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GLOBAL EXISTENCE FOR A CLASS OF QUASILINEAR HYPERBOLIC-PARABOLIC EQUATIONS

By

Albert MILANI

Abstract. We prove that classical solutions of the dissipative wave equation

$$\varepsilon u_{tt}+u_{t}-u_{xx}-(f(u_{x}))_{x}=0$$

are globally defined in time, regardless of the size of the initial data, if ε is sufficiently small.

§ 1. Introduction.

1.1. We consider the quasilinear dissipative hyperbolic equation

$$\varepsilon u_{tt}+u_{t}-u_{xx}-(f(u_{x}))_{x}=0 \tag{1.1}$$

where $\varepsilon>0$, $x\in\boldsymbol{R}$, u is a scalar function of (x, t) and f is a given smooth increasing function on $\boldsymbol{R}$. We study the global in time existence of classical solutions of (1.1), corresponding to "large" initial values

$$u(x, 0)=u_{0}(x), \qquad u_{t}(x, 0)=u_{1}(x), \tag{1.2}$$

and show that such solutions are globally defined, regardless of the size of the initial values, if ε sufficiently small. More precisely, given u_{0} and u_{1}, we find $\varepsilon_{0}>0$ such that, if $\varepsilon\leqq\varepsilon_{0}$, the corresponding solutions of (1.1), (1.2) are defined for all $t\geqq 0$; moreover, their derivatives decay to 0 as $t\rightarrow+\infty$. This result is somewhat complementary to our previous result of [7], where we considered a first order system formally equivalent to the equation

$$\varepsilon y_{tt}+y_{t}-(\sigma(y_{x}))_{x}=0 \tag{1.3}$$

(so that here $f(r)=\sigma(r)-r$), and showed that if (1.3) is locally strictly hyperbolic (i.e. if $\sigma'(0)>0$ and $\sigma''(0)\neq 0$), then solutions of (1.3) corresponding to data with small $|y_{0x}|$ do not develop singularities in their higher order derivatives if ε is small. In contrast, here we assume that (1.1) is globally strictly hyperbolic, i.e. $f'(r)\geqq 0$ $\forall r\in\boldsymbol{R}$, and show that no restriction on the size of the data is

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required to prove global existence of the solutions if ε is small. Although this result may not be surprising, since the limitation on the size of $|y_x|$ is needed only to guarantee hyperbolicity of (1.3), we believe it is worth of explicit consideration, because of the simpler, very direct method of proof, and its possible generalization to higher dimensions. Indeed, our proof is based on direct *a priori* estimates on the time derivatives of u, and exploits the presence of the dissipation term u_t in the equation in a way similar to that of Matsumura, [6], replacing his smallness requirements of the data with the smallness of ε (which is equivalent to require that the dissipation is sufficiently large).

In the higher dimensional case, we were able to generalize our result of [7] in [8], where we considered the equation

$$\varepsilon u_{tt}+u_t-\sum_{i,j=1}^{n} a_{ij}(\nabla u)\partial_{ij}u=0, \tag{1.4}$$

under the local strict hyperbolicity assumption $\sum a_{ij}(0)q^i q^j \geqq |q|^2$, and showed that global existence of solutions in the Sobolev spaces $H^{s+1}(\boldsymbol{R}^n)$, $s>(n/2)+1$, follows, if ε is small, under the sole assumption that $\|\nabla u_0\|_{s_0}$ be small, where $s_0=[n/2]+1$ (in which case $|\nabla u_0|_{L^\infty}\leqq\|\nabla u_0\|_{s_0}$). In [9] we tried to remove this restriction, at least for equations analogous to (1.1), i.e. of the form

$$\varepsilon u_{tt}+u_t-\Delta u-\sum_{i,j=1}^{n} \alpha_{ij}(\nabla u)\partial_{ij}u=0\,, \tag{1.5}$$

under the global hyperbolicity assumption $\sum \alpha_{ij}(p)q^i q^j \geqq 0$ for all p, $q\in\boldsymbol{R}^n$, but that proof contains an error that, so far, we have not been able to correct. Thus, by considering the simpler model (1.1), we have tried to gain a better understanding of the mutual balancing effect between the nonlinear and the dissipative terms in the equation, with the hope to be able to generalize this global existence result to equation (1.5), at least if the nonlinear operator is in the conservative form $-\operatorname{div} F(\nabla u)$, with F monotone.

1.2. One of our main motivations in this study stems from the associated singular perturbation problem, consisting in considering equation (1.1) as a perturbation of the limit parabolic equation

$$u_t-u_{xx}-(f(u_x))_x=0\,. \tag{1.6}$$

Indeed, a lot of attention has recently been devoted to the general question of the validity of modelling propagation phenomena by means of "parabolic" equations, such as the heat or the porous media equations, which give rise to such inconsistencies as for instance the "instant propagation with infinite speed" of the heat flow. Already in 1948, for instance, Cattaneo ([1]) proposed equation

(1.1) as a better model for the nonlinear heat equation, with the remark that the "thermal relaxation" parameter ε is very small, but not negligible.

A similar model is provided by Maxwell's equations for the electromagnetic potential A, which can be written as

$$\varepsilon A_{tt}+\sigma A_t+\operatorname{curl}\zeta(\operatorname{curl} A)-\nabla\operatorname{div} A=0\,; \tag{1.7}$$

in this model, ε and σ measure respectively the displacement and eddy currents. In many situations, one has that $\varepsilon\ll\sigma$, so that the reduced equations

$$\sigma A_t+\operatorname{curl}\zeta(\operatorname{curl} A)-\nabla\operatorname{div} A=0 \tag{1.8}$$

are considered instead. The reason for this is of course that equation (1.8) is much easier to study, both theoretically and numerically; for instance, when ζ is monotone (so that (1.7) is of type (1.1)), a suitable weak solution theory can be established for (1.8), with quite robust finite element methods for its numerical treatment, while the same is not available, as far as we know, for (1.7), except of course for its one-dimensional version (1.3). In this case, with the usual substitutions $y_t=u$, $y_x=v$, (1.3) is formally equivalent to the first order system

