TSUKUBA J. MATH.
Vol. 10 No. 1 (1986). 73–77

NOWHERE DENSELY GENERATED PROPERTIES IN TOPOLOGICAL MEASURE THEORY

By

Masami SAKAI

1. Introduction.

A topological property $\mathscr{P}$ is said to be nowhere densely generated in a class $\mathcal{C}$ of topological spaces if each $X\in\mathcal{C}$ has $\mathscr{P}$ whenever every nowhere dense closed subset of X has $\mathscr{P}$ [5]. For example, Katětov showed in [4] that the subspace of nonisolated points of a T_1-space X is compact if each nowhere dense closed subset is compact. This means that compactness is a nowhere densely generated property in the class of T_1-spaces without isolated points. More generally, it was showed in [1] [5] that $[\kappa, \lambda]$-compactness is also a nowhere densely generated property in the same class. Other nowhere densely generated properties were investigated in [1], α-closed-completeness, α-compactness and pseudo-(κ, λ)-compactness. The purpose of this paper is to consider nowhere densely generated properties in topological measure theory. In this paper we examine measure-compactness, (weak) Borel measure-completeness, Borel measure-compactness, pre-Radon-ness and Radon-ness.

Terminologies and notations are due to [3]. we denote by $\mathscr{B}(X)$ ($\mathscr{B}^*(X)$) the Borel (Baire) σ-algebra in a space X. A Borel (Baire) measure μ is a σ-additive non-negative real-valued set function on $\mathscr{B}(X)$ ($\mathscr{B}^*(X)$). We assume that all measures are finite (i.e. $\mu(X)<\infty$). A measure μ which is $\mu(X)=1$ is called a probability. A Borel measure μ is called regular (Radon) if for each $B\in\mathscr{B}(X)$ $\mu(B)$ is the supremum of measures of closed (compact) subsets contained to B. A non-empty family $\mathscr{A}$ of sets is called directed upwards if for each $A, B\in\mathscr{A}$ there exists $C\in\mathscr{A}$ such that $A\cup B\subset C$. A Borel measure μ is called weakly τ-additive if for each directed upwards open cover $\mathscr{A}$ of X, $\mu(X)=\sup\{\mu(U): U\in\mathscr{A}\}$. A Borel measure μ is called τ-additive if for given open subset V and a directed upwards open cover $\mathscr{A}$ of V, $\mu(V)=\sup\{\mu(U): U\in\mathscr{A}\}$. Regularity and τ-additivity of Baire measures are also defined by the same way.

We assume all spaces are T_2.

Received July 12, 1985.

2. Nowhere densely generated properties.

DEFINITION 2. 1. [3] A space X is called

(1) measure-compact if each Baire measure in X is τ-additive.

(2) (weakly) Borel measure-complete if each Borel measure in X is (weakly) τ-additive.

(3) Borel measure-compact if each regular Borel measure in X is τ-additive.

These concepts are motivated by the characterization of real-compactness associated with 2-valued Baire measures. A space is realcompact if and only if each 2-valued Baire muasure in the space is τ-additive.

DEFINITION 2. 2. [3] A space X is called

(1) pre-Radon if each τ-additive Borel measure in X is Radon.

(2) Radon if each Borel measure in X is Radon.

Compact spaces are pre-Radon [3, 11. 3].

A cardinal κ is called real-valued measurable if there is a discrete space X with $|X|=\kappa$ and a diffused Borel probablity μ in X, where a diffused measure is a measure such that $\mu(\{x\})=0$ for any $x\in X$. By (*) (cl(*)) we denote the condition that the cardinality of each (closed) discrete subspace is not real-valued measurable.

THEOREM 2. 3. Let X be a Tychonoff space satisfying cl(*), and assume that $\mathscr{B}^*(Y)=\{B\cap Y: B\in\mathscr{B}^*(X)\}$ for each closed subset Y in X. Then X is measure-compact if each nowhere dense closed subset of X is measure-compact.

