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A METATHEORY OF NONSTANDARD ANALYSIS

By

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In the previous work [11] of Yasugi, we set up a formal system IR of infinitary logic, by whose proof-theoretical properties the infinitesimal calculus can be justified. Such an attempt was started due to the first author's wish to single out the essence of the metatheory of nonstandard mathematics as a "trick of the language", and was concluded with the "linkage principles" which support the nonstandard theory.

We have since pushed that thought forward and extended the result to general nonstandard analysis, with the theory of Loeb measure in mind.

Let V be the universe of analysis with the real numbers as individuals and let C be the set of constants representing V. Let A be a collection of set theoretical axioms (those necessary for analysis) and let B be a set of mathematical axioms, which will be expressed in terms of "elementary" formulas involving the constants in C. Let C be the collection of specification axioms on the domains:

$$\forall x\in d\vee[c\in d](x=c),$$

where $\vee[c\in d]$ expresses the disjunction over the domain d.

With the axioms in C, the elementary quantifiers and the "restricted" conjunctions and disjunctions becomes equivalent:

$$\forall x\in dF(x)\longleftrightarrow\wedge[c\in d]F(c).$$

C specifies the mathematical objects to the "standard" ones.

Our basic logic is an infinitary logic with "elementary" quantifiers. The "standard" analysis SA is a consequence of A and B, and C regulates the meaning of quantifiers. If one lifts the regulation C, then we obtain the subsystem GA, in which the existence of nonstandard objects will become consistent. The "internality" is characterized by "elementariness". The bridge between SA and GA is the first group of "linkage" principles stated in §3. See also §3 of [11]. By virtue of these principles, we can define a system NA, nonstandard analysis, as an enlargement of GA. In order to develop ex-

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ternal theory, such as Loeb measure, one needs to introduce the "external sets". This can be realized by adding the external variables and by interpreting the axioms of comprehension in the direct limit of power set operations.

The "standardization" property is also important. That is, the standard elements of an external subset of a standard set form another standard set. This allows one to introduce *inf/sup* of a bounded (external) subset of standard reals, and hence also the "standard parts". It is based on the "informal" comprehension axiom. We owe to N. Motohashi of Tsukuba in this regard.

We are thus able to sum up the basis of the external theory, EA, as consisting of the axioms of the external comprehension and the "comprehension in the real world". (The linkage principles between NA and EA will be stated in § 6.)

Our work is "another view" of Kawai's model construction in [5]. He sets up an axiomatic system of nonstandard set theory WNST, which is sufficient to develop external analysis, and justifies it by constructing a model of it. We recommend the reader to look up [5].

For the details of the homogeneous, infinitary logic, one can, for example, refer to [2] or [10]. Our basic knowledge in nonstandard analysis has been taken from [1], [6], [7] and [9]. We have referred to [3], [4], [5] and [8] for the metatheories.

The first author got some ideas here during her visit to H. Sayeki in Montreal, to whose hospitality she is grateful. She is also in debt to S. Szabo for his interest in this line of work.

§ 1. The universe of analysis.

DEFINITION 1.1. 1) Let $\boldsymbol{R}$ be the set of all reals, which are regarded as individuals. Then, the universe of discourse, $\boldsymbol{V}$, is defined as follows.

$$V_0=\boldsymbol{R}, \qquad V_{n+1}=V_n \cup P(V_n),$$
$$\boldsymbol{V}=\boldsymbol{V}(\boldsymbol{R})=\bigcup_n V_n .$$

2) C(=C($\boldsymbol{V}$)) consists of all the constant symbols for the elements of $\boldsymbol{V}$. If c represents a $v \in \boldsymbol{V}$, then we may write c_v for c and v_c for v.

3) We shall employ the notations $\boldsymbol{\omega}$ and $\boldsymbol{Q}$ for the actual set of natural numbers and that of rationals respectively; N, Q and R for the constants representing $\boldsymbol{\omega}$, $\boldsymbol{Q}$, and $\boldsymbol{R}$ respectively.

4) The well-ordering property for each set in $\boldsymbol{V}$ will be assumed.