$$\begin{aligned}\varepsilon u_t&=(\sigma(v))_x-u\,,\\ v_t&=u_x\,;\end{aligned} \tag{1.9}$$

when $r\sigma''(r)>0$ $\forall r\in\boldsymbol{R}$, (1.9) represents a model, in nonlinear isothermal elasticity, for the vibrations of an elastic string influenced by a linear damping term; when $\sigma(r)=-r^{-\gamma}$, $1<\gamma<3$, (1.9) describes instead a model for the evolution of a polytropic gas (in Lagrangean coordinates). In both instances, ε is a measure of the internal inertial forces; note that, when $\varepsilon=0$ in (1.9), we formally derive the porous media equation

$$v_t-(\sigma(v))_{xx}=0\,. \tag{1.10}$$

For this model, the singular convergence of weak solutions is described in [5]; for the general n-dimensional case, we refer to [10] where, however, the global existence of smooth solutions of (1.4), at least when ε is small, is explicitly assumed. Hence, we believe, the importance of global existence results of the type we propose to present.

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§ 2. Notations and Results.

As in [9], by the change of variable $t\to t/\varepsilon$ we transform (1.1), (1.2) into the initial value problem

(2.1) $$u_{tt}+u_t-\varepsilon u_{xx}-\varepsilon(f(u_x))_x=0\,,$$

(2.2) $$u(x,\,0)=u_0(x),\qquad u_t(x,\,0)=\varepsilon u_1(x).$$

For integer $m\geqq 0$ we consider the Sobolev spaces $H^m=H^m(\boldsymbol{R})$, with norm $\|\cdot\|_m$ and scalar product $(\cdot,\,\cdot)_m$; we omit the index $m=0$ for $H^0=L^2(\boldsymbol{R})$; also, $|\cdot|$ denotes the L^∞ norm. Following Kato, [3], we look for solutions of (2.1) in the space $X_{s+1}(T)\doteq\bigcap_{k=0}^{s+1}C^k([0,\,T]\,;\,H^{s+1-k})$, where $T>0$ is arbitrary and $s\geqq 2$ an integer (to conform with Kato's theory, which requires $s>(n/2)+1$ if $x\in\boldsymbol{R}^n$; here, $n=1$).

We assume that $f:\boldsymbol{R}\to\boldsymbol{R}$ is a C^m function, $m\geqq 3$, satisfying

(2.3) $$f(0)=0,\qquad f'(r)\geqq 0\ \ \forall r\in\boldsymbol{R},$$

and that, for $j=1,\,\cdots,\,m$, there are continuous increasing functions $h_j:\boldsymbol{R}^+\to\boldsymbol{R}^+$ such that

(2.4) $$\forall r\in\boldsymbol{R},\qquad |f^{(j)}(r)|\leqq h_j(|r|)$$

(as, for example, the one-dimensional version of the p-Laplacian considered by Lions in Chapter I.8 of [4], i.e. $f(r)=|r|^{p-2}r$, $p>2$: then (2.4) holds with $m=3$ if $p\geqq 4$).

Under the said assumptions, a straightforward application of Kato's results of [3] yields the local existence result

THEOREM 1. *Let $m\geqq 3$, and s be such that $2\leqq s\leqq m-1$. Given any $u_0\in H^{s+1}$, $u_1\in H^s$ and $\varepsilon>0$, there exist $\tau>0$ and a unique $u\in X_{s+1}(\tau)$, solution of* (2.1), (2.2).

Our goal is now to show that, if ε is sufficiently small, such a local solution can be extended to any interval $[0,\,T]$ (and, in fact, to all of $\boldsymbol{R}^+$): setting, for integer $k,\,l$,

$$C_b^k(\boldsymbol{R}_0^+\,;\,H^l)\doteq\{f\in C^k([0,\,+\infty[\,;\,H^l)\,|\,\exists M>0\ \forall t\geqq 0,\ \forall i=0,\,\cdots\,k,\ \|\partial_t^i f(t)\|_l\leqq M\},$$

we claim

THEOREM 2. *Let* $m \geqq 3$, *and* s *be such that* $2 \leqq s \leqq m-1$. *Given any* $u \in H^{s+1}$ *and* $u_1 \in H^s$, *there exists* $\varepsilon_0 > 0$ *such that, if* $\varepsilon \leqq \varepsilon_0$, *problem* (2.1), (2.2) *has a unique solution* $u \in X_{s+1}(+\infty) \doteq \bigcap_{k=0}^{s+1} C_b^k(\boldsymbol{R}_0^+; H^{s+1-k})$.

We remark that, as will be evident from the proof of Theorem 2, we could consider equations of type (1.3) directly, provided we assume global strict hyperbolicity, i.e. that $\exists \nu > 0 \mid \forall r \in \boldsymbol{R}, \ \sigma'(r) \geqq \nu$.

As is to expected, such global solutions will be uniformly bounded as $t \to +\infty$, and their derivatives will decay to 0 (although, as far as we can show, with a rate of decay not uniform with respect to ε). At least in the case $s=0$ (and, we believe, for $s>2$ as well, but we have not checked the details of the proof), this is described by

THEOREM 3. *Let* $s=2$, $\varepsilon \leqq \varepsilon_0$, *and* $u \in X_3(+\infty)$ *be the solution of* (2.1), (2.2) *assured by Theorem* 2. *There exists* $M>0$ *such that*

$$\forall t \geqq 0, \qquad \sum_{i=0}^{3} \|\partial_t^i u(t)\|_{3-i}^2 \leqq M, \tag{2.5}$$

$$\lim_{t \to +\infty} \left(\|u_x(t)\|_2^2 + \sum_{i=1}^{3} \|\partial_t^i u(t)\|_{3-i}^2 \right) = 0. \tag{2.6}$$

§ 3. Proof of Theorem 2.

