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COMPLETELY CELL SOLUBLE SPACES

(Dedicated to Professor Yukihiro Kodama on his 60th birthday)

By

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1. Introduction.

A topological space X is called homogeneous if for arbitrary points $x, y\in X$ there exists a homeomorphism f from X onto itself such that $f(x)=y$. Is every compact T_2-space the continuous image of a homogeneous compact T_2-space (Arhangel'skii [2])? Particularly, is a compact T_2-space nonhomogeneous if it can be mapped continuously onto βN (van Douwen [3])? These interesting problems remain unsolved. Related to these problems, Motorov showed that there exists a metrizable compact T_2-space which is not a retract of any homogeneous compact T_2-space. In the specific idea of Motorov, Arhangel'skii ([1], [2]) found an interesting topological property called cell solubility which every retract of an arbitrary homogeneous compact T_2-space posesses. He raised some problems related to this topological property. We solved already one of his problems [6]. In this paper we will answer to some other problems of Arhangel'skii.

2. Definitions.

The following definitions were introduced by Arhangel'skii [1], [2].

2.1. DEFINITION. Let X be a topological space. A map F of X into the set of all closed subsets of X is called a *cellularity* on X if the following conditions are satisfied:

1) $x\in F(x)$,

2) if $y\in F(x)$ then $F(y)\subset F(x)$,

3) if f is a homeomorphism from X onto itself such that $f(x)=y$ then $f(F(x))=F(y)$.

The sets $F(x)$ are called the *terms* of the cellularity F. A cellularity F on a space X is called *disjoint* if for any $x, y\in X$ either $F(x)=F(y)$ or $F(x)\cap F(y)$

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$=\emptyset$. Arhangel'skii showed that a topological space X is homogeneous if and only if every cellularity on X having a compact term is disjoint.

Let $q=(Y, Z, \mathcal{E})$ be a fixed triple, where Y is a topological space, Z a subspace of Y, and $\mathcal{E}$ a family of subsets of Y. Let X be an arbitrary topological space. A closed subset P of X is said to be *q-saturated* if for any continuous map $f: Y \to X$ such that $f(E) \cap P \neq \emptyset$ for all $E \in \mathcal{E}$ we have $f(Z) \subset P$. For an arbitrary point $x \in X$ we denote $F_q(x)$ the intersection of all q-saturated subsets containing x. Then F_q is a cellularity on X. This cellularity is called the cellularity induced by the triple q.

2.2. DEFINITION. A topological space X is called *cell soluble* if for any triple q as above its induced cellularity is disjoint, provided that at least one of its terms is compact.

Arhangel'skii proved that every retract of an arbitrary homogeneous compact T_2-space is cell soluble. We showed that every zero-dimensional space is cell soluble [6].

Let us denote by $\boldsymbol{T}$ the category of all Tychonoff spaces and all continuous maps between such spaces.

2.3. DEFINITION. An *abstract cellularity* is a rule that associates with each space $X \in \boldsymbol{T}$ some callularity F^X on X in such a way that the following condition is satisfied: for any map $f: X \to Y$ in the category $\boldsymbol{T}$,

$$f(F^X(x)) \subset F^Y(f(x))$$

is held for all $x \in X$.

The above abstract cellularity is expressed by

$$\boldsymbol{F}=\{F^X : X \in \boldsymbol{T}\}.$$

Let $q=(Y, Z, \mathcal{E})$ be a triple as before. For each space $X \in \boldsymbol{T}$, let $F_q{}^X$ be the cellularity induced on X by q. Then $\boldsymbol{F}_q=\{F_q{}^X : X \in \boldsymbol{T}\}$ is an abstract cellularity. This abstract cellularity is called a *representable* cellularity.

2.4. DEFINITION. A compact T_2-space X is called *completely cell soluble* if for every abstract cellularity $\boldsymbol{F}$ the cellularity F^X on X is disjoint.

3. A non-representable abstract cellularity.

Arhangel'skii posed the following problem in [2].

3.1. PROBLEM. Is there an example of a non-representable abstract cellularity?

We will give an affirmative answer to this problem. For any space $X \in \boldsymbol{T}$ let F_c^X be the map of X into the set of closed subsets of X defined by

$$F_c^X(x) \text{ is the connected component of } x$$

for all $x \in X$. Then $\boldsymbol{F}_c = \{F_c^X : X \in \boldsymbol{T}\}$ is obviously an abstract cellularity.