Note. 1) V is adequate for developing ordinary analysis. In particular, V is closed with respect to the basic set theoretical operations. That is,

$$a,\ b\in V_n \text{ implies } \{a,\ b\}\in V_{n+1},\ a\cup b,\ a\cap b,\ a\setminus b\in V_n,$$

$$\cup a\in V_{n-1},\ P(a)\in V_{n+1} \text{ and } a\times b\in V_{n+1}.$$

2) Throughout the subsequent context, we shall abuse some notations. That is, for the "real" conditions such as $v_c \in v_d$ we may write $c\in d$.

§2. Formalization of analysis.

DEFINITION 2.1. 1) LA ($\equiv LA(\mathrm{C})$), the language of analysis over C, consists of the following.

1.1) The symbols in C, the predicates $=$ and $\in$, and the language of first order predicate calculus over these.

1.2) Other constants (function and predicate symbols) necessary for analysis. We do not list them, and appeal to the common sense of mathematicians.

2) LA-terms are defined as usual.

3) LA-elementary formulas are defined as the usual formulas, except that quantifiers are bounded: if ϕ is an elementary formula and s is a term, then $\forall x\in s\phi$ and $\exists x\in s\phi$ are elementary formulas.

4) LS, the language of 'standard' analysis, is LA plus some infinitary $\wedge$ and $\vee$.

The introduction of infinitary connectives require many more variables. (See [2] or [10], for instance.)

5) LS-formulas are defined similarly to LA-elementary formulas by allowing infinitary conjunctions and disjunctions such as $\bigwedge_{\lambda<r} A_\lambda$ and $\bigvee_{\lambda<r} A_\lambda$, which may also be expressed as $\wedge[\lambda<\gamma]A_\lambda$ and $\vee[\lambda<\gamma]A_\lambda$, where γ is an ordinal below the cardinality of V. (We may write $\wedge[c\in d]\ A(c)$ for $\wedge[v_c\in v_d]\ A(c)$.) A variable-free LS-formula will be called a standard formula.

6) Our statements will be expressed in the sequential forms:

$$\{A_\alpha\}_{\alpha<\gamma} \longrightarrow \{B_\beta\}_{\beta<\delta},$$

A_α and B_β being LS-formulas. (See also Definition 1.1 of [11].)

Note. "$x\in c$" may be read as "x is of sort c".

DEFINITION 2.2. The logical system IL of the LS-sequential calculus is defined similarly to EIL in Definition 1.2 of [11]. The initial sequents are of the form $D\rightarrow D$. Let us list some of the rules of inference.

$\wedge$ left $\qquad \dfrac{\{A_{\lambda,\mu};\ \mu<\beta_\lambda,\ \lambda<\gamma\},\ \Gamma\to\Delta}{\{\wedge[\mu<\beta_\lambda]A_{\lambda,\mu};\ \lambda<\gamma\},\ \Gamma\to\Delta}$;

$\wedge$ right $\qquad \dfrac{\Gamma\to\Delta,\ \{A_{\lambda,\mu_\lambda};\ \lambda<\gamma\},}{\Gamma\to\Delta,\ \{\wedge[\mu<\beta_\lambda]A_{\lambda,\mu};\ \lambda<\gamma\}}$ for all $\{\mu_\lambda\}_{\lambda<\gamma}$, $(\mu_\lambda<\beta_\lambda;\ \lambda<\gamma)$,

where $(\mu_\lambda<\beta_\lambda;\ \lambda<\gamma)$ means "for all possibilities of $\langle\mu_\lambda;\ \lambda<\gamma\rangle\in\Pi[\lambda<\gamma]\beta_\lambda$;

$\forall$ left $\qquad \dfrac{\{t_\lambda\in s_\lambda\vdash A_\lambda(t_\lambda);\ \lambda<\gamma\},\ \Gamma\to\Delta}{\{\forall x_\lambda\in s_\lambda A_\lambda(x_\lambda);\ \lambda<\gamma\},\ \Gamma\to\Delta}$;

$\forall$ right $\qquad \dfrac{\Gamma\to\Delta,\ \{x_\lambda\in s_\lambda\vdash A_\lambda(x_\lambda);\ \lambda<\gamma\}}{\Gamma\to\Delta,\ \{\forall x_\lambda\in s_\lambda A_\lambda(x_\lambda);\ \lambda<\gamma\}}$,

where $\{x_\lambda\}_\lambda$ are eigenvariables;

Cut $\qquad \dfrac{\{\Gamma\to\Delta,\ A_\lambda;\ \lambda<\gamma\}\ \{A_\lambda;\ \lambda<\gamma\},\ \Pi\to\Lambda}{\Gamma,\ \Pi\to\Delta,\ \Lambda}$.

