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THE H_{∞}-WELLPOSED CAUCHY PROBLEM FOR SCHRÖDINGER TYPE EQUATIONS

By

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§1. Introduction.

We study the Cauchy problem for a Schrödinger type operator

$$L=L(t, x, D_t, D_x)=D_t+\frac{1}{2}\sum_{j=1}^{n}(D_j-a_j(t, x))^2+c(t, x),$$

where $a_j(t, x)$, $c(t, x)\in C_t^0([0, T]; \mathcal{B}^{\infty}(R^n))$, $(T>0)$, $a_j(t, x)=a_j^R(t, x)+ia_j^I(t, x)$ ($a_j^R(t, x)$ and $a_j^I(t, x)$ are real valued functions) for $j=1, \cdots, n$ and $D_t=-i\partial/\partial t$, $D_j=-i\partial/\partial x_j$. Here $\mathcal{B}^{\infty}(R^n)$ denotes the set of C^{∞}-functions whose derivatives of any order are all bounded in R^n and $g(t, x)\in C_t^k([0, T]; X)$ $(k=0, 1, 2, \cdots)$ means that the mapping: $[0, T]\ni t\to g(t)\in X$ is k-times continuously differentiable in the topology of X.

In this paper we give a sufficient condition for the Cauchy problem

$$(1.1)\qquad \begin{cases} L(t, x, D_t, D_x)u(t, x)=f(t, x), & (t, x)\in[0, T]\times R^n, \quad (T>0), \\ u(0, x)=u_0(x), & x\in R^n \end{cases}$$

to be H_{∞}-wellposed in $[0, T]$ $(T>0)$, where H_s denotes the Sobolev space of order s and $H_{\infty}\equiv\bigcap_{-\infty<s<\infty}H_s$.

We say that the Cauchy problem (1.1) is H_{∞}-wellposed in $[0, T]$ if for any initial data $u_0\in H_{\infty}$ and $f(t, x)\in C_t^0([0, T]; H_{\infty})$ there exists a unique solution $u(t, x)\in C_t^1([0, T]; H_{\infty})$, and for any $s\in R^1$ there exist constants $C(s, T)>0$ and $s'\in R^1$ such that the energy inequality

$$(1.2)\qquad \|u(t, \cdot)\|_{(s)}\leqq C(s, T)\left\{\|u_0\|_{(s')}+\int_0^t\|f(\tau, \cdot)\|_{(s')}d\tau\right\}$$

holds for $t\in[0, T]$. Here, $\|u(t, \cdot)\|_{(s)}$ denotes the H_s norm.

Let us briefly recall some known facts. In [3], Ichinose obtained a necessary condition of the Cauchy problem (1.1) to be H_{∞}-wellposed. The sufficient conditions for the Cauchy problem (1.1) to be H_{∞}-wellposed are given by Ichinose [2] and Takeuchi [6].

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The following theorem, which is the main result of the present paper, gives a sufficient condition for the Cauchy problem (1.1) to be H_∞-wellposed.

THEOREM 1.1. *Assume that the coefficients* $a_j(t, x)=a_j^R(t, x)+ia_j^I(t, x)$ *satisfy*

$$\begin{cases} |a_j^I(t, x)| \leqq C\langle x\rangle^{-1}, \\ |D_x^\alpha a_j(t, x)| \leqq C_\alpha \langle x\rangle^{-1} \end{cases} \tag{1.3}$$

for $(t, x)\in[0, T]\times R^n$ *and for any multi-indices* α $(|\alpha|\geqq 1)$ *and* $j=1, \cdots, n$, *where* C *and* C_α *are positive constants and* $\langle x\rangle=(1+|x|^2)^{1/2}$. *Then the Cauchy problem for* (1.1) *is* H_∞*-wellposed in* $[0, T]$.

REMARK 1.2. If there is $\varepsilon>0$ such that $a_j(t, x)$ satisfy

$$\begin{cases} |a_j^I(t, x)| \leqq C\langle x\rangle^{-1-\varepsilon}, \\ |D_x^\alpha a_j(t, x)| \leqq C_\alpha \langle x\rangle^{-1} \end{cases} \tag{1.4}$$

for $(t, x)\in[0, T]\times R^n$ and for any multi-indices α $(|\alpha|\geqq 1)$ and $j=1, \cdots, n$, where C and C_α are positive constants, then the Cauchy problem (1.1) is L^2-wellposed in $[0, T]$.

REMARK 1.3. In the case of $T<0$, we can prove Theorem 1.1 in the same way.

To prove Theorem 1.1 we modify the method given in [5], Ch. 7, §3. Conjugating L by a pseudo-differential operator $K(t, x, D_x)$ with its symbol $\sigma(K)(t, x, \xi)=\exp \Lambda(t, x, \xi)$, where $\Lambda(t, x, \xi)$ is a solution of the following equation

$$\begin{cases} \left(D_t+\sum_{j=1}^n \xi_j D_j\right)\Lambda(t, x, \xi)+\dfrac{iM\langle\xi\rangle}{\langle x\rangle}=0, \\ \Lambda(0, x, \xi)=0, \end{cases} \tag{1.5}$$

we reduce L to $K^{-1}\circ L\circ K=D_t-P(t)$. Then taking a parameter $M>0$ sufficiently large, we can make the imaginary part of $P(t)$ nonnegative in $L^2(R^n)$ and therefore can obtain an energy estimate in L^2-sense for the operator $D_t-P(t)$.

REMARK 1.4. If (1.4) is valid, we replace $\langle x\rangle$ in (1.5) by $\langle x\rangle^{1+\varepsilon}$. Then $K(t, x, D_x)$ becomes a bounded operator from $L^2(R^n)$ to $L^2(R^n)$.

Let us sum up the contents of the paper briefly. In Section 2 we shall prove that there exists the inverse operator of K as a pseudo-differential operator. In Section 3 we shall give the expression of $P(t)$ and prove Theorem 1.1.

§2. Existence of inverse of e^{Λ}.

In this section we shall show the existence of inverse operator of $K=e^{\Lambda}(t, x, D_x)$. We can solve the solution $\Lambda(t, x, \xi)$ of (1.5) as follows

$$\begin{aligned}
(2.1)\qquad \Lambda(t, x, \xi) &= \int_0^t \frac{M\langle\xi\rangle ds}{\langle x-s\xi\rangle} = \frac{\langle\xi\rangle}{|\xi|}\int_0^{t|\xi|} \frac{M ds}{\langle x-s\omega\rangle} \\
&= \frac{\langle\xi\rangle}{|\xi|}\int_0^{t|\xi|} \frac{M ds}{\sqrt{(s-x\cdot\omega)^2+|x|^2-(x\cdot\omega)^2+1}} \\
&= \frac{\langle\xi\rangle}{|\xi|}\int_{-x\cdot w}^{t|\xi|-x\cdot\omega} \frac{M ds}{\sqrt{s^2+|x|^2-(x\cdot\omega)^2+1}} \\
&= M\frac{\langle\xi\rangle}{|\xi|}\log\left\{\frac{\langle x-t\xi\rangle+t|\xi|-x\cdot\omega}{\langle x\rangle-x\cdot\omega}\right\} \\
&= M\log\left\{\frac{\langle x-t\xi\rangle+t|\xi|-x\cdot\omega}{\langle x\rangle-x\cdot\omega}\right\} \\
&\quad + M\Big(\frac{\langle\xi\rangle}{|\xi|}-1\Big)\log\left\{\frac{\langle x-t\xi\rangle+t|\xi|-x\cdot\omega}{\langle x\rangle-x\cdot\omega}\right\} \\
&\equiv \Lambda_0(t, x, \xi)+\Lambda_1(t, x, \xi)\,.
\end{aligned}$$

where $\omega=\xi/|\xi|$ $(\xi\neq 0)$.