We start by remarking that it is sufficient to extend the local solution to a global one in X_3: in fact, higher order derivatives can be bounded in terms of the norm in X_3 by means of standard estimates (see e.g. [8] or [9]). Given then integers $m, r \geqq 0$ and a smooth function $u(x, t)$, we introduce the functions

$$E_{m,r}(u, \cdot) \doteq \|\partial_t^r u_t\|_m^2 + (\partial_t^r u, \partial_t^r u_t)_m + \frac{1}{2} \|\partial_t^r u\|_m^2 + \varepsilon \|\partial_t^r u_x\|_m^2$$

(the "energy" norms), and the seminorms

$$S_{m,r}(u, \cdot) \doteq \|\partial_t^r u_t\|_m^2 + \varepsilon \|\partial_t^r u_x\|_m^2 ;$$

indeed, by Schwartz' inequality we easily check that, for instance, $\sup_{t \geq 0} E_{0,0}(u, t)$ is the square of a norm in X_1, and in particular

$$\|u_t\|^2 \leqq 2E_{0,0}(u, \cdot), \quad \|u\|^2 \leqq 4E_{0,0}(u, \cdot), \quad \varepsilon \|u_x\|^2 \leqq E_{0,0}(u, \cdot); \tag{3.1}$$

similar inequalities hold for the other norms $E_{m,r}$.

Exploiting the different behavior of the time and space derivatives of u

with respect to the rescaling $t\to t/\varepsilon$, we propose at first to establish a direct *a priori* estimate on the norm

$$E_{0,1}(u, t)+E_{0,2}(u, t)+\frac{1}{2}\int_0^t(S_{0,1}(u, \theta)+S_{0,2}(u, \theta))d\theta$$

and then, by a sort of "elliptic" procedure, to use this estimate to provide an analogous estimate for the norm

$$E_{2,0}(u, t)+\frac{1}{2}\int_0^t S_{2,0}(u, \theta)d\theta .$$

In the course of these estimates we shall also have to consider the functions

$$\phi(u)=\int_R F(u(x, \cdot))dx, \qquad \text{where } F(r)=\int_0^r f(s)ds ,$$

$$F_{m,r}(u, \cdot)=(f'(u_x)\partial_x^m\partial_t^r u, \partial_x^m\partial_t^r u);$$

note that (2.3) implies that F, and therefore ϕ and $F_{m,r}$, are all nonnegative.

We start from the local existence Theorem 1, from the proof of which we know that

PROPOSITION 1. *Let* $\Delta_0^2=E_{2,0}(u, 0)+\varepsilon(2\phi(u_{0x})+F_{2,0}(u, 0)+F_{3,0}(u, 0))$. *For all* $\Delta>\Delta_0$, *there exists* $T>0$ *such that*

$$\forall t\in[0, T], \qquad E_{2,0}(u, t)+\frac{1}{2}\int_0^t S_{2,0}(u, \theta)d\theta\leqq\Delta^2. \tag{3.2}$$

(The proof is standard, and we actually obtain the estimate

$$\begin{aligned}E_{2,0}(u, t)&+\frac{1}{2}\int_0^t S_{2,0}(u, \theta)d\theta\\&+\varepsilon(2\phi(u_x(t))+F_{2,0}(u, t)+F_{3,0}(u, t))\\&+\frac{\varepsilon}{2}\int_0^t(2\phi(u_x(\theta))+F_{2,0}(u, \theta)+F_{3,0}(u, \theta))d\theta\leqq\Delta^2 ,\end{aligned} \tag{3.3}$$

which, however, we shall not need). We now claim:

PROPOSITION 2. *There exists* $M>\Delta_0$, *independent of* ε, *such that, for all* $\Delta>\Delta_0$, *there exists* $\varepsilon_\Delta>0$ *such that, for all* $\varepsilon\leqq\varepsilon_\Delta$, *for all* $T>0$ *such that* (3.2) *holds,*

$$\forall t\in[0, T], \quad E_{2,0}(u, t)+\frac{1}{2}\int_0^t S_{2,0}(u, \theta)d\theta\leqq M^2; \tag{3.4}$$

We shall prove this Proposition in the next section; assuming its validity, we choose $\Delta=2M$ and, by Proposition 1, we first find $T_1>0$ such that $\forall t\in$

$[0, T_1]$, $E_{2,0}(u, t)+\frac{1}{2}\int_0^t S_{2,0}(u, \theta)d\theta\leqq 4M^2$. Then, Proposition 2 ensures that, if $\varepsilon\leqq\varepsilon_{2M}$, we have in fact $\forall t\in[0, T_1]$, $E_{2,0}(u, t)+\frac{1}{2}\int_0^t S_{2,0}(u, \theta)d\theta\leqq M^2$. This means that the energy norm does not increase in $[0, T_1]$, so that we can repeat the same argument to extend the local solution to a global one in the usual way. Thus, Theorem 2 follows from Propositions 1 and 2, with the choice $\varepsilon_0=\varepsilon_{2M}$. □

§ 4. Proof of Proposition 2.

4.1. We shall obtain the *a priori* bounds on the space derivatives of u described in Proposition 2 by means of analogous bounds on the time derivatives of u, provided by

PROPOSITION 3. *For all $\Delta>\Delta_0$, there exists $\varepsilon_\Delta>0$ such that, for all $\varepsilon\leqq\varepsilon_\Delta$, for all $T>0$ such that* (3.2) *holds in* $[0, T]$, *for all* $t\in[0, T]$,

$$E_{01}(u, t)+E_{0,2}(u, t)+\varepsilon F_{1,1}(u, t)+\varepsilon F_{1,2}(u, t)$$
$$+\frac{1}{2}\int_0^t(S_{0,1}(u, \theta)+S_{0,2}(u, \theta))d\theta+\frac{\varepsilon}{2}\int_0^t(F_{1,1}(u, \theta)+F_{1,2}(u, \theta))d\theta$$
$$\leqq E_{0,1}(u, 0)+E_{0,2}(u, 0)+\varepsilon F_{1,1}(u, 0)+\varepsilon F_{1,2}(u, 0)+\varepsilon^2 . \tag{4.1}$$