IL is a portion of the general infinitary logic with homogeneous quantifiers, and so the following hold. (See [2] and [10]; see also Proposition 1.1 and Theorem 1 in [11].) These are the crucial facts for determining the metatheory of nonstandard analysis.

MAIN PROPOSITION. 1) *The consistency and completeness together with the cut elimination theorem hold for IL.*

2) *The compactness theorem holds for IL. That is, if $\Gamma\to\Delta$ is IL-provable, where Γ and Δ consist of elementary formulas, then there exist finite subsets $\Gamma_0\subset\Gamma$ and $\Delta_0\subset\Delta$ such that $\Gamma_0\to\Delta_0$ is finitarily provable.*

DEFINITION 2.3. 1) A will denote a collection of axioms on sets, sufficient for the development of analysis and expressed in elementary sentences, In particular, it includes the comprehension axiom:

$$\forall x\in c\exists z\in e\forall y\in d(y\in z\equiv y\in x\wedge\phi(y,\ x)),$$

where ϕ is elementary, $v_d=\cup v_c$ and $v_e=P(v_d)$. B will denote a collection of mathematical, elementary axioms as needed. It includes the attributes of reals and the defining formulas of various constants. (See also Definition 2.3 in [11].)

2) C will denote the collection of specification-axioms on the domains:

$$\forall x\in d\vee[c\in d](x=c), \qquad \text{for each } d \text{ in } C.$$

3) $\Gamma\to\Delta$, a sequent of LS, is said to be a theorem of SA (standard analysis) if IL proves A, B, C, $\Gamma\to\Delta$.

4) eSA will denote the elementary part of SA, and sSA will denote the standard (variable-free) part of SA.

NOTE. A, B, C are true in V with the natural interpretation of the constants and connectives, and hence the consistency of the axiom set can be assumed.

The facts below, though easily proved, characterize the "standard" analysis.

THEOREM 1. (See Theorems 2 and 3 in [11].) 1) *The following is a theorem of SA for every F and c.*

$$\forall x \in c F(x) \longleftrightarrow \wedge [d \in c] F(d).$$

Similarly for ($\exists$, $\vee$).

2) *The completeness of SA holds. That is, for any LS-sentence A, exactly one of A and $\neg A$ is a theorem of SA.*

NOTE. $\rightarrow$ in 1) above is called the "standardization" of a bounded quantifier, and $\leftarrow$ the "internalization" of a bounded conjunction.

§ 3. Generalized theory of analysis and the linkage principles.

DEFINITION 3.1. 1) The language we work with here is LS.

2) D will denote the collection of elementary, closed theorems of SA.

3) $\Gamma \rightarrow \Delta$ is said to be a theorem of GA (general analysis) if IL proves $D, \Gamma \rightarrow \Delta$.

4) eGA will denote the elementary part of GA, and sGA will denote the standard (variable-free) part of GA.

The facts below are nearly the restatements of the results in the corollary of Definition 3.1 and Theorems 4 and 5 in [11].

THEOREM 2. 1) *The compactness of GA holds. That is, if $\Gamma \rightarrow \Delta$ is a theorem of eGA, then there are finite $D_0 \subset D$, $\Gamma_0 \subset \Gamma$ and $\Delta_0 \subset \Delta$ such that D_0, $\Gamma_0 \rightarrow \Delta_0$ is finitarily provable.*

2) *The subtheory property $GA \subset SA$ holds.*

3) *SA is consistent with GA.*

4) *The axioms in C are not provable in GA.*

5) *The completeness of eGA as well as that of sGA holds.*

6) *The transfer principle holds. That is, $eGA = eSA$ and $sGA = sSA$.*

The first group of linkage principles (of GA with SA) can now be stated. (See § 3 of [11].)

