.

2.1. For the case where the inference rule is $\dfrac{\mathfrak{A}, \Gamma\to\Delta}{\Gamma\to\Delta, \neg\mathfrak{A}}$: For the lower sequent take the same Φ as for the upper one.

2.2. Similar to 2.1 for the cases where the inference rules are $\neg$-left, $\supset$-right, $\vee$-right, $\wedge$-left, $\forall$-left, $\exists$-right, structural rules and (I-1)-(I-6), (I-8)-(I-10).

2.3. For the case where the inference rule is $\dfrac{\Gamma\to\Delta, \mathfrak{A} \quad \Gamma\to\Delta, \mathfrak{B}}{\Gamma\to\Delta, \mathfrak{A}\wedge\mathfrak{B}}$: Let $\Phi=\mathfrak{C}_1, \cdots, \mathfrak{C}_\kappa$ for the left upper sequent and $\Phi=\mathfrak{D}_1, \cdots, \mathfrak{D}_\lambda$ for the right upper one. Take $\Phi=\mathfrak{C}_1\wedge\mathfrak{D}_1, \cdots, \mathfrak{C}_1\wedge\mathfrak{D}_\lambda, \cdots, \mathfrak{C}_\kappa\wedge\mathfrak{D}_1, \cdots, \mathfrak{C}_\kappa\wedge\mathfrak{D}_\lambda$ for the lower sequent.

2.4. Similar to 2.3 for the cases where the inference rules are $\vee$-left and $\supset$-left.

2.5. For the case where the inference rule is $\dfrac{\Gamma\to\Delta, s=t \quad \mathfrak{A}(t), \Pi\to\Sigma}{\mathfrak{A}(s), \Gamma, \Pi\to\Delta, \Sigma}$: Let $\Phi=\Phi_1$ for the left upper sequent and $\Phi=\Phi_2$ for the right upper one. Take $\Phi=\Phi_1, \Phi_2$ for the lower sequent.

2.6. For the case where the inference rule is $\dfrac{\Gamma\to\Delta, \mathfrak{A}(a)}{\Gamma\to\Delta, \forall x\mathfrak{A}(x)}$: Let $\Phi=\mathfrak{B}_1(a), \cdots, \mathfrak{B}_\kappa(a)$ $(\kappa\geqq 1)$ for the upper sequent. Fix an enumeration $\mathfrak{s}_1, \cdots, \mathfrak{s}_\lambda$ $(\lambda=2^\kappa-1)$ without repetition of all non-empty subsets of $\{1, \cdots, \kappa\}$. For the lower sequent take $\Phi=\mathfrak{C}_1, \cdots, \mathfrak{C}_\lambda$, where $\mathfrak{C}_i$ is defined as follows: Let $\mathfrak{s}_i=\{j_1, \cdots, j_\mu\}\subseteqq\{1, \cdots, \kappa\}$. Put $\mathfrak{C}_i=\forall x\{\mathfrak{B}_{j_1}(x)\vee\cdots\vee\mathfrak{B}_{j_\mu}(x)\}\wedge\exists x\mathfrak{B}_{j_1}(x)\wedge\cdots\wedge\exists x\mathfrak{B}_{j_\mu}(x)$.

2.7. Similar to 2.6 for the case where the inference rule is $\exists$-left.

SUBLEMMA 3.2. *There is a function $f:\omega\to\omega$ with the following property* (*A*).
(*A*) *For each proof $\mathfrak{P}$ of $\mathfrak{N}_1^+$ and each sequent $\mathfrak{S}$ in $\mathfrak{P}$, if* $\mathrm{lh}(\mathfrak{P})\leqq m$, $\Theta(\mathfrak{P}, \mathfrak{S})=\Theta_0$ *and* $\Phi(\mathfrak{P}, \mathfrak{S})=\mathfrak{B}_1, \cdots, \mathfrak{B}_\kappa$ *then* 1) $\kappa\leqq f(m)$, 2) *$\Theta_0\to\mathfrak{B}_1\vee\cdots\vee\mathfrak{B}_\kappa$ is provable in $\mathfrak{N}_0$ without induction with length $\leqq f(m)$*, 3) *for each i $(1\leqq i\leqq\kappa)$ and each element $\mathfrak{A}$ of Θ_0, $\mathfrak{B}_i\to\mathfrak{A}$ is provable in $\mathfrak{N}_0$ without induction with length $\leqq f(m)$, and* 4) *for*

each pair i, j $(1 \leqq i < j \leqq \kappa)$, $\mathfrak{B}_i, \mathfrak{B}_j \rightarrow$ *is provable in* $\mathfrak{N}_0$ *without induction with length* $\leqq f(m)$.

PROOF OF SUBLEMMA 3.2. By 1) and 2) of Sublemma 3.1.

In the remainder of this paper $\mathfrak{P}$ ranges over cut-free proofs of $\mathfrak{N}_1^+$ of $\mathfrak{L}_0$-formulas and $\mathfrak{B}_0$ ranges over elements of $\Phi(\mathfrak{P}, \mathfrak{S}_0)$, where $\mathfrak{S}_0$ is the end-sequent of $\mathfrak{P}$.

Step 3. In this step we define, for $\mathfrak{P}$, $\mathfrak{B}_0$ and a sequent $\mathfrak{S}$ in $\mathfrak{P}$, a set $\mathfrak{K}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$ and a function $\mathfrak{T}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$ satisfying the following conditions.

(Condition 1) 1) $\mathfrak{K}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$ is a set of sequences of variables with the same length as that of $\alpha(\mathfrak{P}, \mathfrak{S})$. 2) $\mathfrak{T}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$ is a function whose domain is $\mathfrak{K}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$. 3) For each element β of $\mathfrak{K}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$, $\mathfrak{T}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})(\beta)$ is an element of $\Phi(\mathfrak{P}, \mathfrak{S})$.

These definitions are done by induction from the end-sequent up to begining sequents.

To simplify notations, we say sometimes that $\mathfrak{K}=\mathfrak{K}_0$ for $\mathfrak{S}$ (respectively, $\mathfrak{T}=\mathfrak{T}_0$ for $\mathfrak{S}$) if $\mathfrak{K}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})=\mathfrak{K}_0$ (respectively, $\mathfrak{T}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})=\mathfrak{T}_0$).