PROOF. We start by remarking that the right side of (4.1) is $O(\varepsilon^2)$: in fact, from (2.1) and (2.2) we compute that

$$u_{tt}(0)=(\varepsilon u_{xx}+\varepsilon f'(u_x)u_{xx}-u_t)(0)\equiv\varepsilon u_2\in H^1 ,$$

and, from the differentiated equation

$$u_{ttt}+u_{tt}-\varepsilon u_{xxt}-\varepsilon(f'(u_x)u_{xt})_x=0 , \tag{4.2}$$
$$u_{ttt}(0)=(\varepsilon u_{xxt}+\varepsilon(f'(u_x)u_{xt})_x-u_{tt})(0)\equiv\varepsilon u_3\in L^2 .$$

Next, we multiply equation (2.1) in L^2 by $2u_t$, obtaining

$$\frac{d}{dt}(\|u_t\|^2+\varepsilon\|u_x\|^2+2\varepsilon\phi(u_x))+2\|u_t\|^2=0 ; \tag{4.3}$$

from this we deduce that, for suitable $K_1>0$ independent of ε,

$$(\|u_t\|^2+\varepsilon\|u_x\|^2+2\varepsilon\phi(u_x))(t)+2\int_0^t\|u_t\|^2\leqq K_1^2\varepsilon . \tag{4.4}$$

Multiplying then (2.1) by u as well, adding to (4.3) and integrating, we obtain

$$(4.5)\qquad E_{0,0}(u,\ t)+2\varepsilon\phi(u_x(t))+\int_0^t(S_{0,0}(u,\ s)+\varepsilon(f(u_x),\ u_x)(s))ds \leqq E_{0,0}(u,\ 0)+2\varepsilon\phi(u_{0x});$$

since $\forall r\in\boldsymbol{R}$, $r(f(r))\geqq 0$ by (2.3), (4.5) means in particular that the norm of $\{u,\ u_t\}$ in $H^1\times L^2$ is conserved.

4.2. Next, we multiply the differentiated equation (4.2) in L^2 by $2u_{tt}+u_t$, obtaining

$$(4.6)\qquad \frac{d}{dt}\left\{\|u_{tt}\|^2+\varepsilon\|u_{xt}\|^2+\varepsilon(f'(u_x)u_{xt},\ u_{xt})+(u_{tt},\ u_t)+\frac{1}{2}\|u_t\|^2\right\} +\|u_{tt}\|^2+\varepsilon\|u_{xt}\|^2+\varepsilon(f'(u_x)u_{xt},\ u_{xt})=\varepsilon(f''(u_x)u_{xt}u_{xt},\ u_{xt})\equiv I\,.$$

We estimate I by means of (2.4), using Nirenberg's interpolation inequalities and (3.2), (4.4), noting that, by (3.1), these imply that $\|u_{xx}\|\leqq 2\Delta$, $\|u_{xxt}\|\leqq\sqrt{2}\Delta$, $\|u_t\|\leqq\sqrt{\varepsilon}K_1$ and $\|u_x\|\leqq K_1$: we obtain that, for suitable constant $C>0$ independent of u,

$$(4.7)\qquad \begin{aligned}|u_x|&\leqq C\|u_{xx}\|^{1/2}\|u_x\|^{1/2}\leqq C\sqrt{2\Delta K_1}\,,\\ |u_{xt}|&\leqq C\|u_{xxt}\|^{3/4}\|u_t\|^{1/4}\leqq 2^{3/8}C\Delta^{3/4}K_1^{1/4}\varepsilon^{1/8}\,,\end{aligned}$$

and therefore, denoting here and in the sequel by R a generic positive constant depending only on Δ,

$$(4.8)\qquad I\leqq\varepsilon h_2(|u_x|)|u_{xt}|\,\|u_{xt}\|^2\leqq R\varepsilon^{9/8}\|u_{xt}\|^2\,.$$

Consequently, if ε is so small (in dependence of Δ) that

$$(4.9)\qquad 2R\varepsilon^{1/8}\leqq\frac{1}{2}\,,$$

we obtain from (4.6) that, in particular,

$$(4.10)\qquad \frac{d}{dt}\{E_{0,1}(u,\ t)+\varepsilon F_{1,1}(u,\ t)\}+\frac{1}{2}(S_{0,1}(u,\ t)+\varepsilon F_{1,1}(u,\ t))\leqq 0\,;$$

and since $\forall r$, $\forall s\in\boldsymbol{R}$, $f'(r)s^2\geqq 0$ because of (2.3), (4.10) means that the norm of $\{u_t,\ u_{tt}\}$ in $H^1\times L^2$ is also conserved. In particular, (4.10) implies that there exists $K_2>0$, depending only on the norm of the initial values, but not on ε nor on Δ if ε satisfies (4.9), such that

$$(4.11)\qquad \|u_{tt}\|\leqq K_2\varepsilon,\qquad \|u_t\|\leqq K_2\varepsilon\qquad \|u_{xt}\|\leqq K_2\sqrt{\varepsilon}\,,$$

$$(4.12)\qquad \int_0^t\|u_{tt}\|^2\leqq K_2\varepsilon^2,\qquad \int_0^t\|u_{xt}\|^2\leqq K_2\varepsilon\,;$$

note that estimate (4.11b) allows us to improve estimate (4.7b): indeed, we now have

$$|u_{xt}| \leqq 2^{3/8} C \Delta^{3/4} K_2^{1/4} \varepsilon^{1/4} . \tag{4.13}$$

In the sequel, we shall indicate by K_i a generic positive constant with the same properties as K_1 and K_2.