Linkage Principles I

[SPC: specification principle]	1) of Theorem 1
[CML: completeness of SA]	2) of Theorem 1
[SBT: subtheory property]	2) of Theorem 2
[CMP: compactness of GA]	1) of Theorem 2
[TRF: transfer principle]	6) of Theorem 2

These are fundamental principles in the foundations of nonstandard analysis and will be freely used. Let us emphasize that the cut-elimination theorem for IL lies at the basis of all. The roles of these principles are outlined in § 3 of [11]. Some typical applications of these are seen throughout [11].

DEFINITION 3.2. Various notions and notations concerning nonstandard objects can be defined as in Definition 3.2 of [11]. The following are some examples.

st(x, d) (x is a standard element of sort d.): $\bigvee[c\in d]\ (x=c)$
nst(x, d) (x is a nonstandard object of sort d.): $x\in d\wedge\neg\bigvee[c\in d]\ (x=c)$
inf(x, R) (x is an infinite real.): $x\in R\wedge\bigwedge[c\in R]|x|>c$
infl(x, R) (x is an infinitesimal real.): $x\in R\wedge\bigwedge[c\in R]\ (c>0\vdash|x|<c)$
$x\approx y$ (x and y are infinitely close reals.): infl$(x-y, R)$.

§ 4. Internality and Saturation.

In [11], the internality (as well as saturation) did not play much role explicitly, but here it does.

DEFINITION 4.1. itl(t, c) (t is internal of sort c): $t\in c$, where t is any term of LA.

This explains that the internality means being sorted.

PROPOSITION 4.1. *The following are theorems of GA with c and d constants.*

1) $y\in x,\ itl(x, c)\rightarrow y\in d$, *where* $v_d=\bigcup v_c$.
2) $itl(x, c),\ itl(y, d)\rightarrow itl(x\cap y, e)$, *where* $v_e=P(\bigcup v_c\cap\bigcup v_d)$.
3) $itl(x, c),\ \neg x\in d\rightarrow itl(x, e)$, *where* $v_e=v_c\setminus v_d$.
4) $itl(c, d)$ *where* $v_d=\{v_c\}$.
5) *Suppose* $\phi(x_1, \cdots, x_n, x)$ *is elementary with at most* $x_1, \cdots, x_n, x$ *free. Then*

$$\{itl(x_i, c_i)\}_{i=1,\cdots,n}\rightarrow\exists z\in d\ \forall x\in c\ (x\in z\equiv x\in x_1\wedge\phi(x_1, \cdots, x_n, x)),$$

where $v_d = P(v_c)$ *and* $v_c = \cup v_{c_1}$.

The succeedent formula will be denoted as $itl(\{x \in x_1/\phi(x_1, \cdots, x_n, x)\}, d)$.

NOTE. By virtue of 5) and the obvious fact $itl(x, c) \to x \in c \wedge x = x$, the internal sets are characterized by elementary formulas, and hence the definition below.

DEFINITION 4.2. An LS-formula $A(x_1, \cdots, x_n, x)$ is said to be internal if, for an elementary $\phi(x_1, \cdots, x_n, x)$, and a constant c,

$$\{x_i \in c_i\}_{i=1,\dots,n} \to \forall x \in c\ (A(x_1, \cdots, x_n, x) \equiv x \in x_1 \wedge \phi(x_1, \cdots, x_n, x))$$

is a theorem of GA, where $v_c = \cup v_{c_1}$.

By 5) above, this implies that

$$\{x_i \in c_i\}_{i=1,\dots,n} \to itl(\{x \in x_1/A(x_1, \cdots, x_n, x)\}, d)$$

for some d.

As a consequence of the compactness, we have a simple counterexample.

Let $A(x)$ be $\vee[n \in \omega]x = n$. $A(x)$ is not internal. (See Proposition 3.3 in [11].)

DEFINITION 4.3. Consider an elementary formula $B(x, y)$. (B may contain other free variables $x_1, \cdots, x_n$, in which case $\{x_i \in c_i\}_{i=1,\dots,n}$ should be placed in the premise in the subsequent argument.) Suppose J is a constant. $\{B(x, i); v_i \in v_J\}$ (abbreviated to $\{B_i; i \in J\}$) will be said to be uniformly internal.