3.2. THEOREM. *F_c is a non-representable abstract cellularity.*

Before we give the proof of this theorem, let us recall the long line. Let τ be an arbitrary uncountable ordinal. The extended τ-long line L_τ is constructed from the ordinal space τ by placing between each ordinal α and its successor $\alpha+1$ a copy of the unit interval $I=(0, 1)$. L_τ is then linearly ordered and we give it the order topology. L_τ has the following properties:

(1) L_τ is connected.

(2) If τ is a successor ordinal than L_τ is compact.

PROOF OF THEOREM. This proof is much the same as that of our theorem in [6]. For any triple $q=(Y, Z, \mathcal{E})$ as before, we will show that $\boldsymbol{F}_q \neq \boldsymbol{F}_c$. We can assume that $Z \neq \emptyset$ since if $Z=\emptyset$ then $F_q^X(x)=\{x\}$ for any $X \in \boldsymbol{T}$ and any $x \in X$. Further we can assume that $\mathcal{E} \neq \emptyset$ ($Z \neq \emptyset$ has been assumed) since if $\mathcal{E}=\emptyset$ then $F_q^X(x)=X$ for any $X \in \boldsymbol{T}$ and any $x \in X$.

Suppose that there exists a clopen subset G of Y such that $G \cap Z \neq \emptyset$ and $E-G \neq \emptyset$ for every $E \in \mathcal{E}$. Then we can show that $F_q^X(x)=X$ for any $X \in \boldsymbol{T}$ and any $x \in X$, and hence $F_q^X \neq F_c^X$ for non-connected spaces X. In fact, if a non-empty subset P of X satisfies $P \neq X$, then there exists a continuous map $f: Y \to X$ such that $f(G) \subset X-P$ and $f(Y-G) \subset P$. This shows that P is not q-saturated.

The case remained is the following: For any clopen subset G of Y, if $G \cap Z \neq \emptyset$ then there exists some member E of $\mathcal{E}$ such that $E \subset G$. In this case, it will be proved that there exists a space $L \in \boldsymbol{T}$ such that $F_q^L \neq F_c^L$. Let κ be the cardinality of Y. Let L be the $(\kappa^+ +1)$-extended long line. The linearly order relation of L is expressed by $\leqq$. Since L is connected, $F_c^X(x)=L$ for any $x \in L$. On the other hand, for the last point κ^+ of L it will be shown that $F_q^L(\kappa^+)=\{\kappa^+\}$. In fact, let $f: Y \to L$ be a continuous map such that $f(E) \cap \{\kappa^+\} \neq \emptyset$ for any $E \in \mathcal{E}$. It suffices to show that $f(Z)=\{\kappa^+\}$. Assume that $f(Z) \neq \{\kappa^+\}$. Let y be a point of $f(Z)$ which is distinct from the point κ^+. Then, since the cardinality of $\{x \in L \mid y \leqq x \leqq \kappa^+\}$ is greater than the cardinally of Y, there exists a point $z \in L - f(Y)$ between y and κ^+. Let $U = \{x \in L \mid x \leqq z\}$. Then $f^{-1}(U)$ is a clopen subset of Y satisfying $f^{-1}(U) \cap Z \neq \emptyset$ and $E - f^{-1}(U) \neq \emptyset$ for any $E \in \mathcal{E}$. This contradicts the assumption of the last case. It follows that $f(Z) = \{\kappa^+\}$.

In connection with the above result the following problem arises.

3.3. PROBLEM. Is every cell soluble compact T_2-space completely cell soluble?

4. Complete cell solubility of zero-dimensional spaces.

Arhangel'skii posed also the following problems in [2].

4.1. PROBLEM. Is every zero-dimensional compact T_2-space completely cell soluble?

4.2. PROBLEM. Is it true that $\beta N - N$ is completely cell soluble ?

We will show that these problems are solved affirmatively. It suffices to give the affirmative answer to 4.1.