4.3. We would now like to differentiate (4.2) once more with respect to time; however, if $s=2$, we are prevented to do so by the fact that, in general, u_{ttt} and $u_{xtt} \notin C^1([0, T]; L^2)$. Thus, we shall regularize by means of Ikawa's mollifiers: denoting by $*$ the convolution with respect to time, for $\alpha>0$ we set $w^\alpha \doteq \phi^\alpha * u_t$, that is, as in [2],

$$w^\alpha(x, t) = \int_{-\infty}^{+\infty} \frac{1}{\alpha} \phi\left(\frac{t-\tau}{\alpha}\right) u_t(x, \tau) d\tau ,$$

where ϕ is a C^∞ function with support in $[-2, -1]$, such that $\phi \geqq 0$, $\int_{-\infty}^{+\infty} \phi(t) dt = 1$; we recall that if $z \in L^2(\Omega \times]0, T+\alpha_0[)$ for some $\alpha_0>0$, then for $0<\alpha<(1/2)\alpha_0$ $\phi^\alpha * z \in C^\infty([0, T]; L^2)$, and $\phi^\alpha *$ commutes with $\partial/\partial t$. Applying $\phi^\alpha *$ to (4.2), we see that w^α solves the equation

$$\begin{aligned} & w_{tt}^\alpha + w_t^\alpha - \varepsilon w_{xx}^\alpha - \varepsilon (f'(u_x) w_y^\alpha)_x \\ & \quad = \varepsilon(\phi^\alpha * (f'(u_x) u_{xt}) - f'(u_x) w_x^\alpha)_x \equiv \varepsilon R_x^\alpha . \end{aligned}$$

We can now differentiate this equation in time; multiplying then by $2w_{tt}^\alpha + w_t^\alpha$, we obtain

$$\begin{aligned} & \frac{d}{dt}\left\{\|w_{tt}^\alpha\|^2 + \varepsilon \|w_{xt}^\alpha\|^2 + \varepsilon (p'(u_x) w_{xt}^\alpha, w_{xt}^\alpha) + (w_{tt}^\alpha, w_t^\alpha) + \frac{1}{2}\|w_t^\alpha\|^2\right\} \\ & \quad + \|w_{tt}^\alpha\|^2 + \varepsilon \|w_{xt}^\alpha\|^2 + \varepsilon (f'(u_x) w_{xt}^\alpha, w_{xt}^\alpha) \\ & \quad = \varepsilon (f''(u_x) u_{xt} w_{xt}^\alpha, w_{xt}^\alpha) + 2\varepsilon (f''(u_x) u_{xxt} w_x^\alpha, w_{tt}^\alpha) \\ & \quad\quad + 2\varepsilon (f''(u_x) u_{xt} w_{xx}^\alpha, w_{tt}^\alpha) + 2\varepsilon (f'''(u_x) u_{xx} u_{xt} w_x^\alpha, w_{tt}^\alpha) \\ & \quad\quad - e (f''(u_x) u_{xt} w_x^\alpha, w_{xt}^\alpha) + \varepsilon (R_{xt}^\alpha, 2 w_{tt}^\alpha + w_t^\alpha) \\ & \quad \equiv A + B_1 + B_2 + D_1 + D_2 + E_1 + E_2 . \end{aligned} \tag{4.14}$$

We estimate A as in (4.8):

$$A \leqq \varepsilon h_2(|u_x|) |u_{xt}| \, \|w_{xt}^\alpha\|^2 \leqq R \varepsilon^{9/8} \|w_{xt}^\alpha\|^2 ;$$

again by interpolation inequalities, (2.4), and recalling that, as we have remarked, $\|u_{xtt}\| \leqq \sqrt{2}\Delta$, we have

$$B_1 \leqq 2\varepsilon h_2(|u_x|)\|u_{xxt}\|\,|w_x^\alpha|\,\|w_{tt}^\alpha\|$$
$$\leqq \varepsilon R|w_x^\alpha|\,\|w_{tt}^\alpha\| \leqq \varepsilon R\|w_x^\alpha\|^{1/2}\|w_{xx}^\alpha\|^{1/2}\|w_{tt}^\alpha\|$$
$$\leqq \varepsilon^2 R\|w_x^\alpha\|\,\|w_{xx}^\alpha\| + \eta\|w_{tt}^\alpha\|^2,$$

for any $\eta>0$; similarly, recalling (4.13),

$$B_2 \leqq 2\varepsilon h_2(|u_x|)|u_{xt}|\,\|w_{xx}^\alpha\|\,\|w_{tt}^\alpha\| \leqq \varepsilon R\varepsilon^{1/4}\|w_{xx}^\alpha\|\,\|w_{tt}^\alpha\|$$
$$\leqq R\varepsilon^{5/2}\|w_{xx}^\alpha\|^2 + \eta\|w_{tt}^\alpha\|^2;$$

also, recalling (4.11c), for suitable $c>0$:

$$D_1 \leqq 2\varepsilon h_3(|u_x|)\|u_{xx}\|\,|u_{xt}|\,|w_x^\alpha|\,\|w_{tt}^\alpha\|$$
$$\leqq 2c\varepsilon h_3(|u_x|)\|u_{xx}\|\,\|u_{xxt}\|^{1/2}\|u_{xt}\|^{1/2}\|w_x^\alpha\|^{1/2}\|w_{xx}^\alpha\|^{1/2}\|w_{tt}^\alpha\|$$
$$\leqq \varepsilon R\varepsilon^{1/4}\sqrt{K_2}\|w_x^\alpha\|^{1/2}\|w_{xx}^\alpha\|^{1/2}\|w_{tt}^\alpha\|$$
$$\leqq \varepsilon^{5/2}R\|w_x^\alpha\|\,\|w_{xx}^\alpha\| + \eta\|w_{tt}^\alpha\|^2;$$

$$D_2 \leqq \varepsilon h_2(|u_x|)|u_{xt}|\,\|w_x^\alpha\|\,\|w_{xt}^\alpha\|$$
$$\leqq \varepsilon c h_2(|u_x|)\|u_{xxt}\|^{1/2}\|u_{xt}\|^{1/2}\|w_x^\alpha\|\,\|w_{xt}^\alpha\|$$
$$\leqq \varepsilon^{5/4}R\|w_x^\alpha\|\,\|w_{xt}^\alpha\| \leqq \varepsilon^{3/2}R\|w_x^\alpha\|^2 + \eta\varepsilon\|w_{xt}^\alpha\|^2;$$

$$E_1 = 2\varepsilon(R_{xt}^\alpha, w_{tt}^\alpha) \leqq R\varepsilon^2\|R_{xt}^\alpha\|^2 + \eta\|w_{tt}^\alpha\|^2,$$

$$E_2 = \varepsilon(R_{xt}^\alpha, w_t^\alpha) = -\varepsilon(R_t^\alpha, w_{xt}^\alpha) \leqq R\varepsilon\|R_t^\alpha\|^2 + \eta\varepsilon\|w_{xt}^\alpha\|^2.$$