THEOREM 3 *Suppose* $\{B_i; i \in J\}$ *is uniformly internal. If, for every finite* $v_{i_1}, \cdots, v_{i_m} \in v_J$,

$$\exists x \in s \bigwedge_{k=1}^{m} B(x, i_k)$$

is consistent with D *(in* IL*), then so is*

$$\exists x \in c \wedge [i \in J] B(x, i).$$

PROOF. Suppose otherwise:

$$D,\ \exists x \in c \wedge [i \in J] B(x, i) \to$$

is IL-provable. Then

$$D,\ x \in c,\ \{B(x, i)\}_{i \in J} \to$$

is also. By the compactness,

$$D_0,\ x \in c,\ \{B(x, i_k)\}_{k=1,\dots,m} \to$$

for some finite $D_0 \subset D$, $i_1, \cdots, i_k \in J$, or

$$D_0,\ \exists x \in c \bigwedge_{k=1}^{m} B(x, i_k) \rightarrow ,$$

contradicting the hypothesis.

§ 5. Nonstandard analysis.

DEFINITION 5.1. 1) Let J be a constant denoting a nonempty set, and let v_K be the collection of the finite subsets of v_J. Then $\wedge[v_k \in v_K]A_k$ may be symbolically expressed as

$$\wedge[\{i_1, \cdots, i_m\} \subset J] A_{(i_1, \cdots, i_m)} .$$

With this notation, gst will denote the collection of the following formulas for any uniformly internal $\{B(x, i)\}$:

$$\wedge[\{i_1, \cdots, i_m\} \subset J] \exists x \in c \bigwedge_{k=1}^{m} B(x, i_k) \vdash \exists x \in c \wedge [i \in J] B(x, i) .$$

gst stands for the "general saturation" axioms.

2) An LS-sequent $\Gamma \rightarrow \Delta$ is said to be a theory of NA (nonstandard analysis) if

$$D, \text{gst}, \Gamma \rightarrow \Delta$$

is IL-provable.

By the compactness and a generalization of Theorem 3, we obtain the

THEOREM 4. *NA is consistent.*

PROPOSITION 5.1. *The following are some examples of the theorems of NA.*

1) $\exists x \in N \wedge [n \in \omega]\ (x > n)$

2) $\exists x \in R\ (x \neq 0 \wedge \wedge [n \in \omega] |x| < 1/n)$

3) $\vee[c \in \boldsymbol{R}] x \approx c \rightarrow \vee[r \in \boldsymbol{Q}]\ (r > 0 \wedge |x| < r)$

4) $\{c_i \in c\}_{i \in \omega},\ \{c_i \neq c_j\}_{i, j \in \omega, i \neq j} \rightarrow \exists x \in c \wedge [d \in c] x \neq d .$

These claim successively the existence of an infinite natural number, that of a nontrivial infinitesimal real, the boundedness of a near-standard real and the existence of a nonstandard element in any infinite set.

The instruments which are crucial for nonstandard analysis, such as the extension property, the least number principle and the preservation theorem, are straightforward consequences of NA. We take up the first one as an example.

PROPOSITION 5.2 (Extension). *Let J and c be constants, $v_d = P(v_J \times \cup v_c)$, and suppose a and f are variable parameters. Then*

$$itl(b, c),\ \{f`i \in b\,;\ i \in J\} \to \exists F \in d\ \ \forall x \in J(F`x \in b \wedge \wedge[i \in J]\ (F`i = f`i))$$

is a theorem of NA, where $f`i$ means the application of f to i.