PROOF. It is known that a space X is measure-compact if and only if each Baire probability in X has a nonempty support [3, 14. 4], where the support of a Baire (Borel) measure μ in X is the set of all $x\in X$ such that $\mu(U)>0$ for each cozero (open) neighborhood U of x. We assume that there exists a Baire probability μ in X having the empty support. For each $x\in X$ we take a cozero neighborhood U_x of x such that $\mu(U_x)=0$. Let $\mathscr{U}=\{U_\alpha: \alpha\in A\}$ be a maximal disjoint collection of nonempty open subsets refining $\{U_x: x\in X\}$. Then $F=X-\cup\mathscr{U}$ is nowhere densely closed, so F is measure-compact. Since we can extend Baire set of F to a Baire set of X, we can consider the restricted Baire measure μ_F in F [3, 3. 2], where μ_F is constructed in the following manner: $\mu_F(E)=\inf\{\mu(B): E\subset B\in\mathscr{B}^*(X)\}$ for each $E\in\mathscr{B}^*(F)$. Obviously the support of μ_F is empty, hence $\mu_F=0$. Therefore there exists $B\in\mathscr{B}^*(X)$ such that $F\subset B$ and $\mu(B)<1$. Since $\mu(X-B)>0$ and a Baire measure is always regular [3, 14. 2], there exists a zero set

Z in X such that $Z\subset X-B$ and $\mu(Z)>0$. Set $A'=\{\alpha\in A: Z\cap U_\alpha\neq\phi\}$ and take $x\in Z\cap U_\alpha$ for $\alpha\in A'$. Then $D=\{x_\alpha:\alpha\in A'\}$ is closed discrete in X. We define a Borel measure ν in D by the following equation, $\nu(E)=1/\mu(Z).\ \mu_Z(Z\cap(\bigcup_{x_\alpha\in E} U_\alpha))$ for $E\subset D$. It is easy to show that ν is a diffused Borel probability. Hence $|D|$ is real-valued measurable. This is a cotradiction.

The following two theorems are similarly proved.

THEOREM 2.4. Let X be a space satisfying $(*)$. Then X is weakly Borel measure-complete if each nowhere dense closed subset of X is weakly Borel measure-complete.

COROLLARY 2.5. A space X is Borel measure-complete if and only if each nowhere dense closed subset of X is Borel measure-complete and $(*)$ is satisfied.

PROOF. Note that Borel measure-completeness is equivalent to be hereditarily weakly Borel measure-complete [3, 7.4].

THEOREM 2.6. Let X be a space satisfying cl $(*)$. Then X is Borel measure-compact if each nowhere dense closed subset of X is Borel measure-compact.

Theorem 2.3 and 2.6 generalize Corollary 2.5 in [1]. In fact, it is known that a space X is closed-complete if and only if each 2-valued regular Borel measure in X is τ-additive.

LEMMA 2.7. If X is a countable union of pre-Radon subspaces, then X is pre-Radon.

PROOF. Let μ be a τ-additive Borel measure in X and $B\in\mathcal{B}(X)$. Put $X=\bigcup_{i=1}^{\infty} X_i$, where X_i is pre-Radon. For any $\varepsilon>0$, since the restricted Borel measure μ_{X_i} in X_i is τ-additive, we can take a compact suset K_i such that $K_i\subset B\cap X_i$ and $\mu_{X_i}(B\cap X_i-K_i)<\varepsilon/2^{i+1}$. Then $\mu(B-\bigcup_{i=1}^{\infty} K_i)\leq\Sigma\mu_{X_i}(B\cap X_i-K_i)\leq\varepsilon/2<\varepsilon$. Since μ is σ-additive, $\mu(B-\bigcup_{i=1}^{n} K_i)<\varepsilon$ for some n. This shows that X is pre-Radon.

A space X is called locally pre-Radon if each point of X has a pre-Radon neighborhood.