Let B be a countable set of new free variables. Variables in B are called to be *B-variables*. Let F be a function such that i) the domain of F is the set of eigen-variables of $\mathfrak{P}$, ii) for each eigen-variable a of $\mathfrak{P}$ $F(a)$ is a countable subset of B and iii) if a and b are distinct then $F(a) \cap F(b)=\emptyset$.

3.0. For the end-sequent $\rightarrow \mathfrak{A}_0$: Define $\mathfrak{K}=\{\langle\,\rangle\}$ and $\mathfrak{T}(\langle\,\rangle)=\mathfrak{B}_0$.

3.1. For the case where the inference rule is $\dfrac{\mathfrak{A}, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg \mathfrak{A}}$: Let $\mathfrak{K}=\mathfrak{K}_0$ and $\mathfrak{J}=\mathfrak{T}_0$ for the lower sequent. Take $\mathfrak{K}=\mathfrak{K}_0$ and $\mathfrak{T}=\mathfrak{T}_0$ also for the upper one.

3.2. Similar to 3.1 for the cases where the inference rules are $\neg$-left, $\supset$-right, $\vee$-right, $\wedge$-left, $\forall$-left, $\exists$-right, structural rules and (I-1)-(I-6), (I-8)-(I-10).

3.3. For the case where the inference rule is $\dfrac{\Gamma \rightarrow \Delta, \mathfrak{A} \quad \Gamma \rightarrow \Delta, \mathfrak{B}}{\Gamma \rightarrow \Delta, \mathfrak{A} \wedge \mathfrak{B}}$: Let $\mathfrak{K}=\mathfrak{K}_0$ and $\mathfrak{T}=\mathfrak{T}_0$ for the lower sequent. Assume that $\Phi=\mathfrak{C}_1, \cdots, \mathfrak{C}_\kappa$ for the left upper sequent and $\Phi=\mathfrak{D}_1, \cdots, \mathfrak{D}_\lambda$ for the right upper one. Hence $\Phi=\mathfrak{C}_1 \wedge \mathfrak{D}_1, \cdots, \mathfrak{C}_1 \wedge \mathfrak{D}_\lambda, \cdots, \mathfrak{C}_\kappa \wedge \mathfrak{D}_1, \cdots, \mathfrak{C}_\kappa \wedge \mathfrak{D}_\lambda$ for the lower one. For the left upper sequent take $\mathfrak{K}=\mathfrak{K}_0$ and $\mathfrak{T}=\mathfrak{T}_1$, where $\mathfrak{T}_1$ is the function with domain $\mathfrak{K}_0$ defined by; if $\beta \in \mathfrak{K}_0$ and $\mathfrak{T}_0(\beta)=\mathfrak{C}_i \wedge \mathfrak{D}_j$ define $\mathfrak{T}_1(\beta)=\mathfrak{C}_i$. For the right upper sequent take $\mathfrak{K}=\mathfrak{K}_0$ and $\mathfrak{T}=\mathfrak{T}_2$, where $\mathfrak{T}_2$ is the function with domain $\mathfrak{K}_0$ defined by; if $\beta \in \mathfrak{K}_0$ and $\mathfrak{T}_0(\beta)=\mathfrak{C}_i \wedge \mathfrak{D}_j$ define $\mathfrak{T}_2(\beta)=\mathfrak{D}_j$.

3.4. Similar to 3.3 for the cases where the inference rules are $\vee$-left and $\supset$-left.

3.5. For the case where the inference rule is $\frac{\Gamma\to\Delta,\ s=t\quad \mathfrak{A}(t),\ \Pi\to\Sigma}{\mathfrak{A}(s),\ \Gamma,\ \Pi\to\Delta,\ \Sigma}$: Let $\mathfrak{R}=\mathfrak{R}_0$ and $\mathfrak{T}=\mathfrak{T}_0$ for the lower sequent. Assume that $\Phi=\Phi_1$ for the left upper sequent and $\Phi=\Phi_2$ for the right upper one. Hence $\Phi=\Phi_1,\ \Phi_2$ for the lower one.

For the left upper sequent take $\mathfrak{R}=\{\beta/\beta\in\mathfrak{R}_0$ and $\mathfrak{T}_0(\beta)$ is an element of $\Phi_1\}$ and $\mathfrak{T}=\mathfrak{T}_0\upharpoonright\mathfrak{R}$. For the right upper one take $\mathfrak{R}=\{\beta/\beta\in\mathfrak{R}_0$ and $\mathfrak{T}_0(\beta)$ is an element of $\Phi_2\}$ and $\mathfrak{T}=\mathfrak{T}_0\upharpoonright\mathfrak{R}$.

3.6. For the case where the inference rule is $\frac{\Gamma\to\Delta,\ \mathfrak{A}(a)}{\Gamma\to\Delta,\ \forall x\mathfrak{A}(x)}$: Let $\mathfrak{R}=\{\beta_1, \cdots, \beta_\nu\}$ and $\mathfrak{T}=\mathfrak{T}_0$ for the lower sequent. Assume that $\Phi=\mathfrak{B}_1(a), \cdots, \mathfrak{B}_\kappa(a)$ for the upper one. Fix ν subsets $C_1, \cdots, C_\nu$ of $F(a)$ such that 1) $F(a)=C_1\cup\cdots\cup C_\nu$, 2) if i is distinct from j then $C_i\cap C_j=\emptyset$ and 3) each C_i is countable. For each i $(1\leqq i\leqq\nu)$ we define $\mathfrak{R}_i$ and $\mathfrak{T}_i$ as follows: Let $\mathfrak{T}_0(\beta_i)=\forall x[\mathfrak{B}_{j_1}(x)\vee\cdots\vee\mathfrak{B}_{j_\mu}(x)]\wedge\exists x\mathfrak{B}_{j_1}(x)\wedge\cdots\wedge\exists x\mathfrak{B}_{j_\mu}(x)$. Define $\mathfrak{R}_i=\{\beta_i, b_1; \cdots; \beta_i, b_\mu\}$, where $b_1, \cdots, b_\mu$ are the first μ elements of C_i. $\mathfrak{T}_i$ is the function with domain $\mathfrak{R}_i$ defined by; $\mathfrak{T}_i(\beta, b_k)=\mathfrak{B}_{j_k}(a)$ $(k=1, \cdots, \mu)$. Now take $\mathfrak{R}=\mathfrak{R}_1\cup\cdots\cup\mathfrak{R}_\nu$ and $\mathfrak{T}=\mathfrak{T}_1\cup\cdots\cup\mathfrak{T}_\nu$ for the upper sequent.