Choosing η small enough, we deduce then from (4.14) that

$$\frac{d}{dt}\{E_{0,1}(w^\alpha, t) + \varepsilon F_{1,1}(w^\alpha, t)\} + \frac{1}{2}S_{0,1}(w^\alpha, t) + \varepsilon F_{1,1}(w^\alpha, t)$$
$$\leqq R\varepsilon^{5/2}\|w_{xx}^\alpha\|^2 + R\varepsilon^{3/2}\|w_x^\alpha\|^2 + \varepsilon^2\|R_{xt}^\alpha\|^2 + R\varepsilon\|R_t^\alpha\|^2, \tag{4.15}$$

from which, integrating,

$$E_{0,1}(w^\alpha, t) + \varepsilon F_{1,1}(w^\alpha, t) + \frac{1}{2}\int_0^t \{S_{0,1}(w^\alpha, \theta) + \varepsilon F_{1,1}(w^\alpha, \theta)\}\,d\theta$$
$$\leqq E_{0,1}(w^\alpha, 0) + \varepsilon F_{1,1}(w^\alpha, 0) + R\varepsilon^{5/2}\int_0^t \|w_{xx}^\alpha\|^2$$
$$+ R\varepsilon^{3/2}\int_0^t \|w_x^\alpha\|^2 + R\varepsilon^2\int_0^t \|R_{xt}^\alpha\|^2 + R\varepsilon\int_0^t \|R_t^\alpha\|^2. \tag{4.16}$$

To estimate the right side of (4.16), we recall the following results on the mollifier, whose proof can be obtained by adapting the arguments of Ikawa, [2]:

LEMMA 1. *Let* $u \in X_3(T)$: *then, as* $\alpha \downarrow 0$:

(4.17) $$\int_0^t \|R_t^\alpha\|_1^2 \longrightarrow 0, \qquad \int_0^t \|w_x^\alpha\|_1^2 \longrightarrow \int_0^t \|u_{xt}\|_1^2,$$

(4.18) $$E_{0,1}(w^\alpha, t) \longrightarrow E_{0,1}(u_t, t), \qquad F_{1,1}(w^\alpha, t) \longrightarrow F_{1,1}(u_t, t),$$
$$S_{0,1}(w^\alpha, t) \longrightarrow S_{0,1}(u_t, t),$$

(4.19) $$\sup_{0\le t\le T} \{\|w_x^\alpha(t)\|_1^2 + \|R_t^\alpha(t)\|_1^2\} = O(1).$$

Thus, letting $\alpha \downarrow 0$ in (4.16), we obtain

$$E_{0,1}(u_t, t) + \varepsilon F_{1,1}(u_t, t) + \frac{1}{2}\int_0^t \{S_{0,1}(u_t, \theta) + \varepsilon F_{1,1}(u_t, \theta)\}\, d\theta$$
$$\leqq E_{0,1}(u_t, 0) + \varepsilon F_{1,1}(u_t, 0) + R\varepsilon^{5/2}\int_0^t \|u_{xxt}\|^2 + R\varepsilon^{3/2}\int_0^t \|u_{xt}\|^2,$$

and therefore, recalling (4.12b) and that (3.2) implies in particular that $\int_0^t \|u_{xxt}\|^2 \leqq 2\Delta^2$,

(4.19) $$E_{0,2}(u, t) + \varepsilon F_{1,2}(u, t) + \frac{1}{2}\int_0^t \{S_{0,2}(u, \theta) + \varepsilon F_{1,2}(u, \theta)\}\, d\theta$$
$$\leqq E_{0,2}(u, 0) + \varepsilon F_{1,2}(u, 0) + R\varepsilon^{5/2} 2\Delta^2 + R\varepsilon^{3/2} K_2 \varepsilon.$$

Integrating (4.10) and adding to (4.19) yields then

$$E_{0,1}(u, t) + E_{0,2}(u, t) + \varepsilon F_{1,1}(u, t) + \varepsilon F_{1,2}(u, t)$$
$$+ \frac{1}{2}\int_0^t \{S_{0,1}(u, \theta) + S_{0,2}(u, \theta)\}\, d\theta + \frac{\varepsilon}{2}\int_0^t \{F_{1,1}(u, \theta) + F_{1,2}(u, \theta)\}\, d\theta$$
$$\leqq E_{0,1}(u, 0) + E_{0,2}(u, 0) + \varepsilon F_{1,1}(u, 0) + \varepsilon F_{1,2}(u, 0) + R\varepsilon^{5/2} 2\Delta^2 + R\varepsilon^{3/2} K_2 \varepsilon,$$

so that we obtain (4.1) if ε is so small (again, in dependence of Δ) that, in addition to (4.9),

(4.20) $$2\Delta^2 R\sqrt{\varepsilon} + K_2 R\sqrt{\varepsilon} \leqq 1:$$

this ends the proof of Proposition 3. □

4.4. In particular, (4.1) implies that, for suitable $K_3 > 0$,

(4.21) $$\|u_{ttt}\| \leqq K_3\varepsilon, \qquad \|u_{tt}\| \leqq K_3\varepsilon, \qquad \|u_{xtt}\| \leqq K_3\sqrt{\varepsilon},$$

(4.22) $$\int_0^t \|u_{ttt}\|^2 \leqq K_3\varepsilon^2, \qquad \int_0^t \|u_{xtt}\|^2 \leqq K_3\varepsilon:$$

with these estimates, we are now ready to prove Proposition 2, for which we still need to estimate

$$\|u_{xx}\|, \qquad \|u_{xxt}\|, \qquad \int_0^t \|u_{xxt}\|^2;$$

$$\sqrt{\varepsilon}\|u_{xxx}\|, \qquad \varepsilon\int_0^t\|u_{xx}\|^2, \qquad \varepsilon\int_0^t\|u_{xxx}\|^2 .$$