REMARK 2.1. If (1.4) is valid, we take $\Lambda(t, x, \xi)$ as

$$(2.2)\qquad \Lambda(t, x, \xi)=\int_0^t \frac{M\langle\xi\rangle}{\langle x-s\xi\rangle^{1+\varepsilon}}ds\,.$$

Then we can see easily $|\Lambda_{(\beta)}^{(\alpha)}(t, x, \xi)|\leqq C_{\alpha,\beta}t^{|\alpha|}$ for any multi-indices α, β, $t\in[0, T]$ and $x, \xi\in R^n$, where $\Lambda_{(\beta)}^{(\alpha)}(t, x, \xi)=\partial_\xi^\alpha D_x^\beta \Lambda(t, x, \xi)$.

LEMMA 2.2. *One can find $C>0$ such that*

$$(2.3)\qquad \Lambda(t, x, \xi)\leqq CM(1+\log\langle t\xi\rangle)$$

for $t\in[0, T]$ and $(x, \xi)\in R^{2n}$.

PROOF. When $\langle x\rangle\geqq 2t|\xi|$, we obtain

$$\begin{aligned}
\Lambda(t, x, \xi) &= \int_0^t \frac{M\langle\xi\rangle ds}{\langle x-s\xi\rangle} \leqq \int_0^t \frac{M\sqrt{2}\langle\xi\rangle ds}{1+|x|-t|\xi|} \\
&\leqq \int_0^t \frac{M\sqrt{2}\langle\xi\rangle}{1+t|\xi|}ds \leqq CM\,.
\end{aligned}$$

When $\langle x\rangle\leqq 2t|\xi|$, since

$$\frac{1}{\langle x\rangle - x\cdot\omega} = \frac{\langle x\rangle + x\cdot\omega}{\langle x\rangle^2 - (x\cdot\omega)^2} \leqq 2\langle x\rangle\,,$$

we have

$$\Lambda(t, x, \xi) = M\frac{\langle\xi\rangle}{|\xi|}\log\left\{\frac{\langle x - t\xi\rangle + t|\xi| - x\cdot\omega}{\langle x\rangle - x\cdot\omega}\right\}$$

$$\leqq M\frac{\langle\xi\rangle}{|\xi|}\log\left\{\frac{2\sqrt{2}\langle t\xi\rangle + 2|x|}{\langle x\rangle - x\cdot\omega}\right\} \leqq CM\log\langle t\xi\rangle\,.$$

This completes the proof.

LEMMA 2.3. *One can find $C>0$ and positive integer l such that*

$$\left\{\begin{array}{l} \exp\Lambda(t, x+y, \xi) \leqq e^{CM}(1+|y|+t|\eta|)^{Ml}\,, \\ \exp\Lambda(t, x, \xi+\eta) \leqq e^{CM}(1+|y|+t|\eta|)^{Ml} \end{array}\right. \tag{2.4}$$

for $|x|\geqq 4t|\xi|$, $x, y, \xi, \eta\in R^n$, $t\in[0, T]$ and $M>0$.

PROOF. When $|x|\geqq 2|y|$, since $|x+y|\geqq|x|-|y|\geqq|x|/2\geqq 2t|\xi|$, we obtain

$$\Lambda(t, x+y, \xi) = \int_0^t \frac{M\langle\xi\rangle}{\langle x+y-s\xi\rangle}ds \leqq \int_0^t \frac{\sqrt{2}M\langle\xi\rangle}{1+|x+y|-t|\xi|}ds \tag{2.5}$$

$$\leqq \int_0^t \frac{\sqrt{2}M\langle\xi\rangle}{1+t|\xi|}ds \leqq CM\,.$$

When $|x|\leqq 2|y|$, since $|y|\geqq|x|/2\geqq 2t|\xi|$, (2.3) implies

$$\Lambda(t, x+y, \xi) \leqq CM\log\langle t\xi\rangle \leqq C'M\log\langle y\rangle\,. \tag{2.6}$$

When $|\xi|\leqq|\eta|$, it follows from (2.3) that

$$\Lambda(t, x, \xi+\eta) \leqq CM\log\langle t(\xi+\eta)\rangle \leqq C'M\log\langle t\eta\rangle\,. \tag{2.7}$$

When $|\xi|\geqq|\eta|$, we have

$$\Lambda(t, x, \xi+\eta) = \int_0^t \frac{M\langle\xi+\eta\rangle}{\langle x-s(\xi+\eta)\rangle}ds \leqq \int_0^t \frac{2\sqrt{2}M\langle\xi\rangle}{1+|x|-t|\xi|-t|\eta|}ds \tag{2.8}$$

$$\leqq \int_0^t \frac{2\sqrt{2}M\langle\xi\rangle}{1+2t|\xi|}ds \leqq CM\,.$$

This completes the proof of Lemma 2.3.

Now we put

$$f(x, \xi) = \langle x\rangle - x\cdot\left(\frac{\xi}{|\xi|}\right). \tag{2.9}$$

LEMMA 2.4. *One can find $C>0$ such that*

$$\left(\frac{f(x', \xi+\eta)}{f(x', \xi)}\right)^{\pm 1} \leqq C\langle\eta\rangle^4 \tag{2.10}$$

for $|x| \leqq 4t|\xi|$, $x, \xi, \eta \in R^n$ *and* $t \in [0, T]$, *where* $x'=x$ *or* $x-t\xi$.

PROOF. Since $1/3 \leqq f(x', \xi) \leqq 3$ for $|x'| \leqq 1$, (2.10) is trivial for $|x'| \leqq 1$. When $x' \cdot \xi \leqq 0$, we have

$$\frac{f(x', \xi+\eta)}{f(x', \xi)} = \frac{\langle x'\rangle - x' \cdot ((\xi+\eta)/|\xi+\eta|)}{\langle x'\rangle - x' \cdot (\xi/|\xi|)} \leqq \frac{2\langle x'\rangle}{\langle x'\rangle} \leqq 2. \tag{2.11}$$

From now on we assume that $x' \cdot \xi \geqq 0$ and $|x'| \geqq 1$. Using the Taylor's formula, we may write

$$f(x', \xi+\eta) = f(x', \xi) + \sum_{j=1}^{n} f_{\xi_j}(x', \xi)\eta_j + 2\sum_{|\alpha|=2}\int_0^1 \frac{1-\theta}{\alpha!} f^{(\alpha)}(x', \xi+\theta\eta)\eta^\alpha d\theta, \tag{2.12}$$

where $f_{\xi_j} = \partial f/\partial \xi_j$ and $f^{(\alpha)} = \partial_\xi^\alpha f$. Since $|x'| \leqq 5t|\xi|$, we have