3.7. Similar to 3.6 for the case where the inference rule is $\exists$-left.

It is easy to see

SUBLEMMA 3.3. 1) *Let* $\mathfrak{R}=\mathfrak{R}_0$ *and* $\mathfrak{R}=\mathfrak{R}_1$ *for some sequents* $\mathfrak{S}_1$ *and* $\mathfrak{S}_2$, *respectively. Assume that* $\beta, b, \beta_1\in\mathfrak{R}_0$, $\beta, c, \beta_2\in\mathfrak{R}_1$ *and* b *is distinct from* c. *Then every variable in* b, β_1 *does not occur in* c, β_2. 2) *Let* $\mathfrak{R}=\mathfrak{R}_0$ *and* $\mathfrak{R}=\mathfrak{R}_1$ *for some sequents* $\mathfrak{S}_1$ *and* $\mathfrak{S}_2$, *respectively. Assume that* $\mathfrak{S}_2$ *stands above* $\mathfrak{S}_1$ *in* $\mathfrak{P}$. *Then, for any* $\gamma\in\mathfrak{R}_1$, *there is an element* β *of* $\mathfrak{R}_0$ *such that* γ *is an extension of* β.

SUBLEMA 3.4. *There is a function* $f:\omega\to\omega$ *with the following property* (*B*). (*B*) *If* $\mathrm{lh}(\mathfrak{P})\leqq m$ *then, for each sequent* $\mathfrak{S}$ *in* $\mathfrak{P}$, *the number of elements of* $\mathfrak{R}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$ *is less than or equal to* $f(m)$.

PROOF OF SUBLEMMA 3.4. By 1) of Sublemma 3.2.

Step 4. In this step we define, for $\mathfrak{P}$, $\mathfrak{B}_0$ and a sequent $\mathfrak{S}$ in $\mathfrak{P}$, functions $\phi(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$, $\psi(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$ and $\mathfrak{G}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$ by induction from begining sequents down to the end-sequent. These functions satisfy the following conditions.

(Condition 2) 1) $\phi(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$, $\psi(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$ and $\mathfrak{G}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$ are functions whose domains are $\mathfrak{R}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$. 2) For each element β of $\mathfrak{R}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$, $\phi(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})(\beta)$ is a set of equations, $\psi(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})(\beta)$ is a set whose elements are one of the forms $r\approx s$ and $r\approx s\oplus 1$ (r and s are terms) and $\mathfrak{G}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})(\beta)$ is a subset of B.

As usual we say sometimes that $\phi=\phi_0$ for $\mathfrak{S}$ (respectively, $\psi=\psi_0$ for $\mathfrak{S}$,

$\mathfrak{G}=\mathfrak{G}_0$ for $\mathfrak{S}$) if $\phi(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})=\phi_0$ (respectively, $\psi(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})=\psi_0$, $\mathfrak{G}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})=\mathfrak{G}_0$).

4.0. For a begining sequent $\mathfrak{A}\to\mathfrak{A}$: Let $\Theta=\Theta_0$ and $\alpha=\alpha_0$ for $\mathfrak{A}\to\mathfrak{A}$. Define $\phi(\beta)=\{s(\beta)=t(\beta)/s(\alpha_0)=t(\alpha_0)$ is an element of $\Theta_0\}$, $\psi(\beta)=\emptyset$ and $\mathfrak{G}(\beta)=\emptyset$.

4.1. For the case where the inference rule is $\dfrac{\mathfrak{A},\ \Gamma\to\Delta}{\Gamma\to\Delta,\ \neg\mathfrak{A}}$: Let $\mathfrak{K}=\mathfrak{K}_0$ for the lower sequent. Hence $\mathfrak{K}=\mathfrak{K}_0$ also for the upper one. Let $\phi=\phi_1$, $\psi=\psi_1$ and $\mathfrak{G}=\mathfrak{G}_1$ for the upper one. Take $\phi=\phi_1$, $\psi=\psi_1$ and $\mathfrak{G}=\mathfrak{G}_1$ also for the lower one.

4.2. Similar to 4.1 for the cases where the inference rules are $\neg$-left, $\supset$-right, $\vee$-right, $\wedge$-left, $\exists$-right, $\forall$-left, structural rules and (I-1)-(I-6), (I-8)-(I-10).

4.3. For the case where the inference rule is $\dfrac{\Gamma\to\Delta,\ \mathfrak{A}\quad \Gamma\to\Delta,\ \mathfrak{B}}{\Gamma\to\Delta,\ \mathfrak{A}\wedge\mathfrak{B}}$: Let $\mathfrak{K}=\mathfrak{K}_0$ for the lower sequent. Hence $\mathfrak{K}=\mathfrak{K}_0$ also for the upper ones. Let $\phi=\phi_1$, $\psi=\psi_1$ and $\mathfrak{G}=\mathfrak{G}_1$ for the left upper one, and let $\phi=\phi_2$, $\psi=\psi_2$ and $\mathfrak{G}=\mathfrak{G}_2$ for the right upper one. For the lower one take $\phi=\phi_0$, $\psi=\psi_0$ and $\mathfrak{G}=\mathfrak{G}_0$, where ϕ_0, ψ_0 and $\mathfrak{G}_0$ are defined by: 1) ϕ_0, ψ_0 and $\mathfrak{G}_0$ are functions with domain $\mathfrak{K}_0$. 2) If $\beta\in\mathfrak{K}_0$, define $\phi_0(\beta)=\phi_1(\beta)\cup\phi_2(\beta)$, $\psi_0(\beta)=\psi_1(\beta)\cup\psi_2(\beta)$ and $\mathfrak{G}_0(\beta)=\mathfrak{G}_1(\beta)\cup\mathfrak{G}_2(\beta)$.