From equation (2.1) we have

$$\varepsilon u_{xx}=\frac{u_{tt}+u_t}{1+f'(u_x)},$$

and since $f'(u_x)\geqq 0$, recalling (4.11) we deduce that

$$\varepsilon\|u_{xx}\|\leqq\|u_{tt}\|+\|u_t\|\leqq 2K_3\varepsilon . \tag{4.23}$$

From equation (4.2) we also have

$$\varepsilon u_{xxt}=\frac{u_{ttt}+u_{tt}-\varepsilon f'(u_x)u_{xx}u_{xt}}{1+f'(u_x)}, \tag{4.24}$$

from which

$$\varepsilon\|u_{xxt}\|\leqq\|u_{ttt}\|+\|u_{tt}\|+\varepsilon\|f'(u_x)u_{xx}u_{xt}\| . \tag{4.25}$$

Noting that (4.23) allows us to modify (4.7a) into $|u_x|\leqq C\sqrt{2K_2K_1}$, recalling (4.21) we proceed from (4.25) with

$$\begin{aligned}\varepsilon\|u_{xxt}\| &\leqq\|u_{ttt}\|+\|u_{tt}\|+\varepsilon h_1(|u_x|)|u_{xt}|\,\|u_{xx}\|\\ &\leqq 2\varepsilon K_3+\varepsilon Ch_1(C\sqrt{2K_1K_2})\|u_{xt}\|^{1/2}\|u_{xxt}\|^{1/2}\|u_{xx}\|\\ &\leqq 2\varepsilon K_3+\varepsilon Ch_1(C\sqrt{2K_1K_2})\varepsilon^{1/4}K_2^{1/2}\|u_{xxt}\|^{1/2}2K_2\\ &=2\varepsilon K_3+\varepsilon^{3/4}K_4(\varepsilon\|u_{xxt}\|)^{1/2}\\ &\leqq 2\varepsilon K_3+\frac{1}{2}\varepsilon^{3/2}K_4^2+\frac{1}{2}\varepsilon\|u_{xxt}\| ;\end{aligned}$$

therefore, we obtain that

$$\varepsilon\|u_{xxt}\|\leqq 4\varepsilon K_3+\varepsilon^{3/2}K_4^2\leqq\varepsilon K_5 . \tag{4.26}$$

From (4.24) we also have, using (4.23), (4.22a), (4.12a), that

$$\begin{aligned}\varepsilon^2\int_0^t\|u_{xxt}\|^2 &\leqq 2\int_0^t\|u_{ttt}+u_{tt}\|^2+2\int_0^t\|\varepsilon f'(u_x)u_{xx}u_{xt}\|^2\\ &\leqq 4\int_0^t\|u_{ttt}\|^2+4\int_0^t\|u_{tt}\|^2+2N\varepsilon^2\int_0^t\|u_{xxt}\|\,\|u_{xt}\|\\ &\leqq 4K_3\varepsilon^2+4K_2\varepsilon^2+2N\varepsilon^2\Big(\int_0^t\|u_{xt}\|^2\Big)^{1/2}\Big(\int_0^t\|u_{xxt}\|^2\Big)^{1/2}\\ &\leqq 4K_3\varepsilon^2+4K_2\varepsilon^2+CN^2\varepsilon^2\int_0^t\|u_{xt}\|^2+\frac{1}{2}\varepsilon^2\int_0^t\|u_{xxt}\|^2,\end{aligned}$$

where $N=4CK_2^2h_1(C\sqrt{2K_1K_2})^2$; thus, using (4.12b), we have

(4.27)
$$\frac{1}{2}\varepsilon^2\int_0^t\|u_{xxt}\|^2\leqq 4K_3\varepsilon^2+4K_2\varepsilon^2+CN^2K_2\varepsilon^3\leqq K_6\varepsilon^2\,.$$

From (4.23), (4.12a) and (4.4) we also have

$$\varepsilon^2\int_0^t\|u_{xx}\|^2\leqq 2\int_0^t\|u_{tt}\|^2+2\int_0^t\|u_t\|^2\leqq 2K_2\varepsilon^2+K_1^2\varepsilon\,,$$

so that

(4.28)
$$\varepsilon\int_0^t\|u_{xx}\|^2\leqq K_7\,.$$

Differentiating equation (2.1) with respect to x we obtain that

(4.29)
$$\varepsilon(1+f'(u_x))u_{xxx}=u_{ttx}+u_{tx}-\varepsilon f''(u_x)u_{xx}u_{xx}\,;$$

from this we obtain, as before, that

$$\begin{aligned}\varepsilon\|u_{xxx}\|&\leqq\|u_{xtt}\|+\|u_{xt}\|+\varepsilon\|f''(u_x)u_{xx}u_{xx}\|\\&\leqq\|u_{xtt}\|+\|u_{xt}\|+\varepsilon h_2(|u_x|)|u_{xx}|_{L^4}^2\\&\leqq\|u_{xtt}\|+\|u_{xt}\|+\varepsilon Ch_2(|u_x|)\|u_{xx}\|^{3/2}\|u_{xxx}\|^{1/2}\\&\leqq\sqrt{\varepsilon}K_3+\sqrt{\varepsilon}K_2+\varepsilon Ch_2(C\sqrt{2K_1K_2})(2K_2)^{3/2}\|u_{xxx}\|^{1/2}\\&\leqq\sqrt{\varepsilon}K_3+\sqrt{\varepsilon}K_2+\varepsilon C_K+\frac{1}{2}\varepsilon\|u_{xxx}\|\,,\end{aligned}$$

for suitable $C_K>0$; hence,

(4.30)
$$\varepsilon\|u_{xxx}\|\leqq\sqrt{\varepsilon}K_8\,.$$

Finally, from (4.29) we estimate

$$\begin{aligned}\varepsilon^2\int_0^t\|u_{xxx}\|^2&\leqq 4\int_0^t\|u_{xtt}\|^2+4\int_0^t\|u_{xt}\|^2+2\int_0^t\|\varepsilon f''(u_x)u_{xx}u_{xx}\|^2\\&\leqq 4K_3\varepsilon+4K_2\varepsilon+2\varepsilon^2Ch_2(C\sqrt{2K_1K_2})\int_0^t\|u_{xxx}\|\|u_{xx}\|^3\\&\leqq 4K_3\varepsilon+4K_2\varepsilon+C_K\varepsilon^2\int_0^t\|u_{xx}\|^2+\frac{1}{2}\varepsilon^2\int_0^t\|u_{xxx}\|^2,\end{aligned}$$

and therefore, recalling (4.28),

(4.31)
$$\varepsilon^2\int_0^t\|u_{xxx}\|^2\leqq 8K_8\varepsilon+K_2\varepsilon+\varepsilon C_KK_7=\varepsilon K_9\,.$$