$$\begin{aligned}
&\left|\sum_{|\alpha|=2}\int_0^1 \frac{1-\theta}{\alpha!} f^{(\alpha)}(x', \xi+\theta\eta)\eta^\alpha d\theta / f(x', \xi)\right| \\
&\leqq \int_0^1 \frac{C|x'|}{(\langle x'\rangle - x' \cdot (\xi/|\xi|))} \frac{\langle\eta\rangle^2}{\langle\xi+\theta\eta\rangle^2} d\theta \\
&\leqq \int_0^1 \frac{C(\langle x'\rangle + x' \cdot (\xi/|\xi|))}{(\langle x'\rangle^2 - (x' \cdot (\xi/|\xi|))^2)} \frac{|x'|\langle\eta\rangle^2}{\langle\xi+\theta\eta\rangle^2} d\theta \\
&\leqq \int_0^1 \frac{C\langle x'\rangle^2\langle\eta\rangle^4}{\langle\xi+\theta\eta\rangle^2\langle\theta\eta\rangle^2} d\theta \leqq \frac{C\langle x'\rangle^2\langle\eta\rangle^4}{\langle\xi\rangle^2} \\
&\leqq \frac{C\langle 5t\xi\rangle^2\langle\eta\rangle^4}{\langle\xi\rangle^2} \leqq C'\langle\eta\rangle^4
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
\frac{f_{\xi_j}(x', \xi)}{f(x', \xi)} &= \frac{((x' \cdot \xi)\xi_j - x'_j|\xi|^2)(\langle x'\rangle + (x' \cdot \xi)/|\xi|)}{|\xi|^3(\langle x'\rangle^2 - ((x' \cdot \xi)/|\xi|)^2)} \\
&= \frac{((x' \cdot \omega)\omega_j - x'_j)(\langle x'\rangle + x' \cdot \omega)}{|\xi|(|x'|^2 + 1 - (x' \cdot \omega)^2)} \\
&\leqq \frac{2((x' \cdot \omega)\omega_j - x'_j)\langle x'\rangle}{|\xi|\{(|x'| + x' \cdot \omega)(|x'| - x' \cdot \omega) + 1)\}} \\
&\leqq \frac{2\langle x'\rangle|h_j(x', \omega)|}{|\xi|\{|x'|(|x'| - x' \cdot \omega) + 1)\}}
\end{aligned} \tag{2.14}$$

where $\omega_j = \xi_j/|\xi|$ and $h_j(x', \omega) = (x' \cdot \omega)\omega_j - x'_j$. Since $h_j(\omega, \omega) = 0$, we have $|x'|h_j(\omega, \omega) = h_j(|x'|\omega, \omega) = 0$. From the mean value theorem it follows that

$$h_j(x', \omega)=h_j(x', \omega)-h_j(|x'|\omega, \omega)$$
$$=\sum_{k=1}^{n} h_{jx'_k}(x'+\theta(x'-|x'|\omega), \omega)(x'_k-|x'|\omega_k).$$

Since $h_j(x', \omega)$ is a homogeneous functions of degree 1 with respect to x', there exists a constant $c>0$ such that $|h_{jx'_k}(x'+\theta(x'-|x'|\omega), \omega)|\leqq c$. Hence we obtain

$$|h_j(x', \omega)|^2\leqq c^2|x'-\omega|x'||^2=c^2(|x'|^2-2(x'\cdot\omega)|x'|+|\omega|^2|x'|^2)$$
$$=2c^2|x'|(|x'|-(x'\cdot\omega)).$$

Consequently we have

$$|h_j(x', \omega)|\leqq\sqrt{2}c|x'|^{1/2}(|x'|-(x'\cdot\omega))^{1/2}. \tag{2.15}$$

Noting that $1\leqq|x'|\leqq 5t|\xi|$, from (2.14) and (2.15) we have

$$\frac{f_{\xi_j}(x', \xi)}{f(x', \xi)}\leqq\frac{2\sqrt{2}c(|x'|(|x'|-x'\cdot\omega))^{1/2}\langle x'\rangle}{|\xi|\{|x'|(|x'|-(x'\cdot\omega))+1)\}}\leqq C. \tag{2.16}$$

By (2.11), (2.12), (2.13) and (2.16), we obtain

$$\frac{f(x', \xi+\eta)}{f(x', \xi)}\leqq C\langle\eta\rangle^4. \tag{2.17}$$

Moreover if we put $\xi+\eta=z$, it follows from (2.17) that

$$\frac{f(x', \xi)}{f(x', \xi+\eta)}=\frac{f(x', z-\eta)}{f(x', z)}\leqq C\langle-\eta\rangle^4=C\langle\eta\rangle^4. \tag{2.18}$$

This proves Lemma 2.4.

LEMMA 2.5. *One can find $C>0$ such that*

$$\left(\frac{f(x+y, \xi)}{f(x, \xi)}\right)^{\pm 1}\leqq C\langle y\rangle^3 \tag{2.19}$$

for $x, y, \xi\in R^n$.

PROOF. (2.19) is trivial for $|x|\leqq 1$. When $x\cdot\xi\leqq 0$, we have

$$\frac{f(x+y, \xi)}{f(x, \xi)}=\frac{\langle x+y\rangle-(x+y)\cdot(\xi/|\xi|)}{\langle x\rangle-x\cdot(\xi/|\xi|)} \tag{2.20}$$
$$\leqq\frac{2\langle x+y\rangle}{\langle x\rangle}\leqq 2^2\langle y\rangle.$$

From now on we assume that $x\cdot\xi\geqq 0$ and $|x|\geqq 1$. Using Taylor's formula, we may write

$$f(x+y, \xi)=f(x, \xi)+\sum_{j=1}^{n} f_{x_j}(x, \xi)y_j+2\sum_{|\alpha|=2}\int_0^1\frac{1-\theta}{\alpha!}(iy)^\alpha f_{(\alpha)}(x+\theta y, \xi)d\theta, \tag{2.21}$$

where $f_{x_j}=\partial f/\partial x_j$ and $f_{(\alpha)}=D_x^\alpha f$. If follows that

(2.22)
$$\Big|\sum_{|\alpha|=2}\int_0^1\frac{1-\theta}{\alpha!}(iy)^\alpha f_{(\alpha)}(x+\theta y,\xi)d\theta/f(x,\xi)\Big|$$
$$\leqq\int_0^1\frac{C|y|^2}{\langle x+\theta y\rangle(\langle x\rangle-x\cdot(\xi/|\xi|))}d\theta$$
$$\leqq\int_0^1\frac{C\langle x\rangle\langle y\rangle^2}{\langle x+\theta y\rangle}d\theta\leqq\int_0^1\frac{C\langle x\rangle\langle y\rangle^3}{\langle x+\theta y\rangle\langle\theta y\rangle}d\theta\leqq C'\langle y\rangle^3,$$

and

(2.23)
$$\frac{f_{x_j}(x,\xi)}{f(x,\xi)}=\frac{x_j-\omega_j\langle x\rangle}{\langle x\rangle f(x,\xi)}=\frac{(\langle x\rangle+(x\cdot\omega))(x_j-\omega_j\langle x\rangle)}{\langle x\rangle(|x|^2-(x\cdot\omega)^2+1)}$$
$$\leqq\frac{2|x-\omega\langle x\rangle|}{|x|^2-(x\cdot\omega)^2+1}\leqq\frac{2(|x-\omega|x||+|\omega|x|-\omega\langle x\rangle|)}{(|x|+x\cdot\omega)(|x|-x\cdot\omega)+1}$$
$$\leqq\frac{2p(x,\omega)}{|x|(|x|-x\cdot\omega)+1}+\frac{2}{\langle x\rangle+|x|}$$

where $\omega_j=\xi_j/|\xi|$ and $p(x,\omega)=|x-\omega|x||$. Since

$$p(x,\omega)^2=|x-\omega|x||^2=|x|^2-2|x|x\cdot\omega+|x|^2$$
$$=2(|x|-x\cdot\omega)|x|,$$

we have

(2.24)
$$p(x,\omega)=\sqrt{2}((|x|-x\cdot\omega)|x|)^{1/2}.$$

It follows from (2.23) and (2.24) that

(2.25)
$$\frac{f_{x_j}(x,\xi)}{f(x,\xi)}\leqq\frac{2\sqrt{2}(|x|(|x|-x\cdot\omega))^{1/2}}{|x|(|x|-x\cdot\omega)+1}+\frac{2}{\langle x\rangle+|x|}\leqq C,$$

By (2.20), (2.21), (2.22) and (2.25), we have

(2.26)
$$\frac{f(x+y,\xi)}{f(x,\xi)}\leqq C\langle y\rangle^3.$$

Moreover if we put $x+y=z$, it follows from (2.26) that

(2.27)
$$\frac{f(x,\xi)}{f(x+y,\xi)}=\frac{f(z-y,\xi)}{f(z,\xi)}\leqq C\langle -y\rangle=C\langle y\rangle^3.$$

This completes the proof of Lemma 2.5.