4.4. Similar to 4.3 for the cases where the inference rules are $\vee$-left and $\supset$-left.

4.5. For the case where the inference rule is $\dfrac{\Gamma\to\Delta,\ s(\alpha_0)=t(\alpha_0)\quad \mathfrak{A}(t(\alpha_0)),\ \Pi\to\Sigma}{\mathfrak{A}(s(\alpha_0)),\ \Gamma,\ \Pi\to\Delta,\ \Sigma}$, where $\alpha=\alpha_0$ for the upper and lower sequents: Let $\mathfrak{K}=\mathfrak{K}_1$ and $\mathfrak{K}=\mathfrak{K}_2$ for the left upper sequent and the right upper one, respectively. Hence $\mathfrak{K}=\mathfrak{K}_0$ for the lower one, where $\mathfrak{K}_0=\mathfrak{K}_1\cup\mathfrak{K}_2$. Let $\phi=\phi_1$, $\psi=\psi_1$ and $\mathfrak{G}=\mathfrak{G}_1$ for the left upper sequent. Let $\phi=\phi_2$, $\psi=\psi_2$ and $\mathfrak{G}=\mathfrak{G}_2$ for the right upper one. For the lower one take $\phi=\phi_0$ and $\mathfrak{G}=\mathfrak{G}_0$, where ϕ_0 and $\mathfrak{G}_0$ are defined by: 1) ϕ_0 and $\mathfrak{G}_0$ are functions with domain $\mathfrak{K}_0$, 2) if $\beta\in\mathfrak{K}_1$, define $\phi_0(\beta)=\phi_1(\beta)$; if $\beta\in\mathfrak{K}_2$, define $\phi_0(\beta)=\phi_2(\beta)$, and 3) if $\beta\in\mathfrak{K}_1$, define $\mathfrak{G}_0(\beta)=\mathfrak{G}_1(\beta)$; if $\beta\in\mathfrak{K}_2$, define $\mathfrak{G}_0(\beta)=\mathfrak{G}_2(\beta)$. For the lower sequent, take $\psi=\psi_0$, where ψ_0 is defined as follows: 1) ψ_0 is a function with domain $\mathfrak{K}_0$, and 2) if $\beta\in\mathfrak{K}_1$, define $\psi_0(\beta)=\psi_1(\beta)$; if $\beta\in\mathfrak{K}_2$, define $\psi_0(\beta)=\psi_2(\beta)\cup\{u(s(\beta),\ \beta)\approx u(t(\beta),\ \beta)/P(r,\ u(s(\alpha_0),\ \alpha_0),\ v)$ is a subformula of $\mathfrak{A}(s(\alpha_0))$ for some r and $v\}$.

4.6. For the case where the inference rule is $\dfrac{\Gamma\to\Delta,\ \mathfrak{A}(a)}{\Gamma\to\Delta,\ \forall x\mathfrak{A}(x)}$: In this case $\forall x\mathfrak{A}(x)$ is P-eliminable and hence, for any subformula $P(r, s, t)$ of $\mathfrak{A}(a)$, s does not contain a. Let $\mathfrak{K}=\mathfrak{K}_0$ and $\mathfrak{K}=\mathfrak{K}_1$ for the upper sequent and the lower one, respectively. Let $\alpha=\alpha_0$ for the lower one and so $\alpha=\alpha_0$, a for the upper one.

Now assume that $\phi=\phi_1$, $\psi=\psi_1$ and $\mathfrak{G}=\mathfrak{G}_1$ have been defined for the upper sequent. For the lower one take $\phi=\phi_0$, $\psi=\psi_0$ and $\mathfrak{G}=\mathfrak{G}_0$, where ϕ_0, ψ_0 and $\mathfrak{G}_0$ are defined in the following manner: 1) ϕ_0, ψ_0 and $\mathfrak{G}_0$ are functions with domain $\mathfrak{K}_1$. 2) Assume $\beta\in\mathfrak{K}_1$. Let $\beta, c_1; \cdots; \beta, c_\kappa$ be all elements of $\mathfrak{K}_0$ which are extensions of β. Define $\phi_0(\beta)=\phi_1(\beta, c_1)\cup\cdots\cup\phi_1(\beta, c_\kappa)$, $\psi_0(\beta)=\psi_1(\beta, c_1)\cup\cdots\cup\psi_1(\beta, c_\kappa)$ and $G_0(\beta)=G_1(\beta, c_1)\cup\cdots\cup G_1(\beta, c_\kappa)\cup\{c_1, \cdots, c_\kappa\}$.

4.7. Similar to 4.6 for the case where the inference rule is $\exists$-left.

4.8. For the case where the inference is

$$\frac{P(r(\alpha_0), s(\alpha_0), t(\alpha_0))\supset P(r(\alpha_0), s(\alpha_0)', t(\alpha_0)+r(\alpha_0)), \Gamma\to\Delta}{\Gamma\to\Delta},$$ where $\alpha=\alpha_0$ for the upper sequent: Let $\mathfrak{K}=\mathfrak{K}_0$ for the upper sequent. Then $\mathfrak{K}=\mathfrak{K}_0$ also for the lower one. Let $\phi=\phi_1$, $\psi=\psi_1$ and $\mathfrak{G}=\mathfrak{G}_1$ for the upper one. For the lower sequent take $\phi=\phi_1$, $\psi=\psi_0$ and $\mathfrak{G}=\mathfrak{G}_1$, where ψ_0 is defined by; 1) ψ_0 is a function whose domain is $\mathfrak{K}_0$, and 2) for any element β of $\mathfrak{K}_0$, define $\psi_0(\beta)=\psi_1(\beta)\cup\{s(\beta)'\approx s(\beta)\oplus 1\}$.