PROOF. Let $B(y, F)$ $(\equiv B(b, f, y, F))$ be the elementary formula

$$\forall x \in J(F`x \in b \wedge F`y = f`y)\,.$$

For each $q \equiv \{i_1, \cdots, i_n\} \subset J$, define g_q to be a term satisfying

$$g_q`i_k = f`i_k, \qquad k = 1, \cdots, n\,,$$

$$g_q`u = f`i_1, \qquad u \neq i_1, \cdots, i_n\,.$$

Then

$$itl(b, c),\ \{f`i \in b\,;\ i \in J\} \to g_q \in d \wedge \bigwedge_{k=1}^{n} B(i_k, g_q)\,.$$

So, writing the premise as Γ,

$$\Gamma \to \exists g \in d \bigwedge_{k=1}^{n} B(i_k, g)\,.$$

By gst, we then obtain

$$\Gamma \to \exists F \in d \wedge [i \in J] B(i, F)\,,$$

from which follow the proposition.

The well-known theorems of nonstandard analysis can be proved in NA, and also the interrelations between the standard and the nonstandard formulations of various properties can be established by virtue of the linkage principles I similarly to the case of the elementary calculus in [11].

§6. External theory.

DEFINITION 6.1. 1) The external language ex$L$ is LS plus $X, Y, \cdots$, external variables. (In contrast, the original variables may be called "internal" ones.)

2) Terms are those of LS. (That is, external variables do not occur in compound terms.)

3) Atomic ex$L$-formulas are of the form $t \in s$ where one or both of t and s may be external variables. The ex$L$-formulas are defined similarly to LS-formulas where the quantifications over external variables are unbounded.

4) The external logical system exIL is defined as the system IL (See Definition 2.2) with the following additions.

$\forall$ex right

$$\frac{\Gamma \to \Delta,\ \{A_\alpha(X_\alpha)\}_{\alpha<\gamma}}{\Gamma \to \Delta,\ \{\forall X_\alpha A_\alpha(X_\alpha)\}_{\alpha<\gamma}},$$

where X_α is an external eigenvariable.

$\forall$ex left

$$\frac{\{A_\alpha(t_\alpha)\}_{\alpha<\gamma},\ \Gamma \to \Delta}{\{\forall X_\alpha A_\alpha(X_\alpha)\}_{\alpha<\gamma},\ \Gamma \to \Delta},$$

where X_α is external, t_α is either an external variable or an LS-term, in the latter case of which t_α is a maximal term.

Similarly for $\exists$ex.

Similarly to the Main Proposition in §3 we obtain the external parallel.

MAIN exPROPOSITION. *exIL is a two-sorted, homogeneous logic, and hence the consistency and the completeness together with the cut elimination hold. (The external domain contains the internal domain.)*

PROPOSITION 6.1. *The following are exIL-provable under the premise of the symmetry of the equality*: $t=t$.

1) $\exists X(x=X)$ *(An internal element is external.)*
2) $\exists X(c=X)$ *(A standard element is external.)*
3) $\vee[c \in C]X=c \to \exists x(X=x)$ *(A standard element is internal.)*

We owe an informal account of the following lemma, which is necessary for the standardization property, to N. Motohashi.

DIAGONAL LEMMA. *Let A be any* ex$L$*-formula, and let v_e be any infinite set. For any map v_i from $P(v_e)$ to v_e,*

$$\to \{i`d \in d \equiv A(i`d)\,;\, v_d \in P(v_e)\}$$

is provable in GA extended to the external language ex$L$.

PROOF. Define (in the real world)

$$p = \{v_i(v_d)\,;\, v_i(v_d) \notin v_d\}.$$

By the comprehension axiom in V, p is a set, and $p \in P(v_e)$ and $v_i(p) \in p$. But then there is a $q \in v_e$ such that $v_i(p)=v_i(q)$ and $v_i(q) \notin q$. Put $b=i`c_p$. Notice that $i`c_p \in c_p$ and $i`c_q \notin c_q$ can be regarded as axioms (in D).

$$A(b) \to i`c_p \in c_p \equiv A(i`c_p)$$

and

$$\neg A(b) \to i`c_q \in c_q \equiv A(i`c_q)$$

are provable, and hence

$$A(b) \to \{i`d \in d \equiv A(i`d)\,;\, v_d \in P(v_e)\}$$

and

$$\neg A(b) \to \{i`d \in d \equiv A(i`d)\,;\, v_d \in P(v_e)\}$$

are also.

From these follows the claim.

NOTE. One should keep in mind that the lemma relies on the fact that p is a set, which is a consequence of the comprehension in the real world.