LEMMA 2.6. *One can find $C>0$ and a positive integer l such that*

(2.28)
$$\exp\{-\Lambda(t,x,\xi+\eta)+\Lambda(t,x+y,\xi)\}\leqq e^{CM}(|y|+t|\eta|+1)^{Ml}$$

for $x, y, \xi, \eta\in R^n$, $t\in[0,T]$ and $M>0$.

PROOF. When $|x|\geqq 4t|\xi|$, we have (2.28) from Lemma 2.3. When $|x|\leqq 4t|\xi|$, it follows from (2.1) that

$$\begin{aligned}
&\exp\{-\Lambda(t,x,\xi+\eta)+\Lambda(t,x+y,\xi)\}\\
&=\left\{\frac{\langle x\rangle-x\cdot((\xi+\eta)/|\xi+\eta|)}{\langle x-t(\xi+\eta)\rangle+t|\xi+\eta|-x\cdot((\xi+\eta)/|\xi+\eta|)}\times\frac{\langle x+y-t\xi\rangle+t|\xi|-(x+y)\cdot(\xi/|\xi|)}{\langle x+y\rangle-(x+y)\cdot(\xi/|\xi|)}\right\}^M\\
&\quad\times\exp\{\Lambda_1(t,x,\xi+\eta)+\Lambda_1(t,x+y,\xi)\}\\
&=\left\{\frac{\langle x\rangle-x\cdot((\xi+\eta)/|\xi+\eta|)}{\langle x\rangle-x\cdot(\xi/|\xi|)}\times\frac{\langle x-t\xi\rangle-(x-t\xi)\cdot(\xi/|\xi|)}{\langle x-t\xi\rangle-(x-t\xi)\cdot((\xi+\eta)/|\xi+\eta|)}\right.\\
&\quad\times\frac{\langle x-t\xi\rangle-(x-t\xi)\cdot((\xi+\eta)/|\xi+\eta|)}{\langle x-t(\xi+\eta)\rangle-(x-t(\xi+\eta))\cdot((\xi+\eta)/|\xi+\eta|)}\times\frac{\langle x\rangle-x\cdot(\xi/|\xi|)}{\langle x+y\rangle-(x+y)\cdot(\xi/|\xi|)}\\
&\quad\left.\times\frac{\langle x+y-t\xi\rangle-(x+y-t\xi)\cdot(\xi/|\xi|)}{\langle x-t\xi\rangle-(x-t\xi)\cdot(\xi/|\xi|)}\right\}^M\times\exp\{-\Lambda_1(t,x,\xi+\eta)+\Lambda_1(t,x+y,\xi)\}\\
&=\left\{\frac{f(x,\xi+\eta)}{f(x,\xi)}\times\frac{f(x-t\xi,\xi)}{f(x-t\xi,\xi+\eta)}\times\frac{f(x-t\xi,\xi+\eta)}{f(x-t(\xi+\eta),\xi+\eta)}\times\frac{f(x,\xi)}{f(x+y,\xi)}\right.\\
&\quad\left.\times\frac{f(x+y-t\xi,\xi)}{f(x-t\xi,\xi)}\right\}^M\times\exp\{-\Lambda_1(t,x,\xi+\eta)+\Lambda_1(t,x+y,\xi)\}.
\end{aligned}$$

Because $\Lambda_1(t,x,\xi)$ is bounded when $|\xi|\geqq 1$, (2.28) can be obtained by using Lemmas 2.4 and 2.5.

LEMMA 2.7. *For any multi-indices α, β ($|\alpha+\beta|\geqq 1$), we have*

$$|\Lambda_{(\beta)}^{(\alpha)}(t,x,\xi)|\leqq C_{\alpha,\beta}t^{|\alpha|} \tag{2.29}$$

for $x,\xi\in R^n$ and $t\in[0,T]$.

PROOF. For any multi-indices α, β ($|\alpha+\beta|\geqq 1$) we can estimate

$$\left|\partial_\xi^\alpha D_x^\beta\left(\frac{\langle\xi\rangle}{\langle x-s\xi\rangle}\right)\right|\leqq\begin{cases}C_{\alpha,\beta}\dfrac{s^{|\alpha|-1}}{\langle x-s\xi\rangle^{|\alpha+\beta|}}+C'_{\alpha,\beta}\dfrac{\langle\xi\rangle s^{|\alpha|}}{\langle x-s\xi\rangle^{|\alpha+\beta|+1}} & \text{for } |\alpha|\geqq 1,\\[2ex] C''_{\alpha,\beta}\dfrac{\langle\xi\rangle}{\langle x-s\xi\rangle^{|\beta|+1}} & \text{for } |\alpha|=0\end{cases}$$

Therefore we have

$$|\Lambda_{(\beta)}^{(\alpha)}(t,x,\xi)|=\left|\partial_\xi^\alpha D_x^\beta\int_0^t\frac{M\langle\xi\rangle ds}{\langle x-s\xi\rangle}\right|\leqq C_{\alpha,\beta}t^{|\alpha|}.$$

This proves Lemma 2.7.

Let $\sigma(K)(t,x,D_x)$ and $\sigma(\tilde{K})(t,x,D_x)$ be pseudo-differential operators with its symbols $\sigma(K)(t,x,\xi)=\exp(\Lambda(t,x,\xi))$ and $\sigma(\tilde{K})(t,x,\xi)=\exp(-\Lambda(t,x,\xi))$ respectively. Then the symbol of the product of $\tilde{K}(t,x,D_x)$ and $K(t,x,D_x)$ is given by

$$\sigma(\tilde{K}\circ K)(t, x, \xi)=1-\sigma(R)(t, x, \xi), \tag{2.30}$$

where

$$\sigma(R)(t, x, \xi)=\sum_{|\gamma|=1}\int_0^1 Os-\iint e^{-iy\cdot\eta}\{D_\xi^\gamma \Lambda(t, x, \xi+\eta)\}e^{-\Lambda(t,x,\xi+\eta)} \tag{2.31}$$
$$\times\{\partial_x^\gamma \Lambda(t, x+\theta y, \xi)\}e^{\Lambda(t,x+\theta y,\xi)}dy\,\bar{d}\eta\,d\theta \quad (\bar{d}\eta=(2\pi)^{-n}d\eta).$$

Here an oscillatory integral of a symbol $a(x, \xi)$ means

$$Os-\iint e^{-iy\cdot\eta}a(y, \eta)dy\,\bar{d}\eta=\lim_{\varepsilon\to 0}\iint e^{-iy\cdot\eta}\chi(\varepsilon y, \varepsilon\eta)a(y, \eta)dy\,\bar{d}\eta$$

for $\chi\in\mathcal{S}$ in R^{2n} such that $\chi(0, 0)=1$.

LEMMA 2.8. *Assume that $\Lambda(t, x, \xi)$ satisfies (2.28) and (2.29). Let $R_\pm(t, x, D_x)$ be a pseudo-differential operators with its symbol $r_\pm(t, x, \xi)$ satisfying*

$$|r_{\pm}{}^{(\alpha)}_{(\beta)}(t, x, \xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{m_\pm}\langle x\rangle^{k_\pm}e^{\pm\Lambda(t,x,\xi)} \tag{2.32}$$

for $t\in[0, T]$ and $x, \xi\in R^n$, where $m_\pm$ and $k_\pm$ are real numbers. Then $q(t, x, \xi)=\sigma(R_-\circ R_+)(t, x, \xi)$ satisfies

$$|q^{(\alpha)}_{(\beta)}(t, x, \xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{m_-+m_+}\langle x\rangle^{k_-+k_+}$$

for $t\in[0, T]$ and $x, \xi\in R^n$.