SUBLEMMA 3.5. 1) *Let* $\mathfrak{K}=\mathfrak{K}_0$ *and* $\psi=\psi_0$ *for a sequent* $\mathfrak{S}_1$. *Let* $\mathfrak{K}=\mathfrak{K}_1$ *and* $\psi=\psi_1$ for *a sequent* $\mathfrak{S}_2$. *Assume that* $\mathfrak{S}_2$ *stands above* $\mathfrak{S}_1$ *in* $\mathfrak{P}$. *Further assume that* $\beta\in\mathfrak{K}_0$, $\gamma\in\mathfrak{K}_1$ *and* γ *ts an extension of* β. *Then* $\psi_1(\gamma)\subseteq\psi_0(\beta)$. 2) *Let* $\mathfrak{K}=\mathfrak{K}_0$ *and* $\mathfrak{G}=\mathfrak{G}_0$ *for a sequent* $\mathfrak{S}$. *Assume that* $\beta_0\in\mathfrak{K}_0$ *and* $b\in\mathfrak{G}_0(\beta_0)$. *Then, for some sequent* $\mathfrak{S}_1$ *above* $\mathfrak{S}$ *and some sequence* β_1 *of B-variables,* β_0, β_1, $b\in\mathfrak{K}_1$, *where* $\mathfrak{K}=\mathfrak{K}_1$ *for* $\mathfrak{S}_1$. 3) *Let* $\mathfrak{K}=\mathfrak{K}_0$, $\phi=\phi_0$ *for a sequent* $\mathfrak{G}$. *Assume that* $\beta\in\mathfrak{K}_0$. *Then every B-variable occurring in* $\phi_0(\beta)$ *occurs in* β *or* $\mathfrak{G}_0(\beta)$. 4) *Let* $\mathfrak{K}=\mathfrak{K}_0$ *and* $\mathfrak{G}=\mathfrak{G}_0$ *for a sequent* $\mathfrak{S}_1$. *Let* $\mathfrak{K}=\mathfrak{K}_1$ *and* $\mathfrak{G}=\mathfrak{G}_1$ *for a sequent* $\mathfrak{S}_2$. *Assume that* $\mathfrak{S}_1$ *does not stand above* $\mathfrak{S}_2$ *and* $\mathfrak{S}_2$ *does not stand above* $\mathfrak{S}_1$. *Further assume that* $\beta\in\mathfrak{K}_0$ *and* $\gamma\in\mathfrak{K}_1$. *Then* $\mathfrak{G}_0(\beta)\cap\mathfrak{G}_1(\gamma)=\emptyset$. 5) *Let* $\mathfrak{K}=\mathfrak{K}_0$ *and* $\mathfrak{G}=\mathfrak{G}_0$ *for some sequent. Assume that* β, $\gamma\in\mathfrak{K}_0$ *and* β *is distinct from* γ. *Then* $\mathfrak{G}_0(\beta)\cap\mathfrak{G}_0(\gamma)=\emptyset$. 6) *Let* $\Theta=\Theta_0$, $\alpha=\alpha_0$, $\mathfrak{K}=\mathfrak{K}_0$ *and* $\phi=\phi_0$ *for some sequent. Assume that* $\beta\in\mathfrak{K}_0$ *and* $r(\alpha_0)=s(\alpha_0)$ *occurs in* Θ_0. *Then* $r(\beta)=s(\beta)\in\phi_0(\beta)$. 7) *Let* $\mathfrak{K}=\mathfrak{K}_0$ *and* $\psi=\psi_0$ *for some sequent. Assume that* $\beta\in\mathfrak{K}_0$ *and* $r\approx s\oplus 1\in\psi_0(\beta)$. *Then* $\to r=s'$ *is provable in* $\mathfrak{N}_0$ *without induction with length* 3. 8) *Let* $\mathfrak{K}=\mathfrak{K}_0$, $\phi=\phi_0$ *and* $\psi=\psi_0$ *for some sequent. Assume that* $\beta\in\mathfrak{K}_0$ *and* $r\approx s\in\psi_0(\beta)$. *Then* $u=v\to r=s$ *is provable in* $\mathfrak{N}_0$ *without induction with length* 4 *for some element* $u=v$ *of* $\phi_0(\beta)$.

PROOF OF SMBLEMMA 3.5.

1) It is easily proved by induction corresponding to the inductive definition of ψ. In the induction step, we use 2) of Sublemma 3.3 when we consider the cases where the inference rules are equality inferences, $\forall$-right or $\exists$-left.

2) and 3). Easily proved by induction corresponding to the inductive definitions of $\mathfrak{G}$ and ϕ.

4) By the stipulation on F and 2) of Sublemma 3.5, it is easily proved by induction corresponding to the inductive definition of $\mathfrak{G}$.

5) Easily proved by induction corresponding to the inductive definition of $\mathfrak{G}$.

In the induction step, we use 1) of Sublemma 3.3 and 2) and 4) of Sublemma 3.5.

6) Easily proved by induction corresponding to the inductive definition of ϕ. We use 2) of Sublemma 3.1 for the cases where the inferences are $\forall$-right or $\exists$-left.

7) Immediate from the definition of ψ.

8) By the definition of ψ and 6) of Sublemma 3.5.

SUBLEMMA 3.6. 1) *There is a function* $g: \omega \to \omega$ *with the following property* (C). (C) *For each sequent* $\mathfrak{S}$ *in a proof* $\mathfrak{P}$ *and each element* β *of* $\mathfrak{K}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$, *if* $\mathrm{lh}(\mathfrak{P}) \leqq m$ *then the number of elements of* $\phi(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})(\beta)$ *is less than or equal to* $g(m)$. 2) *There is a function* $h: \omega^2 \to \omega$ *with the following property* (D). (D) *For each sequent* $\mathfrak{S}$ *in a proof* $\mathfrak{P}$ *and each element* β *of* $\mathfrak{K}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$, *if* $\mathrm{lh}(\mathfrak{P}) \leqq m$ *and every variable in* $\mathfrak{G}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})(\beta)$ *does not occur in sequences of formulas* Π, Σ *and* $\phi(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$, $\Pi \to \Sigma$ *is provable in* $\mathfrak{N}_0$ *without induction with length* $\leqq k$ *then* $\mathfrak{C}(\beta)$, $\Pi \to \Sigma$ *is provable in* $\mathfrak{N}_0$ *without induction with length* $\leqq h(m, n)+k$, *where* n *is the length of the subproof of* $\mathfrak{S}$ *in* $\mathfrak{P}$, $\alpha(\mathfrak{P}, \mathfrak{S}) = \alpha_0$ *and* $\mathfrak{T}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})(\beta) = \mathfrak{C}(\alpha_0)$.