4.3. THEOREM. *Every zero-dimensional compact T_2-space is completely cell soluble.*

PROOF. Let $\boldsymbol{F}$ be an arbitrary abstract cellularity. Let $2 = \{0, 1\}$ be the two-point discrete space. Now, let us consider the cellularty F^2 on 2 which is associated with $\boldsymbol{F}$. Since 2 is a compact homogeneous space, the following two case occur:

(1) $F^2(0) = \{0\}$ and $F^2(1) = \{1\}$.

(2) $F^2(0) = F^2(1) = 2$.

Case (1). Let X be an arbitrary zero-dimensional compact T_2-space. Then it will be shown that $F^X(x) = \{x\}$ for any $x \in X$, and hence the cellularity F^X on X is disjoint. In fact, assume that $F^X(x)$ contains more than one point. Then there exists a continuous map $f: X \to 2$ such that $f(F^X(x)) = 2$. This contradicts one of the conditions of abstract cellularity since $f(F^X(x)) \supsetneqq F^2(f(x))$.

Case (2). It will be proved that $F^X(x) = X$ for any $X \in \boldsymbol{T}$ and any $x \in X$. In fact, let x be an arbitrary point of X. Then for any another point $y \in X$ there exists a continuous map $f: 2 \to X$ such that $f(0) = x$ and $f(1) = y$. From

$$f(F^2(0))\subset F^X(f(0))=F^X(x)$$

it follows that $y\in F^X(x)$. This implies that $F^X(x)=X$.

5. Weakly homogeneous spaces.

As a result of Arhangel'skii every retract of a homogeneous compact T_2-space is completely cell soluble. And we showed that every zero-dimensional compact T_2-space is completely cell soluble. Hence there arises the following natural question.

5.1. QUESTION. Is every zero-dimensional compact T_2-space the retract of a homogeneous compact T_2-space?

In fact, the answer to the following problem of Arhangel'skii [2] is also unknown.

5.2. PROBLEM. For every zero-dimensional compact T_2-space X, does there exist a homogeneous compact T_2-space Y such that $X\times Y$ is homogeneous?

We do not have perfect solutions for these problems. But we will give partial answers to these problems.

5.3. DEFINITION. A topological space X is called *weakly homogeneous* if, for arbitrary $x, y\in X$ and any neighborhoods U, V of x, y respectively, there exists a homeomorphism f from X onto itself such that $f(x)\in V$ and $f^{-1}(y)\in U$.

Every homogeneous space is clearly weakly homogeneous.

5.4. PROPOSITION. *A zero-dimesional T_1-space X is weakly homogeneous if and only if, for arbitrary $x, y\in X$ and any neighborhoods U, V of x, y respectively, there exist homeomorphic clopen neighborhoods U', V' of x, y respectively such that $U'\subset U$, $V'\subset V$.*

PROOF. (if) Let x, y be distinct points of X. Let U, V be neighborhoods of x, y respectively. We can assume that U and V are disjoint clopen subsets of X. Then there exists a homeomorphism $g: U'\to V'$ for some clopen neighborhoods U', V' of x, y respectively such that $U'\subset U$, $V'\subset V$. Let $f: X\to X$ be the map defined by

$$f(x)=\begin{cases} g(x) & \text{if } x\in U' \\ g^{-1}(x) & \text{if } x\in V' \\ x & \text{otherwise.}\end{cases}$$

then f is a homeomorphism such that $f(x)\in V'\subset V$ and $f^{-1}(y)=g^{-1}(y)\in U'\subset U$.

(only if) We can assume that U and V are clopen subsets. Let $f: X\to X$ be a homeomorphism such that $f(x)\in V$, $f^{-1}(y)\in U$. Now, let $U'=U\cap f^{-1}(V)$ and $V'=V\cap f(U)$. Then U' is a neighborhood of x and V' is a neighborhood of y. Further $f|U': U'\to V'$ is a homeomorphism.

Let us call an infinite, zero-dimensional compact T_2-space X to be *B-homogeneous* if every non-empty clopen subspace of X is homeomorphic to X (cf. [4], [5]). Then every B-homogeneous space is weakly homogeneous. As noted by van Douwen [4], every first countable B-homogeneous space is homogeneous. Similarly, we can show the following.

5.5. PROPOSITION. *Every first countable zero-dimensional weakly homogeneous T_1-space is homogeneous.*

PROOF. We can assume that X has no isolated point. Let x, y be arbitrary points of X.