PROOF. $q(t, x, \xi)$ is written by

$$q(t, x, \xi)=Os-\iint e^{-iy\cdot\eta}r_-(t, x, \xi+\eta)r_+(t, x+y, \xi)dy\,\bar{d}\eta\,.$$

Noting that (2.32), (2.28), (2.29) and $e^{-iy\cdot\eta}=\langle y\rangle^{-2m}\langle D_\eta\rangle^{2m}e^{-iy\cdot\eta}=\langle\eta\rangle^{-2m}\langle D_y\rangle^{2m}e^{-iy\cdot\eta}$ are valid, we get by use of integration by parts

$$\begin{aligned}
&|q^{(\alpha)}_{(\beta)}(t, x, \xi)|\\
&\leqq\sum_{\substack{\alpha'\leqq\alpha\\ \beta'\leqq\beta}}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\iint|\langle y\rangle^{-2m}\langle D_\eta\rangle^{2m}\\
&\quad\times(\langle\eta\rangle^{-2m}\langle D_y\rangle^{2m}(r_-)^{(\alpha')}_{(\beta')}(t, x, \xi+\eta)(r_+)^{(\alpha-\alpha')}_{(\beta-\beta')}(t, x+y, \xi)|dy\,\bar{d}\eta\\
&\leqq C_{\alpha,\beta,m}\langle\xi\rangle^{m_-+m_+}\langle x\rangle^{k_-+k_+}\iint\langle y\rangle^{-2m+k_+}\langle\eta\rangle^{-2m+m_-}\\
&\quad\times\exp\{-\Lambda(t, x, \xi+\eta)+\Lambda(t, x+y, \xi)\}dy\,\bar{d}\eta\\
&\leqq C'_{\alpha,\beta,m}\langle\xi\rangle^{m_-+m_+}\langle x\rangle^{k_-+k_+}\iint\langle y\rangle^{-2m+k_++Ml}\langle\eta\rangle^{-2m+m_-+Ml}dy\,\bar{d}\eta\,.
\end{aligned}$$

Taking $m=\max([(k_++Ml+n)/2+1], [(m_-+Ml+n)/2+1])$, we get Lemma 2.8.

Here $[s]$ denotes the largest integer not greater than s.

It follows from Lemma 2.7 and Lemma 2.8 that $\sigma(R)(t, x, \xi)$ given in (2.31) satisfies $|\sigma(R)_{(\beta)}^{(\alpha)}(t, x, \xi)| \leqq tC_{\alpha,\beta}$ for $t \in [0, T]$ and $x, \xi \in R^n$. So we get the inverse operator of K by the following Lemma 2.9.

LEMMA 2.9 (Ichinose [2], Lemma 2). *If T_0 $(0<T_0\leqq T)$ is small, for $t\in[0, T_0]$ the inverse operator $(I-R)^{-1}(t, x, D_x)$ of $(I-R)(t, x, D_x)$ exists as the continuous map from H_s to H_s space $(s\in R^1)$ and $\sigma((I-R)^{-1})(t, x, \xi)$ belongs to $S^0_{0,0}$ uniformly in $t\in[0, T_0]$. Moreover the inverse operator $K^{-1}(t, x, D_x)$ of $K(t, x, D_x)$ is given by*

$$K^{-1}(t, x, D_x)=(I-R)^{-1}(t, x, D_x)\circ\tilde{K}(t, x, D_x)\,. \tag{2.33}$$

§ 3. Proof of Theorem.

We put $u(t, x)=Kv(t, x)$, where $\sigma(K)(t, x, \xi)=\exp(\Lambda(t, x, \xi))$. Then noting that $\Lambda(t, x, \xi)$ satisfies (1.5) we have

$$\begin{aligned} Lu(t, x)&=L\circ Kv(t, x) \\ &=K\circ\Big(D_t-\frac{1}{2}\Delta\Big)v(t, x) \\ &\quad+\int e^{ix\cdot\xi}\Big(D_t\Lambda(t, x, \xi)-\sum_{j=1}^n a_j(t, x)\xi_j+\sum_{j=1}^n \xi_j D_j\Lambda\Big)e^{\Lambda}\hat{v}(t, \xi)\bar{d}\xi \\ &\quad+\int e^{ix\cdot\xi}\Big\{\frac{1}{2}\sum_{j=1}^n((D_j\Lambda)^2+(D_ja_j)-2a_jD_j\Lambda+a_j^2)-\frac{1}{2}\Delta\Lambda+c(t, x)\Big\} \\ &\quad\times e^{\Lambda}\hat{v}(t, \xi)\bar{d}\xi \\ &=K\circ\Big(D_t-\frac{1}{2}\Delta\Big)v(t, x) \\ &\quad-K\circ\int e^{ix\cdot\xi}\Big(\frac{iM\langle\xi\rangle}{\langle x\rangle}+\sum_{j=1}^n a_j\xi_j\Big)\hat{v}(t, \xi)\bar{d}\xi+\sum_{j=1}^2 K_jv(t, x)=f(t, x), \end{aligned} \tag{3.1}$$

where $\Delta=\sum_{j=1}^n \partial^2/\partial x_j^2$,

$$\begin{aligned} &\sigma(K_1)(t, x, \xi) \\ &=\Big\{\frac{1}{2}\sum_{j=1}^n((D_j\Lambda(t, x, \xi))^2+(D_ja_j(t, x))-2a_jD_j\Lambda+a_j^2\Big)-\frac{1}{2}\Delta\Lambda+c(t, x)\Big\}e^{\Lambda}, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \sigma(K_2)(t, x, \xi)=-\sum_{|\gamma|=1}\int_0^1 Os-\iint e^{-iy\cdot\eta}\{e^{\Lambda(t,x,\xi+\eta)}\}^{(\gamma)} \\ \times\Big\{\sum_{j=1}^n a_j(t, x+\theta y)\xi_j+\frac{iM\langle\xi\rangle}{\langle x+\theta y\rangle}\Big\}_{(\gamma)}dy\bar{d}\eta d\theta\,. \end{aligned} \tag{3.3}$$

LEMMA 3.1. *One can find $C_{\alpha,\beta,M}$ and $C'_{\alpha,\beta,M}$ such that*

$$\left\{\begin{array}{l} |\sigma(K_1)^{(\alpha)}_{(\beta)}(t, x, \xi)| \leqq C_{\alpha,\beta,M} e^{\Lambda(t,x,\xi)}, \\ |\sigma(K_2)^{(\alpha)}_{(\beta)}(t, x, \xi)| \leqq t C'_{\alpha,\beta,M} \langle\xi\rangle\langle x\rangle^{-1} e^{\Lambda(t,x,\xi)} \end{array}\right. \tag{3.4}$$

for $t \in [0, T]$ and $x, \xi \in R^n$.

PROOF. By (2.28) and (2.29), the first estimate of (3.4) can be shown by simple computation. Noting that (2.28), (2.29) and $e^{-iy\cdot\eta} = \langle y\rangle^{-2m}\langle D_\eta\rangle^{2m} e^{-iy\cdot\eta} = \langle\eta\rangle^{-2m}\langle D_y\rangle^{2m} e^{-iy\cdot\eta}$ are valid, we get by use of integration by parts

$$\begin{aligned} |\sigma(K_2)^{(\alpha)}_{(\beta)}(t, x, \xi)| &= \Big| \sum_{\substack{\alpha' \leqq \alpha \\ \beta' \leqq \beta}} \binom{\alpha}{\alpha'}\binom{\beta}{\beta'} \sum_{|\gamma|=1} \int_0^1 Os-\iint e^{-iy\cdot\eta} \{e^{\Lambda(t,x,\xi+\eta)}\}^{(\gamma+\alpha')}_{(\beta')} \\ &\quad \times \Big\{ \sum_{j=1}^n a_j(t, x+\theta y)\xi_j + \frac{iM\langle\xi\rangle}{\langle x+\theta y\rangle} \Big\}^{(\alpha-\alpha')}_{(\gamma+\beta-\beta')} dy\bar{d}\eta d\theta \Big| \\ &\leqq t C_{\alpha,\beta,m} \frac{\langle\xi\rangle}{\langle x\rangle} e^{\Lambda(t,x,\xi)} \iint \langle y\rangle^{-2m+1}\langle\eta\rangle^{-2m} \\ &\quad \times \exp\{\Lambda(t, x, \xi+\eta) - \Lambda(t, x, \xi)\} dy\bar{d}\eta \\ &\leqq t C_{\alpha,\beta,m} \frac{\langle\xi\rangle}{\langle x\rangle} e^{\Lambda(t,x,\xi)} \iint \langle y\rangle^{-2m+1}\langle\eta\rangle^{-2m+Ml} dy\bar{d}\eta, \end{aligned}$$

where l is a positive integer given in Lemma 2.6. Taking $m = \max([n+1)/2+1], [(n+Ml)/2+1])$, we get the second estimate of (3.4).