PROOF OF SUBLEMMA 3.6.

1) By 1) of Sublemma 3.1 and Sublemma 3.4.

2) Using 1) of Sublemma 3.1, Sublemma 3.4 and 1) of Sublemma 3.6, we can define the desired function h.

By the induction on n, we can see that the defined h has the desired properties. In the basis step, we use 1) of Sublemma 3.1. In the induction step; we use 2), 3) and 4) of Sublemma 3.5 for the cases where the inference rules are $\wedge$-right, $\vee$-left and $\supset$-left; we use 2), 3) and 5) of Sublemma 3.5 for the cases where the inference rules are $\forall$-right and $\exists$-left.

Step 5. For a set ξ, terms s, t and integers μ, ν, we write $\langle s, \mu \rangle \underset{\xi}{\Rightarrow} \langle t, \nu \rangle$ if (i) $s \approx t \in \xi$ and μ is ν, (ii) $t \approx s \in \xi$ and μ is ν, (iii) $s \approx t \oplus 1 \in \xi$ and μ is $\nu+1$ or (iv) $t \approx s \oplus 1 \in \xi$ and ν is $\mu+1$. We write $\langle s, \mu \rangle \overset{k}{\underset{\xi}{\Rightarrow}} \langle t, \nu \rangle$ if, for some sequence of terms $s_1, \cdots, s_{k-1}$ and some sequence of integers $\mu_1, \cdots, \mu_{k-1}$, $\langle s, \mu \rangle \underset{\xi}{\Rightarrow} \langle s_1, \mu_1 \rangle, \cdots,$ and $\langle s_{k-1}, \mu_{k-1} \rangle \underset{\xi}{\Rightarrow} \langle t, \nu \rangle$. When $k=0$, $\langle s, \mu \rangle \overset{0}{\underset{\xi}{\Rightarrow}} \langle t, \nu \rangle$ means that s is t and μ is ν. By $\langle s, \mu \rangle \overset{*}{\underset{\xi}{\Rightarrow}} \langle t, \nu \rangle$ we mean that $\langle s, \mu \rangle \overset{k}{\underset{\xi}{\Rightarrow}} \langle t, \nu \rangle$ for some natural number k.

Define $\mathfrak{J}(\mathfrak{P}, \mathfrak{B}_0) = \{\langle s, \nu \rangle / \langle 0, 0 \rangle \overset{*}{\underset{\xi_0}{\Rightarrow}} \langle s, \nu \rangle\}$, where ξ_0 is $\psi(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S}_0)(\langle\,\rangle)$ and $\mathfrak{S}_0$ is the end-sequent of $\mathfrak{P}$. Further define $T(\mathfrak{P}, \mathfrak{B}_0) = \{r(\beta) / \alpha(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S}) = \alpha_0$, $\beta \in \mathfrak{K}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$ and $P(u, r(\alpha_0), v)$ is a subformula of a formula $\mathfrak{A}$ in $\mathfrak{S}$ for some sequent $\mathfrak{S}$ in $\mathfrak{P}\} \cup \{0\}$.

SUBLEMMA 3.7. *Let* ξ_0 *be* $\phi(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S}_0)(\langle\,\rangle)$, *where* $\mathfrak{S}_0$ *is the end-sequent of* $\mathfrak{P}$. 1) *If* $\langle s, \mu\rangle \overset{k}{\underset{\xi_0}{\Rightarrow}} \langle t, \nu\rangle$, *then* $\langle t, \nu\rangle \overset{k}{\underset{\xi_0}{\Rightarrow}} \langle s, \mu\rangle$. 2) *If* $\langle s, \mu\rangle \overset{k}{\underset{\xi_0}{\Rightarrow}} \langle t, \nu\rangle$, *then* $\langle s, \mu+\lambda\rangle \overset{k}{\underset{\xi_0}{\Rightarrow}} \langle t, \nu+\lambda\rangle$ *for each integer* λ. 3) *If* $\langle r_1, \nu_1\rangle \underset{\xi_0}{\Rightarrow} \langle r_2, \nu_2\rangle \cdots$ *and* $\langle r_{k-1}, \nu_{k-1}\rangle \underset{\xi_0}{\Rightarrow} \langle r_k, \nu_k\rangle$ *and* $\nu_1 \geqq 0 \cdots$ *and* $\nu_k \geqq 0$, *then, for some sequence* Γ *of equations in* $\phi(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S}_0)(\langle\,\rangle)$ *with length* $\leqq k$, Γ, $r_1=\bar{\nu}_1 \rightarrow r_k=\bar{\nu}_k$ *is provable in* $\mathfrak{N}_0$ *without induction with length* $\leqq 8+2\times(k-1)$. 4) *If* $\langle 0, 0\rangle \overset{*}{\underset{\xi_0}{\Rightarrow}} \langle s, \nu\rangle$, *then s is an element of* $T(\mathfrak{P}, \mathfrak{B}_0)$.

PROOF OF SUBLMMA 3.7.

1) and 2). Trivial.

3) By 7) and 8) of Sublemma 3.5.

4) By the definition of $\phi(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$.

SUBLEMMA 3.8. 1) *There is a function* $g: \omega \rightarrow \omega$ *with the following property* (E). (E) *If* $\mathrm{lh}(\mathfrak{P}) \leqq m$ *and the number of occurrences of P in* $\mathfrak{P}$ *is less than or equal to m, then the number of elements of* $T(\mathfrak{P}, \mathfrak{B}_0)$ *is less than or equal to* $g(m)$. 2) *There is a function* $h: \omega \rightarrow \omega$ *with the following property* (F). (F) *If* $\mathrm{lh}(\mathfrak{P}) \leqq m$ *and the number of occurrences of P in* $\mathfrak{P}$ *is less than or equal to m and if the range of* $\mathfrak{J}(\mathfrak{P}, \mathfrak{B}_0)$ *is not a subset of* ω, *then* $\mathfrak{B}_0 \rightarrow$ *is provable in* $\mathfrak{N}_0$ *without induction with length* $\leqq h(m)$.