THEOREM 5 (Standard comprehension of external statements.). *For any exL-formula $A(X)$ and any standard set v_e in V,*

$$\to \{\wedge[j \in e]\ (j \in d \equiv A(j))\,;\, v_d \in P(v_e)\}$$

is GA-provable (since the well-ordering property is assumed for the sets in V).

PROOF. By the lemma above and the rule: $\wedge$ right.

DEFINITION 6.2. Z_1 will denote the collection of axioms on sets except the axioms of replacement (and comprehension) and regularity written in terms of the external language. Z_2 will denote the collection of the comprehension axioms on the ex$L$-formulas. That is,

$$\forall Y \exists X \forall Z (Z \in X \equiv Z \in Y \wedge A(Z, Y))$$

for any (external) A.

We need another fact in order to develop the external theory. This is essentially Lemma 2 in [5] of Kawai.

THEOREM 6 (See [5].). *The collection $\Sigma = \{D, gst, Z_1, Z_2\}$ of exL-formulas is consistent with the logic exIL,*

PROOF. Otherwise, $\Sigma \to$, or

$$D, \text{gst}, Z_1, Z_2 \to$$

would be exIL-provable. By the Main exProposition it would be valid, and hence every interpretation of the language ex$L$ makes Σ false. On the other hand, $\{D, \text{gst}\}$ is consistent with IL due to Theorem 4, and hence there is an interpretation $U \equiv \langle U, \phi \rangle$ of LS which makes it true. Define an interpretation of ex$L$, say $\boldsymbol{W} = \langle W, \Phi \rangle$ as follows.

$$W_0 = U, \quad W_{n+1} = P(W_n) \cup W_n, \quad W = \bigcup_n W_n, \quad \Phi(x) \in U,$$

$$\Phi(\in) = \phi(\in) \text{ in } \boldsymbol{U} \text{ and } \Phi(\in) = \text{the real elementhood otherwise.}$$

Then $\boldsymbol{W}$ makes Σ true, and hence $\Sigma \rightarrow$ is impossible.

NOTE. The consistency of Σ relies on the real world of sets, and is not an immediate consequence of metatheory.

DEFINITION 6.3. $\Gamma \rightarrow \Delta$, a sequent of ex$L$, is said to be a 'theorem of EA (external analysis)' if

$$\Sigma, \Gamma \rightarrow \Delta$$

is exIL-provable.

As an immediate consequence of Theorems 5 and 6 and the definition immediately above, we obtain the following conclusion.

THEOREM 7. *The external theory of analysis, EA, is consistent, and proves the axiom of standardization.*

DEFINITION 6.4. $c \overset{s}{=} \sup\{r \in R\,;\, A(r)\}$ (the standard sup of the external set A):

$$(\wedge[r \in R](A(r) \vdash r < c)) \wedge (\wedge[d \in R](d < c \vdash \vee[r \in R](A(r) \wedge d < r))).$$

The statements below relate standard and nonstandard elements, and are consequences of Theorem 7.

PROPOSITION 6.2. 1) *The completeness of standard reals holds in EA.*
2) *The standard part of a bounded (finite) real can be defined in EA.*

PEOOF. 1) Let $A(X)$ be any ex$L$-formula. By virtue of Theorem 5 with $v_e = \boldsymbol{R}$ and $v_r \in \boldsymbol{R}$,

$$\rightarrow \{\wedge[j \in R](j \in d \equiv A(j) \wedge j < r)\,;\, v_d \in P(\boldsymbol{R})\}$$

holds in EA. For each v_d in $P(\boldsymbol{R})$ which is bounded (by a standard real), and for some v_c in $\boldsymbol{R}$,

$$\sup d \overset{s}{=} c$$

is a theorem of GA. So,

$$\vee[c\in R]\ (\sup\{j\in R\ ;\ A(j)\wedge j<r\}\overset{s}{=}c)$$

holds in EA, which means the completeness of standard reals.