Therefore we can transform the Cauchy problem (1.1) to the following problem

$$\left\{\begin{array}{ll} (D_t - P(t))v(t, x) = \tilde{f}(t, x), & (t, x) \in [0, T] \times R^n, \quad (T>0), \\ v(0, x) = v_0(x) \quad (= u_0(x)), & x \in R^n \end{array}\right. \tag{3.5}$$

where $\tilde{f}(t, x) = K^{-1} f(t, x)$ and

$$P(t, x, D_x) = \frac{1}{2}\Delta + \tilde{a}(t, x, D_x) - \sum_{j=1}^{2} K^{-1} \circ K_j(t, x, D_x). \tag{3.6}$$

Here

$$\sigma(\tilde{a})(t, x, \xi) = \frac{iM\langle\xi\rangle}{\langle x\rangle} + \sum_{j=1}^n a_j(t, x)\xi_j. \tag{3.7}$$

Then it follows from Lemma 2.7, Lemma 2.9 and Lemma 3.1 that $\sigma(K^{-1}\circ K_1) \times (t, x, \xi)$ and $\sigma(K^{-1}\circ K_2)(t, x, \xi)$ satisfy

$$\left\{\begin{array}{l} |\sigma(K^{-1}\circ K_1)^{(\alpha)}_{(\beta)}(t, x, \xi)| \leqq C_{\alpha,\beta,M}, \\ |\sigma(K^{-1}\circ K_2)^{(\alpha)}_{(\beta)}(t, x, \xi)| t C'_{\alpha,\beta,M} \langle\xi\rangle\langle x\rangle^{-1} \end{array}\right. \tag{3.8}$$

for $t \in [0, T_0]$ and $x, \xi \in R^n$.

THEOREM 3.2. *Suppose* (1.3) *is valid. Then there is* $0 < T_1 \leqq T$ *such that for any* $v_0 \in H_{s+2}$ *and* $\tilde{f}(t, x) \in C_t^0([0, T_1]; H_{s+2})$ *there exists a unique solution* $v(t, x)$ *of* (3.5) *which belongs to* $C_t^1([0, T_1]; H_s) \cap C_t^0([0, T_1]; H_{s+2})$ *and moreover for any* $s \in R^1$ *there exists a constant* $C(s, T) > 0$ *such that*

$$\|v(t, \cdot)\|_{(s+j)} \leqq C(s, T)\left\{\|v_0\|_{(s+j)} + \int_0^t \|\tilde{f}(\tau, \cdot)\|_{(s+j)} d\tau\right\} \tag{3.9}$$

for $t \in [0, T_1]$, $j = 0, 1, 2$.

The proof of this theorem is the same way as that of Theorem 4.1 in [1].

Following the idea of Kumano-go [4] we introduce the series $\{\zeta_\nu(\xi)\}_{\nu=1}^\infty$ as

$$\zeta_\nu(\xi) = \left(\nu \sin \frac{\xi_1}{\nu}, \cdots, \nu \sin \frac{\xi_n}{\nu}\right) \tag{3.10}$$

and define $P_\nu(t) = p_\nu(t, x, D_x)$ as

$$p_\nu(t, x, \xi) = p(t, x, \zeta_\nu(\xi)) . \tag{3.11}$$

We consider the following Cauchy problem

$$\begin{cases} L_\nu v_\nu = D_t v_\nu - P_\nu(t) v_\nu = \tilde{f}(t) \qquad (t \in [0, T]) , \\ v_\nu|_{t=0} = u_0 . \end{cases} \tag{3.12}$$

We define the series of weight functions $\{\lambda_\nu(\xi)\}_{\nu=1}^\infty$ as

$$\lambda_\nu(\xi) = \langle \zeta_\nu(\xi) \rangle = \left\{1 + \sum_{j=1}^n \left(\nu \sin \frac{\xi_j}{\nu}\right)^2\right\}^{1/2} . \tag{3.13}$$

Then $\{\lambda_\nu(\xi)\}_{\nu=1}^\infty$ satisfies

$$\begin{cases} \text{i)} \ \ 1 \leqq \lambda_\nu(\xi) \leqq \min(\langle \xi \rangle, \sqrt{1 + n\nu^2}) , \\ \text{ii)} \ \ |\partial_\xi^\alpha \lambda_\nu(\xi)| \leqq A_\alpha \lambda_\nu(\xi)^{1-|\alpha|} , \\ \text{iii)} \ \ \lambda_\nu(\xi) \longrightarrow \langle \xi \rangle \qquad (\nu \longrightarrow \infty) \ \ \textit{on} \ R_\xi^n , \\ \qquad\qquad (\textit{uniform convergence in a compact set}). \end{cases} \tag{3.14}$$

Denote by $S^m_{\lambda_\nu, \rho, \delta}$ $(0 \leqq \delta \leqq \rho \leqq 1, \delta < 1)$ the set of symbols $q(x, \xi) \in C^\infty(R^{2n})$ satisfying

$$|q_{(\beta)}^{(\alpha)}(x, \xi)| \leqq C_{\alpha\beta} \lambda_\nu(\xi)^{m - \rho|\alpha| + \delta|\beta|} \tag{3.15}$$

for any multi-index α, β and $S^m_{\rho,\delta} = S^m_{\langle\xi\rangle, \rho, \delta}$. Then we get the following lemma (Kumanogo [4], Ch. 7, Lemma 3.3).

LEMMA 3.3. *For* $q(x, \xi)\in S^m_{\rho,\delta}$ $(0\leqq\delta\leqq\rho\leqq 1,\ \delta<1)$, *we put* $q_\nu(x, \xi)=q(x, \zeta_\nu(\xi))$. *Then* $q_\nu(x, \xi)\in S^m_{\lambda_\nu,\rho,\delta}$ $(0\leqq\delta\leqq\rho\leqq 1,\ \delta<1)$, *and for any* α, β *there is constant* $A_{\alpha,\beta}$ *which is independent of* ν *and* q *such that*

$$\text{(3.16)}\qquad \begin{cases} |q_{\nu(\beta)}^{(\alpha)}(x, \xi)|\leqq(A_{\alpha,\beta}|q|^{(m)}_{|\alpha+\beta|})\lambda_\nu(\xi)^{m-\rho|\alpha|+\delta|\beta|}, \\ q_\nu(x, \xi)\longrightarrow q(x, \xi)\ \ (uniformly)\ (\nu\to\infty)\ in\ R^n_x\times K_\xi \end{cases}$$

where K_ξ *is an arbitrary compact set of* R^n_ξ *and* $|q|^{(0)}_l=\max_{|\alpha+\beta|\leqq l}\sup_{x,\xi\in R^n}\times\{|q^{(\alpha)}_{(\beta)}(x, \xi)|\}$.