PROOF OF SUBLEMMA 3.8.

1) By Sublemma 3.4.

2) By 2) of Sublemma 3.6, Sublemma 3.7 and 1) of Sublemma 3.8.

Step 6. In this step we consider only $\mathfrak{P}$ and $\mathfrak{B}_0$ for which the range of $\mathfrak{J}(\mathfrak{P}, \mathfrak{B}_0)$ is a subset of ω. Hence, by 1) and 2) of Sublemma 3.7, $\mathfrak{J}(\mathfrak{P}, \mathfrak{B}_0)$ is a function.

For a sequent $\mathfrak{S}$ in a proof $\mathfrak{P}$, an element β of $\mathfrak{K}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S})$ and a subformula $\mathfrak{A}$ of a formula in $\mathfrak{S}$, we define $[\mathfrak{A}]^{\mathfrak{s}}_{\beta}$ by induction as follows: (1) If $\mathfrak{A}$ is $r=s$ define $[\mathfrak{A}]^{\mathfrak{s}}_{\beta}$ as $\mathfrak{A}$. (2) If $\mathfrak{A}$ is $P(s, r(\alpha_0), t)$ and the domain of $\mathfrak{J}(\mathfrak{P}, \mathfrak{B}_0)$ does not contain $r(\beta)$ then define $[\mathfrak{A}]^{\mathfrak{s}}_{\beta}$ as $1=0$, where $\alpha_0=\alpha(\mathfrak{P}, \mathfrak{S})$. (3) If $\mathfrak{A}$ is $P(s, r(\alpha_0), t)$ and $\mathfrak{J}(\mathfrak{P}, \mathfrak{B}_0)(r(\beta))=\nu$ then define $[\mathfrak{A}]^{\mathfrak{s}}_{\beta}$ as $t=\underbrace{s+\cdots+s}_{\nu}$. (4) $[\neg\mathfrak{A}]^{\mathfrak{s}}_{\beta}$ is $\neg[\mathfrak{A}]^{\mathfrak{s}}_{\beta}$, $[\mathfrak{A}\supset\mathfrak{B}]^{\mathfrak{s}}_{\beta}$ is $[\mathfrak{A}]^{\mathfrak{s}}_{\beta}\supset[\mathfrak{B}]^{\mathfrak{s}}_{\beta}$, $[\mathfrak{A}\vee\mathfrak{B}]^{\mathfrak{s}}_{\beta}$ is $[\mathfrak{A}]^{\mathfrak{s}}_{\beta}\vee[\mathfrak{B}]^{\mathfrak{s}}_{\beta}$, $[\mathfrak{A}\wedge\mathfrak{B}]^{\mathfrak{s}}_{\beta}$ is $[\mathfrak{A}]^{\mathfrak{s}}_{\beta}\wedge[\mathfrak{B}]^{\mathfrak{s}}_{\beta}$, $[\forall x\mathfrak{A}]^{\mathfrak{s}}_{\beta}$ is $\forall x([\mathfrak{A}]^{\mathfrak{s}}_{\beta})$ and $[\exists x\mathfrak{A}]^{\mathfrak{s}}_{\beta}$ is $\exists x([\mathfrak{A}]^{\mathfrak{s}}_{\beta})$.

SUBLEMMA 3.9. 1) *If* $\mathfrak{A}$ *is an* $\mathfrak{L}_0$*-formula then* $[\mathfrak{A}]^{\mathfrak{s}}_{\beta}$ *is* $\mathfrak{A}$. 2) *Let* $\dfrac{\mathfrak{S}_1}{\mathfrak{S}_2}$ *be a* $\forall$*-left inference in* $\mathfrak{P}$ *whose auxiliary formula is* $\mathfrak{A}(t)$ *and the chief formula is* $\forall x\mathfrak{A}(x)$. *Assume that* $\beta \in \mathfrak{K}(\mathfrak{P}, \mathfrak{B}_0, \mathfrak{S}_1)$. *Then* $[\mathfrak{B}(t)]^{\mathfrak{s}_1}_{\beta}$ *is* $[\mathfrak{B}(x)]^{\mathfrak{s}_2}_{\beta}\binom{x}{t}$ *for each* *subformula* $\mathfrak{B}(x)$ *of* $\mathfrak{A}(x)$. 3) *Similar to* 2 *for* $\exists$*-right inferences.* 4) $[\mathfrak{A}(0)\wedge\forall x(\mathfrak{A}(x)\supset\mathfrak{A}(x'))\supset\forall x\mathfrak{A}(x)]^{\mathfrak{s}}_{\beta}$ *is* $\mathfrak{B}(0)\wedge\forall x(\mathfrak{B}(x)\supset\mathfrak{B}(x'))\supset\forall x\mathfrak{B}(x)$, *where* $\mathfrak{B}(x)$ *is* $[\mathfrak{A}(x)]^{\mathfrak{s}}_{\beta}$. 5) *Let* $\dfrac{\Gamma\to\Delta,\ s=t\ \ \mathfrak{A}(t),\ \Pi\to\Sigma}{\mathfrak{A}(s),\ \Gamma,\ \Pi\to\Delta,\ \Sigma}$ *be an equality inference in* $\mathfrak{P}$. *Let* $\mathfrak{K}=\mathfrak{K}_0$ *and* $\mathfrak{K}=\mathfrak{K}_1$ *for the lower sequent* $\mathfrak{S}_1$ *and the right upper one* $\mathfrak{S}_2$, *respectively. Assume that* $\beta\in\mathfrak{K}_0\cap\mathfrak{K}_1$. *Then* $[\mathfrak{A}(s)]^{\mathfrak{s}_1}_{\beta}$ *is* $\mathfrak{B}(s)$ *and* $[\mathfrak{A}(t)]^{\mathfrak{s}_2}_{\beta}$ *is* $\mathfrak{B}(t)$ *for some* $\mathfrak{B}(x)$. 6) *Let* $\dfrac{\mathfrak{S}_1}{\mathfrak{S}_2}$ *be an* $\exists$*-left inference in* $\mathfrak{P}$ *whose auxiliary formula is* $\mathfrak{A}(a)$ *and the chief formula is* $\exists x\mathfrak{A}(x)$. *Assume that* $\beta\in\mathfrak{K}(\mathfrak{P},\ \mathfrak{B}_0,\ \mathfrak{S}_2)$ *and* $\beta,\ b\in\mathfrak{K}(\mathfrak{P},\ \mathfrak{B}_0,\ \mathfrak{S}_1)$. *Then* $[\mathfrak{B}(a)]^{\mathfrak{s}_1}_{\beta;b}$ *is* $[\mathfrak{B}(x)]^{\mathfrak{s}_2}_{\beta}\binom{x}{a}$ *for each subformula* $\mathfrak{B}(a)$ *of* $\mathfrak{A}(a)$ *and* $[\mathfrak{C}]^{\mathfrak{s}_1}_{\beta;b}$ *is* $[\mathfrak{C}]^{\mathfrak{s}_2}_{\beta}$ *for each subformula* $\mathfrak{C}$ *of each other formula in* $\mathfrak{S}_1$. 7) *Similar to* 6 *for* $\forall$*-right inferences.* 8) *Let* $\dfrac{P(r,\ s,\ t)\supset P(r,\ s',\ t+r),\ \Gamma\to\Delta}{\Gamma\to\Delta}$ *be* (I-9) *in* $\mathfrak{P}$. *Let* $\mathfrak{S}$ *be the upper sequent. Assume that* $\beta\in\mathfrak{K}(\mathfrak{P},\ \mathfrak{B}_0,\ \mathfrak{S})$. *Then* $[P(r,\ s,\ t)\supset P(r,\ s',\ t+r)]^{\mathfrak{s}}_{\beta}$ *is* $t=\underbrace{r+\cdots+r}_{\nu}\supset t+r=\underbrace{r+\cdots r+r}_{\nu+1}$ *for some* ν *or* $[P(r,\ s,\ t)\supset P(r,\ s',\ t+r)]^{\mathfrak{s}}_{\beta}$ *is* $1=0\supset 1=0$. 9) *Let* $\dfrac{P(r,\ s,\ t)\wedge P(r,\ s,\ u)\supset t=u,\ \Gamma\to\Delta}{\Gamma\to\Delta}$ *be* (I-10) *in* $\mathfrak{P}$. *Let* $\mathfrak{S}$ *be the upper sequent. Assume that* $\beta\in\mathfrak{K}(\mathfrak{P},\mathfrak{B}_0,\mathfrak{S})$. *Then* $[P(r,s,t)\wedge P(r,s,u)\supset t=u]^{\mathfrak{s}}_{\beta}$ *is* $t=\underbrace{r+\cdots+r}_{\nu}\wedge u=\underbrace{r+\cdots+r}_{\nu}\supset t=u$ *for some* ν *or* $[P(r,\ s,\ t)\wedge P(r,\ s,\ u)\supset t=u]^{\mathfrak{s}}_{\beta}$ *is* $1=0\wedge 1=0\supset t=u$.