2) For an r in $\boldsymbol{R}$, let $A(x, y)$ abbreviate $x, y\in R\wedge y<x<r$. Then, by 1),

$$\vee[c\in R]\ (\sup\{j\in R\ ;\ A(x, j)\}\overset{s}{=}c).$$

$$\sup\{j\in R\ ;\ A(x, j)\}\overset{s}{=}c\rightarrow x\approx c$$

also holds. So,

$$x\in R\wedge x<r\rightarrow\vee[c\in R](x\approx c).$$

The uniqueness of such c for an x can easily be proven, and hence the c can be expressed as $\mathrm{st}(x)$.

We can now state the second group of linkage principles (of NA with EA) in addition to the first one in § 3.

Linkage Principles II

[STD: standardization theorem]	Theorem 5
[CNS: consistency of external comprehension]	Theorem 6

Since it is now a routine to carry out the external mathematics, we shall explain with just one example how one can express the theorems of Loeb measure in our theory. This will be done in the next section. We do not have set-theoretical operations such as $\{X, Y\}$, $\cup X$, $P(X)$, $X`Y$ and $\{X/X\in Y\wedge A(X)\}$ officially (for external variables). They are abbreviated expressions, which can be restored to finitary formulas.

§ 7. Loeb measure.

We assume henceforth that

$$P\equiv(\Omega,\ \boldsymbol{A},\ \mu)$$

is an internal probability space. That is, P is a variable and we place the following internal premise.

$\Sigma(P, c)$: $P\in c\wedge P$ is of the form $(\Omega, \boldsymbol{A}, \mu)$, Ω is a set, $\boldsymbol{A}$ is a finite algebra over Ω and μ is a finitely additive probability on $\boldsymbol{A}$.

Under this premise, we shall show how an external σ-algebra extending $\boldsymbol{A}$ and the so-called Loeb measure can be expressed in our external theory.

DEFINITION 7.1.

$\boldsymbol{B}(B)$: $B\subset\Omega\wedge\forall a\in\boldsymbol{A}\wedge[\varepsilon>0]\exists x,\ y\in\boldsymbol{A}(x,\ y\subset a\wedge x\subset B\cap a\subset y\wedge\mu(x-y)<\varepsilon)$,

where B is an external variable, the quantifiers are internal and ε is any positive real constant.

Notice that $\Omega\in d\wedge B(B)\rightarrow B\in b$ holds, where $v_b=P(\cup v_a)$.

PROPOSITION 7.1. *$\forall x\in\boldsymbol{AB}(x)$ and B is a σ-algebra, where a 'σ-algebra' is expressed in terms of an infinitary $\wedge$.*

PROOF. One can follow the usual mathematical proof. To show that $\boldsymbol{B}$ is a σ-algebra, one uses the extension theorem, Proposition 5.2, where the b there is here A_a with $a\in\boldsymbol{A}$ and $A_a=\{c\in\boldsymbol{A}\,;\,c\subseteqq a\}$.

Due to the consistency of the external comprehension (CNS), $\boldsymbol{B}$ is an external set, and hence it is meaningful to work on $\boldsymbol{B}$.

DEFINITION 7.2.

$$\lambda(B,\ e):\ e=\inf\{d\,;\,\exists Q\in\boldsymbol{A}(B\subseteqq Q\wedge d\approx\mu(Q))\}.$$

PROPOSITION 7.2. 1) *$B\subset\Omega\rightarrow\vee[e\in R]\lambda(B,\ e)$ is a theorem of EA.*
2) *λ is a countably additive probability measure on $\boldsymbol{B}$.*

PROOF. 1) By Proposition 6.2.
2) For example, the binary additivity of λ is expressed as:

$$B,\ C\subset B,\ B\cap C=\emptyset,\ \lambda(B,\ b),\ \lambda(C,\ c)\rightarrow\lambda(B\cup C,\ b+c).$$

By virtue of 1), the assumptions $\lambda(B,\ b)$ and $\lambda(C,\ c)$ are meaningful.

We can also express and show that "$(\Omega,\ \boldsymbol{B},\ \lambda)$ is a complete, countably additive probability space, which is an extension of $(\Omega,\ \boldsymbol{A},\ \mathrm{st}\mu)$", where $\mathrm{st}\mu$ is the "standard part of μ".

Various properties of Loeb measure and the integration can be dealt with similarly; see also §7 of [11] for the integration.

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