We get the following lemma (Kumano-go [4], Ch. 7, Theorem 1.6).

LEMMA 3.4. $Q=q(x, D_x)\in S^0_{0,0}$ *is a continuous mapping from* L_2 *to* L_2 *and there are* $C>0$ *and a positive integer* l *such that*

$$\text{(3.17)}\qquad \|Qu\|_{L_2}\leqq(C|q|^{(0)}_l)\|u\|_{L_2} \quad for\ u\in L_2(R^n)\,.$$

PROPOSITION 3.5. *Suppose* (1.3) *is valid. Then there is* T_1 $(0<T_1\leqq T_0)$ *such that for any* $v_0\in L_2$ *and any* $\tilde{f}(t, x)\in C^0_t([0, T_1]\,;\,L_2)$ *and there exists a unique solution* $v_\nu(t, x)\in C^1_t([0, T_1]\,;\,L_2)$ *of* (3.12) *which satisfies the energy inequalities*

$$\text{(3.18)}\qquad \|v_\nu(t)\|\leqq C_1(T_1)\left\{\|v_0\|+\int_0^t\|\tilde{f}(\tau)\|d\tau\right\}\quad (t\in[0, T_1)),$$

$$\text{(3.19)}\qquad \|\Lambda^j_\nu v_\nu(t)\|\leqq C_2(T_1)\left\{\|\Lambda^j_\nu v_0\|+\int_0^t\|\Lambda^j_\nu\tilde{f}(\tau)\|d\tau\right\}\quad (t\in[0, T_1]),$$

$$\text{(3.20)}\qquad \left\|\frac{d}{dt}\Lambda^j_\nu v_\nu(t)\right\|\leqq C_3(T_1)\left\{\|\Lambda^{j+2}_\nu v_0\|+\max_{[0,T_1]}\|\Lambda^{j+2}_\nu\tilde{f}(\tau)\|\right\}\quad (t\in[0, T_1]),$$

$$\text{(3.21)}\qquad \|\Lambda^j_\nu(v_\nu(t)-v_\nu(t'))\|\leqq C_4(T_1)|t-t'|\left\{\|\Lambda^{j+2}_\nu v_0\|+\max_{[0,T_1]}\|\Lambda^{j+2}_\nu\tilde{f}(\tau)\|\right\}$$
$$(t, t'\in[0, T_1])$$

where $C_1(T_1)$, $C_2(T_1)$, $C_3(T_1)$ *and* $C_4(T_1)$ *are constants which are independent of* ν, *and* $\Lambda_\nu=\lambda_\nu(D_x)$, $\|\cdot\|=\|\cdot\|_{L_2}$, $j=0, 1, 2$.

PROOF. I) If we fix ν arbitrarily, we have $p_\nu(t, x, \xi)\in\mathcal{B}^0_t([0, T]\,;\,\mathcal{B}^\infty(R^{2n}_{x,\xi}))$. Since $\mathcal{B}^\infty(R^{2n}_{x,\xi})=S^0_{0,0}$, from Lemma 3.4 it follows that $P_\nu(t)$ is an L_2-bounded operator uniformly with respect to t. Therefore there is a unique solution $v_\nu(t)\in C^1_t([0, T_1]\,;\,L^2)$ of the integral equation

$$\text{(3.22)}\qquad v_\nu(t)=v_0+i\int_0^t P_\nu(\tau)v_\nu(\tau)d\tau+i\int_0^t\tilde{f}(\tau)d\tau\,.$$

II) By straitforward computation we have

(3.28) $$\langle x\rangle^{-1}\circ\Lambda_\nu=\Lambda_\nu^*\circ\Lambda_\nu+B_\nu,$$

where $\sigma(\Lambda_\nu)(\xi)=\lambda_\nu(\xi)$, $\Lambda_\nu^*=\Lambda_\nu^{1/2}\circ\langle x\rangle^{-1/2}$, $\Lambda_\nu=\langle x\rangle^{-1/2}\circ\Lambda_\nu^{1/2}$ and $\sigma(B_\nu)(x,\xi)\in S^0_{0,0}$ uniformly in ν. If follows from (3.12) that

(3.24) $$\begin{aligned}\frac{d}{dt}\|v_\nu(t,\cdot)\|^2&=2Re\Big(\frac{d}{dt}v_\nu, v_\nu\Big)\\&=Re(i\Delta_\nu v_\nu, v_\nu)-2M\,Re(\langle x\rangle^{-1}\circ\Lambda_\nu v_\nu, v_\nu)\\&\quad-Re(a_\nu^I(t,x,D_x)v_\nu, v_\nu)+Re(ia_\nu^R(t,x,D_x)v_\nu, v_\nu)\\&\quad-Re(i(K^{-1}\circ K_1)_\nu v_\nu, v_\nu)-Re(i(K^{-1}\circ K_2)_\nu v_\nu, v_\nu)+2Re(i\tilde{f}, v_\nu),\end{aligned}$$

where $\sigma(\Delta_\nu)(\xi)=-\sum_{j=1}^n(\nu\sin(\xi_i/\nu))^2$, $\sigma(a_\nu^I)(t,x,\xi)=\sum_{j=1}^n a_j^I(t,x)\zeta_{\nu j}(\xi)$ and $\sigma(a_\nu^R)(t,x,\xi)=\sum_{j=1}^n a_j^R(t,x)\zeta_{\nu j}(\xi)$. We put

$$J(t,x,D_x)=\langle x\rangle^{1/2}\circ\Lambda_\nu^{-1/2}\circ(K^{-1}\circ K_2)_\nu\circ\Lambda_\nu^{-1/2}\circ\langle x\rangle^{1/2}.$$

Then we have

(3.25) $$|J_{(\beta)}^{(\alpha)}(t,x,\xi)|\leqq tC_{\alpha,\beta}$$

for $t\in[0,T_0]$ and $x,\xi\in R^n$. In fact, it follows from (3.8) and Lemma 3.3 that $W_\nu=(K^{-1}\circ K_2)_\nu$ satisfies

(3.26) $$|W_{\nu(\beta)}^{(\alpha)}(t,x,\xi)|\leqq tC_{\alpha,\beta,M}\lambda_\nu(\xi)\langle x\rangle^{-1}$$

for $t\in[0,T_1]$, $x,\xi\in R^n$ and $\nu=1,2,\cdots$. Moreover we can express

$$\begin{aligned}J(t,x,\xi)=Os-\iint&e^{-i\tilde{y}^3\cdot\tilde{\eta}^3}\langle x+\bar{y}^1\rangle^{1/2}\lambda_\nu^{-1/2}(\xi+\eta^1)W_\nu(t,x+\bar{y}^2,\xi+\eta^2)\\&\times\lambda_\nu^{-1/2}(\xi+\eta^3)\langle x+\bar{y}^3\rangle^{1/2}dy^1dy^2dy^3đ\eta^1đ\eta^2đ\eta^3,\end{aligned}$$

where $\tilde{y}^3\cdot\tilde{\eta}^3=y^1\cdot\eta^1+y^2\cdot\eta^2+y^3\cdot\eta^3$ and $\bar{y}^j=y^1+\cdots+y^j$ $(j=1,2,3)$. By virture of (3.26) we get (3.25) in the same way as the proof of Lemma 3.1. By (3.25), Lemma 3.4, and the Schwartz' inequality, we have

(3.27) $$\begin{aligned}Re((K^{-1}\circ K_2)_\nu v_\nu, v_\nu)&=Re(J(x,D)\langle x\rangle^{-1/2}\Lambda_\nu^{1/2}v_\nu, \langle x\rangle^{-1/2}\Lambda_\nu^{1/2}v_\nu)\\&\leqq\|J(x,D)\langle x\rangle^{-1/2}\Lambda_\nu^{1/2}v_\nu\|\|\langle x\rangle^{-1/2}\Lambda_\nu^{1/2}v_\nu\|\\&\leqq tC_M|J|_l^{(0)}\|\langle x\rangle^{-1/2}\Lambda_\nu^{1/2}v_\nu\|^2.\end{aligned}$$