PROOF OF SUBLEMMA 3.9.

1) It is easily proved by induction on the number of logical symbols of $\mathfrak{A}$.

2) and 3). Easily proved by induction on the number of logical symbols of $\mathfrak{B}(x)$. In the basis step, we use the fact that every formula in $\mathfrak{P}$ is P-eliminable.

4) By the fact that every formula in $\mathfrak{P}$ is P-eliminable.

5) and 8). By the definition of ϕ and $\mathfrak{J}(\mathfrak{P},\ \mathfrak{B}_0)$ and 1) of Sublemma 3.5.

6) and 7). Easily proved by induction on the number of logical symbols of $\mathfrak{B}(x)$ and $\mathfrak{C}$. In the basis step, we use the fact that every formula in $\mathfrak{P}$ is P-eliminable.

9) Trivial.

SUBLEMMA 3.10. *There is a function* $h:\omega^2\to\omega$ *with the following property* (G). (G) *If* $\mathrm{lh}(\mathfrak{P})\leqq m$ *and the range of* $\mathfrak{J}(\mathfrak{P},\ \mathfrak{B}_0)$ *is a subset of* ω *then, for each sequent* $\mathfrak{S}$ *in* $\mathfrak{P}$ *of the form* $\mathfrak{A}_1,\ \cdots,\ \mathfrak{A}_\mu\to\mathfrak{B}_1,\ \cdots,\ \mathfrak{B}_\nu$ *and each* β *in* $\mathfrak{K}(\mathfrak{P},\ \mathfrak{B}_0,\ \mathfrak{S})$, $\mathfrak{T}(\mathfrak{P},\ \mathfrak{B}_0,\ \mathfrak{S})(\beta)$, $[\mathfrak{A}_1]^{\mathfrak{s}}_{\beta},\ \cdots,\ [\mathfrak{A}_\mu]^{\mathfrak{s}}_{\beta}\to[\mathfrak{B}_1]^{\mathfrak{s}}_{\beta},\ \cdots,\ [\mathfrak{B}_\nu]^{\mathfrak{s}}_{\beta}$ *is provable in* $\mathfrak{N}_0$ *with length* $\leqq h(m,\ n)$, *where* n *is the length of the subproof of* $\mathfrak{S}$ *in* $\mathfrak{P}$.

Hence if $\mathfrak{P}$ *is a proof of an* $\mathfrak{L}_0$*-formula* $\mathfrak{A}_0$ *then* $\mathfrak{B}_0\to\mathfrak{A}_0$ *is provable in* $\mathfrak{N}_0$ *with length* $\leqq h(m,\ m)$.

PROOF OF SUBLEMMA 3.10.

Using 1) and 3) of Sublemma 3.2, we can define the desired function h.

By the induction on n, we can see that the defined h has the desired properties. In the induction step: we use 3) of Sublemma 3.2 and 4) of Sublemma 3.9 for the cases where the inference rules are equality inferences; we use 5) and 6) of Sublemma 3.9 for the cases where the inference rules are $\forall$-right or $\exists$-left; and we use 7) and 8) of Sublemma 3.9 for the case where the inference rule is (I-9) or (I-10)]

Now we can derive Lemma 3 from 1) and 2) of Sublemma 3.2, 2) of Sublemma 3.8 and Sublemma 3.10.

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