LEMMA 2.8. Every locally pre-Radon space is pre-Radon.

PROOF. Let μ be a τ-additive Borel measure in a locally pre-Radon space X. It is enough to show that $\mu(U)=\sup\{\mu(K): K$ is a compact subset contained to

$U.\}$ for each open set U in X [3, 6. 4]. Let U be an open set in X and for each $x\in U$, take a pre-Radon neighborhood W_x of x. We may assume that W_x is open in X and $W_x\subset U$ [3, 11. 6]. Since μ is τ-additive, for any $\varepsilon>0$, there exists $x_1, \cdots, x_n\in U$ such that $\mu(U)-\mu(W_{x1}\cup\cdots\cup W_{xn})<\varepsilon/2$. $W=W_{x1}\cup\cdots\cup W_{xn}$ is pre-Radon by Lemma 2. 7 and the restricted Borel measure μ_W is τ-additive, hence μ_W is Radon. So there exists a compact set $K\subset W$ such that $\mu_W(W)-\mu_W(K)<\varepsilon/2$. These facts show that $\mu(U)-\mu(K)<\varepsilon$. Thus μ is Radon.

THEOREM 2. 9. *A space X is pre-Radon if and only if the following (1) and (2) are satisfied.*

(1) Each nowhere dense closed subset of X is pre-Radon.

(2) Each nonempty open subset of X contains a nonempty open pre-Radon set.

PROOF. A pre-Radon space obviously satisfies (1) and (2). Because each Borel subset of a pre-Radon space is pre-Radon [3, 11. 6]. We assume (1) and (2). Let $\mathcal{U}$ be a maximal disjoint collectiion of nonempty open pre-Radon subsets. Since $X-\cup\mathcal{U}$ is nowhere dense closed in X, it is pre-Radon. By Lemma 2. 8, $\cup\mathcal{U}$ is pre-Radon, hence X is pre-Radon by Lemma 2. 7.

There exists a non-pre-Radon space which satisfies (1) in Theorem 2. 9, refer to [3, 5. 11].

THEOREM 2. 10. *A space X is Radon if and only if the following (1), (2) and (3) are satisfied.*

(1) (*) is satisfied.

(2) Each nowhere dense closed subset of X is Radon.

(3) Each nonempty open subset of X contains nonempty open pre-Radon set.

PROOF. Note that Radon-ness is equivalent to be Borel measure-complete and pre-Radon. So each Radon space satisfies (1), (2) and (3). The converse follows from Corollary 2. 5 and Theorem 2. 9.

We give an application of the above theorem. Fremlin proved under $MA+2^\omega<\omega_\omega$ that a first-countable compact space of weight $<2^\omega$ is Radon [2]. We generalize this result.

THEOREM 2. 11. $[MA+2^\omega<\omega_\omega]$ *A space X satisfying (*) is Radon if each nowhere dense closed subset is a first-countable compact set of weight $<2^\omega$.*

PROOF. As mentioned in the first section, the subspace of nonisolated points of X is compact. Hence X is pre-Radon by Lemma 2. 7. By Fremlin's result X satisfies (2) of Theorem 2. 10. Thus X is Radon by Theorem 2. 10.

References

[1] Blair, R. L., Some nowhere densely generated topological properties, **24**, **3** (1983), 465–479.

[2] Fremlin, D. H., Consequences of Martin's axioms, Cambridge University Press, 1984.

[3] Gardner, P. J. and Pffefer, W. F., Borel measures, Handbook of Set-theoretic Topology, North-Holland (1984), 961–1043.

[4] Katětov, M., On the equivalence of certain types of extensions of topological spaces, Časopis Pěst. mat. fys. **72** (1947), 101–106.

[5] Mills, C. F. and Wattel, E., Nowhere densely generated topological properties, Topological structures II, Math. Centre Tracts **116** (1979), 191–198.

Institute of Mathematics
University of Tsukuba
Ibaraki, 305 Japan