Since $a_j^R(t,x)$ $(j=1,\cdots,n)$ are real valued, we have

$$Re(ia_\nu^R(t,x,D_x)v_\nu, v_\nu)\leqq C\|v_\nu\|^2.$$

Putting $M_1=\sup_{t\in[0,T],x\in R^n}\{\langle x\rangle|a^I(t,x)|\}$, we have

(3.28) $$Re(a_\nu^I(t,x,D_x)v_\nu, v_\nu)\leqq M_1\|\langle x\rangle^{-1/2}\Lambda_\nu^{1/2}v_\nu\|^2+C\|v_\nu\|^2.$$

Therefore by (3.24), (3.27) and (3.28) we get

$$\frac{d}{dt}\|v_\nu(t, \cdot)\|^2 \leqq (-2M+M_1+tC_M)\|\langle x\rangle^{-1/2}\Lambda_\nu^{1/2}v_\nu(t, \cdot)\|^2 \tag{3.29}$$

$$+C_M\|v_\nu(t, \cdot)\|^2+C\|\tilde{f}(t, \cdot)\|\|v_\nu(t, \cdot)\| .$$

We take M $(>M_1/2)$ and $T_1=\min((2M-M_1)/C_M, T_0)$. Then

$$-2M+M_1+tC_M \leqq 0 \tag{3.30}$$

for $t\in[0, T_1]$. By (3.29) we get

$$\frac{d}{dt}\|v_\nu(t)\| \leqq C'\{\|v_\nu(t)\|+\|\tilde{f}(t)\|\} \quad (t\in[0, T_1]\,;\ \nu=1, 2, \cdots). \tag{3.31}$$

Therefore we get (3.18). Moreover from (3.12) we have

$$\frac{d}{dt}\Lambda_\nu^j v_\nu = i(P_\nu+[\Lambda_\nu^j, P_\nu]\Lambda_\nu^{-j})\Lambda_\nu^j v_\nu + i\Lambda_\nu^j\tilde{f}\,, \tag{3.32}$$

and moreover

$$|\sigma([\Lambda_\nu^j, P_\nu]\Lambda_\nu^{-j})_{(\beta)}^{(\alpha)}(t, x, \xi)| \leqq C_{\alpha,\beta,M} \tag{3.33}$$

for $t\in[0, T_1]$ and $x, \xi\in R^n$ uniformly with respect to ν. Here, $[A, B]$ denotes the commutator of operators for A and B, that is $A\circ B-B\circ A$. In fact, we have

$$\begin{aligned}
&\sigma([\Lambda_\nu^j, P_\nu])(t, x, \xi)\\
&=Os-\iint e^{-iy\cdot\eta}\lambda_\nu^j(\xi+\eta)P_\nu(t, x+y, \xi)dy\bar{d}\eta-P_\nu(t, x, \xi)\lambda_\nu^j(\xi)\\
&=\sum_{|\gamma|=1}\int_0^1 Os-\iint e^{-iy\cdot\eta}(\lambda_\nu^j)^{(\gamma)}(\xi+\eta)(P_\nu)_{(\gamma)}(t, x+\theta y, \xi)dy\bar{d}\eta d\theta\,.\\
&=\sum_{|\gamma|=1}\iint_0^1 Os-\iint e^{-iy\cdot\eta}(\lambda_\nu^j)^{(\gamma)}(\xi+\eta)\\
&\quad\times\Big\{\sigma(\tilde{a}_\nu)_{(\gamma)}(t, x+\theta y, \xi)-\sum_{j=1}^{2}\sigma((K^{-1}\circ K_j)_\nu)_{(\gamma)}(t, x+\theta y, \xi)\Big\}dy\bar{d}\eta d\theta\,.
\end{aligned}$$

Repeating the same argument as in the proof of (3.25), by use of (3.8), (3.16) and (3.26) we can estimate

$$|\sigma([\Lambda_\nu^j, P_\nu])_{(\beta)}^{(\alpha)}(t, x, \xi)|\, C_{\alpha,\beta,M}\lambda_\nu(\xi)^j\,. \tag{3.34}$$

This implies (3.33). Hence by (3.32) and (3.33), we get (3.19) similarly to (3.18). On the other hand, noting

$$\frac{d}{dt}\Lambda_\nu^j v_\nu = i\{\Lambda_\nu^j\circ P_\nu\circ\Lambda_\nu^{-j-2}\}\circ\Lambda_\nu^{j+2}v_\nu+i\Lambda_\nu^j\tilde{f}$$

and $\sigma(\Lambda_\nu^j\circ P_\nu\circ\Lambda_\nu^{-j-2})(t, x, \xi)\in S_{0,0}^0$ for $t\in[0, T_1]$ (uniformly in ν), by Lemma 3.4

we have

$$\left\| \frac{d}{dt} \Lambda_\nu^j v_\nu(t) \right\| \leqq C \| \Lambda_\nu^{j+2} v_\nu \| + \| \Lambda_\nu^j \tilde{f} \| . \tag{3.35}$$

By (3.29) and (3.19), we get (3.20). Noting $v_\nu(t) - v_\nu(t') = \int_{t'}^{t} (d/d\tau) v_\nu(\tau) d\tau$, we get (3.21) from (3.20). This completes the proof of Proposition 3.5.

From Proposition 3.5, we can prove Theorem 3.2 in the same way as Kumano-go [4], Ch. 7, Theorem 3.2.

PROOF OF THEOREM 1.1. By Theorem 3.2 we can see that there exists a solution $v(t, x) \in C_t^1([0, T_1]; H_\infty)$ of the Cauchy problem (3.5) and by (3.9) we get the energy inequality

$$\| v(t, \cdot) \|_{(s)} \leqq C(s, T) \left\{ \| u_0 \|_{(s)} + \int_0^t \| \tilde{f}(\tau, \cdot) \|_{(s)} d\tau \right\}$$

for any $s \in R^1$ and $t \in [0, T_1]$. Hence, we obtain the unique solution $u(t, x) = Kv(t, x)$ of the equation (1.1) in $[0, T_1]$ and by using Lemma 3.4 we get the energy inequality

$$\| u(t, \cdot) \|_{(s)} \leqq C'(s, T) \left\{ \| u_0 \|_{(s+M')} + \int_0^t \| f(\tau, \cdot) \|_{(s+2M')} d\tau \right\} \tag{3.36}$$

for any $s \in R^1$ and $t \in [0, T_1]$. We can extend the existence interval $[0, T_1]$ of the solution $u(t, x)$ to $[0, T]$ as follows. Consider the Cauchy problem

$$Lw(t, x) = f(t, x) \quad \text{on} \quad [T_1, T_2] \times R_x^n, \quad w(T_1, x) = u(T_1, x).$$

Then, we get the solution $w(t, x) \in C_t^1([T_1, T_2]; H_\infty)$ where $T_2 = \min(2T_1, T)$ in the same way as in the construction of $u(t, x) \in C_t^1([0, T_1]; H_\infty)$. Define

$$\tilde{u}(t, x) = \begin{cases} u(t, x) & \text{for } 0 \leqq t \leqq T_1 , \\ w(t, x) & \text{for } T_1 \leqq t \leqq T_2 . \end{cases}$$

Then $\tilde{u}(t, x)$ belongs to $C^1([0, T_2]; H_\infty)$ and satisfies (1.1) in $[0, T_2]$. Repeating this process, the solution $u(t, x)$ satisfying (1.1) in $[0, T]$ is obtained. The energy estimate (3.36) implies the uniqueness of solution of (1.1).

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