CLAIM 1. Let U, V be homeomorphic clopen neighborhoods of x, y respectively. Then for arbitrary neighborhoods W^x, W^y of x, y respectively there exists a homeomorphism f from U onto V such that $f(x)\in W^y$, $f^{-1}(y)\in W^x$.

In fact, Let $g: U\to V$ be a homeomorphism. If $g(x)=y$, then there is nothing to do. Hence let $g(x)\neq y$. Then there are disjoint homeomorphic clopen neighborhoods $U_{g(x)}$, V_y of $g(x)$, y respectively such that $U_{g(x)}\subset V\cap g(W^x)$, $V_y\subset V\cap W^y$. For a homeomorphism $h: U_{g(x)}\to V_y$ let $k: V\to V$ be the homeomorphism defined by

$$k(x)=\begin{cases} h(x) & \text{if } x\in U_{g(x)} \\ h^{-1}(x) & \text{if } x\in V_y \\ x & \text{otherwise.} \end{cases}$$

Then $f=k\circ g: U\to V$ is a homeomorphism and satisfies $f(x)=k(g(x))\in k(U_{g(x)})\in V_y\subset W^y$, $f^{-1}(y)=g^{-1}(k^{-1}(y))\in g^{-1}(k^{-1}(V_y))=g^{-1}(U_{g(x)})\subset W^x$.

CLAIM 2. There are neighborhood bases $\{U_n\}$, $\{V_n\}$ of x, y respectively, consisting of clopen subsets such that

(a) $U_n\supset U_{n+1}$, $V_n\supset V_{n+1}$;

(b) there is a homeomorphism f_n from U_n-U_{n+1} onto V_n-V_{n+1}

for each $n\in\omega$.

Let $\{W^x{}_n\}$, $\{W^y{}_n\}$ be neighborhood bases of x, y respectively, consisting of clopen subsets of X. Then there are homeomorphic clopen neighborhoods U_0, V_0 of x, y respectively such that $U_0 \subset W^x{}_0$, $V_0 \subset W^y{}_0$. From Claim 1 it follows that there is a homeomorphism $g_1 : U_0 \to V_0$ such that $g_1(x) \in W^y{}_1$, $g_1{}^{-1}(y) \in W^x{}_1$. Let $U_1 = g_1{}^{-1}(W^y{}_1) \cap W^x{}_1$ and $V_1 = g_1(W^x{}_1) \cap W^y{}_1$. Then U_1 and V_1 are homeomorphic clopen neighborhoods of x, y respectively. Further $f_1 = g_1 | (U_0 - U_1)$ is a homeomorphism from $U_0 - U_1$ onto $V_0 - V_1$. Continuing this procedure, we can obtain the desired neighborhood bases of x, y respectively.

Let $f : X \to X$ be the map defined by

$$f(z) = \begin{cases} z & \text{if } z \in X - U_0 \\ f_n(z) & \text{if } z \in U_n - U_{n+1} \\ y & \text{if } z = x . \end{cases}$$

Then f is a homeomorphism such that $f(x) = y$. This completes the proof.

In the next theorem, the cardinal function $w(X)$ means the weight of X.

5.6. THEOREM. *Let X be a zero-dimensional compact T_2-space. Then there exists a zero-dimensional compact T_2-space Y with $w(Y) = w(X)$ such that $X \times Y$ is weakly homogeneous.*

PROOF. Let $\mathscr{B}$ be an open basis of X consisting of clopen subsets. We can assume that $|\mathscr{B}| = w(X)$ and $X \in \mathscr{B}$. Let Y be the topological product

$$\Pi\{B^\omega \mid B \in \mathscr{B}\}.$$

Then the family of clopen subsets of Y which are homeomorphic to Y forms an open basis of Y. Hence Y is weakly homogeneous. Since $X \times Y$ is homeomorphic to Y, the product space $X \times Y$ is weakly homogeneous.

5.7. COROLLARY. *Every zero-dimensional compact T_2-space is a retract of a weakly homogeneous compact T_2-space.*

Since every compact T_2-space is a continuous image of a zero-dimensional compact T_2-space, we can give the following partial answer to the problem of Arhangel'skii stated in the introduction.

5.8. COROLLARY. *Every compact T_2-space is a continuous image of a weakly homogeneous compact T_2-space.*

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