Putting together estimates (4.5), (4.11c), (4.4), (4.23), (4.26), (4.30), (4.12b), (4.28), (4.27) and (4.31), and recalling also (3.1), we deduce that

$$E_{2,0}(u,\,t)+\frac{1}{2}\int_0^tS_{2,0}(u,\,\theta)d\theta\leqq E_{0,0}(u,\,0)+2\phi(u_{0x})+K_{10}\equiv M^2\,.$$

To conclude the proof of Proposition 2, we only need to remark that M depends only on the norm of the initial values; finally, ε_{Δ} is defined by (4.9) and (4.20). □

REMARK. Using the same procedure, it would be possible to prove more than (3.4), namely (compare to (3.3))

$$E_{2,0}(u, t)+\frac{1}{2}\int_0^t S_{2,0}(u, \theta)d\theta$$
$$+\varepsilon(2\phi(u_x(t))+F_{2,0}(u, t)+F_{3,0}(u, t))$$
$$+\frac{\varepsilon}{2}\int_0^t(2\phi(u_x(\theta))+F_{2,0}(u, \theta)+F_{3,0}(u, \theta))d\theta\leqq M^2 .$$

§5. Proof of Theorem 3.

For $\varepsilon\leqq\varepsilon_0$, we consider the global solution u of (2.1) and (2.2) assured by Theorem 2. At first we remark that, since $u\in X_3(+\infty)$, estimate (4.19) holds uniformly with respect to T; consequently, from (4.15) we deduce that, with the same meaning of w^α,

$$\frac{d}{dt}\{E_{0,1}(w^\alpha, t)+\varepsilon F_{1,1}(w^\alpha, t)\}\leqq C\varepsilon ,$$

with $C>0$ independent of t and α; adding this to (4.10) we have then that, in particular,

$$\frac{d}{dt}\{E_{0,1}(u, t)+E_{0,1}(w^\alpha, t)+\varepsilon F_{1,1}(u, t)+\varepsilon F_{1,1}(w^\alpha, t)\}\leqq C\varepsilon . \tag{5.1}$$

Also, from (4.1) we have that, for all $t\geqq 0$:

$$E_{0,1}(u, t)+E_{0,2}(u, t)+\varepsilon F_{1,1}(u, t)+\varepsilon F_{1,2}(u, t) \tag{5.2}$$
$$\leqq E_{0,1}(u, 0)+E_{0,2}(u, 0)+\varepsilon F_{1,1}(u, 0)+\varepsilon F_{1,2}(u, 0)+\varepsilon^2\equiv C_2\varepsilon^2 .$$

Next, we easily see that

$$\int_0^t(E_{0,1}(u, \theta)+E_{0,2}(u, \theta))d\theta\leqq\frac{5}{2}\int_0^t(S_{0,1}(u, \theta)+S_{0,2}(u, \theta))d\theta+\int_0^t\|u_t\|^2 .$$

so that from (4.1) and (4.4) we have that for all $t\geqq 0$

$$\int_0^t\{E_{0,1}(u,\theta)+E_{0,2}(u,\theta)+\varepsilon F_{1,1}(u,\theta)+\varepsilon F_{1,2}(u,\theta)\}d\theta\leqq 5\varepsilon^2C_2+\frac{1}{2}\varepsilon K_1^2\equiv C_3\varepsilon. \tag{5.3}$$

Recalling (4.19), inequalities (5.1), (5.2) and (5.3) show that

$$\lim_{t\to+\infty}\{E_{0,1}(u, t)+E_{0,1}(w^\alpha, t)+\varepsilon F_{1,1}(u, t)+\varepsilon F_{1,1}(w^\alpha, t)\}=0 ;$$

therefore, since $F_{1,1}(u, t)+F_{1,1}(w^\alpha, t)\geqq 0$ and the convergence in (4.18) is uniform in t because $u\in X_3(+\infty)$, letting $\alpha\downarrow 0$ we obtain that

$$\lim_{t\to+\infty}\{E_{0,1}(u,\ t)+E_{0,2}(u,\ t)\}=0\ ,$$

which in turn implies that

$$\lim_{t\to+\infty}\{\|u_{ttt}(t)\|^2+\|u_{tt}(t)\|_1^2+\|u_t(t)\|_1^2\}=0\ . \tag{0.4}$$

Because of (4.23), (4.25) and (4.29), (5.4) also implies that

$$\lim_{t\to+\infty}\{\|u_{xx}(t)\|_1^2+\|u_{xxt}(t)\|^2\}=0\ ; \tag{5.5}$$

finally, decay of $\|u_x(\cdot)\|$ is a consequence of (4.3), (4.4), (4.5) and (5.4). □

REMARK. As we have stated in the Introduction, it should not be difficult to extend this procedure to equations in the conservative form

$$\varepsilon u_{tt}+u_t-\Delta u-\mathrm{div}\,F(\nabla u)=0\ , \tag{5.6}$$

with $F: \boldsymbol{R}^n\to\boldsymbol{R}^n$ monotone; on the other hand, however, extension to equations in the divergence form

$$\varepsilon u_{tt}+u_t-\Delta u-\sum_{i,j=1}^{n}\partial_j(\alpha(\nabla u)\partial_i u)=0$$

seems to be out of our reach. Still, our method would clearly be applicable to the initial boundary value problems corresponding to (1.1) or (5.6), with homogeneous Dirichlet boundary conditions.

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University of Wisconsin-Milwaukee