TSUKUBA J. MATH.
No. 7 No. 1 (1983), 191—202

POLYNOMIAL TYPE HOPF ALGEBRAS

By

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Summary. For any commutative ring K we denote by $K\langle t\rangle$ the free graded K-module $\bigoplus_{n=0}^{\infty} K\cdot t_n$ with grading given by $\deg t_n=2n$. Any commutative and cocommutative graded Hopf K-algebra on $K\langle t\rangle$ is called a polynomial type Hopf algebra over K. The main aim of this paper is to describe all polynomial type Hopf algebras over an arbitrary commutative ring K.

Introduction.

In the study of graded Hopf algebras over commutative rings an important role is played by Hopf algebras whose underlying algebra structures are polynomial ones. This follows from the fact that in many cases interesting graded Hopf algebras can be obtained from polynomial ones by applying standard categorical and ring constructions (see [8, 10, 11, 12, 13, 14]).

In this paper we are concerned with the problem of a description of all commutative and cocommutative graded Hopf K-algebras on the free K-module $K\langle t\rangle=\bigoplus_{n=0}^{\infty} K\cdot t_n$, where K is an arbitrary commutative ring and the grading is given by $\deg t_n=2n$. Any such graded Hopf algebra is called a *polynomial type Hopf algebra* over K.

If K is a field a characterization of polynomial type Hopf algebras over K can be deduced from the results proved in [8, 10, 11, 12, 13, 14]. For the ring of integers the raised above problem has been solved in [5]. In the case of an arbitrary commutative ring there are known two standard polynomial type Hopf algebras: the well known polynomial algebra $K[t]$ and the so-called algebra of polynomials with divided powers (see [5, 9]). Here we give a rather simple description of all polynomial type Hopf algebras over an arbitrary commutative ring K. More precisely we shall show that there is one-to-one correspondence between polynomial type Hopf algebras over K and pairs of sequences $(x_d, x'_d)_{d\in N^+}$, $N^+=\{1, 2, \cdots\}$, of elements in K, which satisfy the following conditions

Received January 18, 1983.

$$x_1 = x_1' = 1\,, \quad x_d x_d' = v_d\,, \quad d \in N^+,$$

where

$$v_d = \begin{cases} 1, & \text{if } d \text{ is not a power of a prime number or } d=1, \\ p, & \text{if } d=p^r \text{ where } p \text{ is a prime number and } r \in N^+ \end{cases}$$

(compare the definition 1.6.4 in [4]). The correspondence is defined as follows. Given such a pair of sequences we define the multiplication

$$m : K\langle t\rangle \otimes K\langle t\rangle \longrightarrow K(t)$$

and the comultiplication

$$m' : K\langle t\rangle \longrightarrow K\langle t\rangle \otimes K\langle t\rangle\,, \quad \otimes = \otimes_K\,,$$

by the following formulas

$$m(t_i \otimes t_j) = m_{i,j} t_{i+j}\,,$$

$$m'(t_n) = \sum_{i+j=n} m'_{i,j} t_i \otimes t_j$$

for any i, j, $n \in N = \{0, 1, 2, \cdots\}$, where

$$m_{i,j} = \prod_{d\in N^+} x_d^{\left[\frac{i+j}{d}\right]-\left[\frac{i}{d}\right]-\left[\frac{j}{d}\right]}$$

$$m'_{i,j} = \prod_{d\in N^+} x_d'^{\left[\frac{i+j}{d}\right]-\left[\frac{i}{d}\right]-\left[\frac{j}{d}\right]}$$

(here $[r]$ denotes the integral part of the real number r). Then $(K\langle t\rangle, m, m')$ with the obvious unity and counity maps is the polynomial type Hopf algebra over K corresponding to the pair $(x_d, x_d')_{d\in N^+}$.

An important role in our proof of the stated above correspondence is played by a generalized version of well-known Lazard's Lemma [2, 4, 6] being the crucial fact in the classification of one-dimensional formal groups.

1. Basic definitions and lemmas.

Let $\mathfrak{A}$ be the category of commutative rings with unity. For any K in $\mathfrak{A}$, $K\langle t\rangle = \bigoplus_{n=0}^{\infty} K\cdot t_n$ is the free graded K-module with free generators t_n, $n \in N$, and with grading given by $\deg t_n = 2n$. We denote by $e : K \to K\langle t\rangle$ the K-linear map such that $e(1) = t_0$, and by $e' : K\langle t\rangle \to K$ the K-linear map such that $e'(t_n) = \delta_{n,0}$, where $\delta_{i,j}$ is the Kronecker index.

A pair of K-linear maps (m, m') such that $(K\langle t\rangle, m, m', e, e')$ is a commutative and cocommutative graded Hopf K-algebra is called a *polynomial type Hopf structure* over K, or equivalently an *h-structure* over K, and $(K\langle t\rangle, m, m', e, e')$ is called a *polynomial type Hopf algebra* over K (see [7] or [8] for the definition of graded Hopf algebra).

If $\varphi: K\rightarrow K'$ is any morphism in $\mathfrak{A}$, then for any h-structure (m, m') over K the pair $(1_{K'}\otimes m, 1_{K'}\otimes m')$ may be considered in a natural way as an h-structure over K' and we denote it by $(m_{\varphi}, m'_{\varphi})$.

Let $\mathfrak{S}$ be the category of sets. We define a covariant functor

$$h: \mathfrak{A} \longrightarrow \mathfrak{S}$$

by assigning to each K from $\mathfrak{A}$ the set $h(K)$ of all h-structures over K and to each $\varphi: K\rightarrow K'$ in $\mathfrak{A}$ the map $h(\varphi): h(K)\rightarrow h(K')$ defined by $h(\varphi)(m, m')=(m_{\varphi}, m'_{\varphi})$ for any (m, m') from $h(K)$.

Now let $(m, m')\in h(K)$. Then

$$m(t_i\otimes t_j)=m_{i,j}t_{i+j}, \quad m'(t_n)=\sum_{i+j=n} m'_{i,j}t_i\otimes t_j, \tag{1.1}$$

for some elements $m_{i,j}$, $m'_{i,j}$ in K and for all i, j, n in N and the following conditions are satisfied:

$$m_{i,0}=m_{0,i}=1, \quad m_{i,j}=m_{j,i}, \quad m_{i,j}m_{i+j,k}=m_{i,j+k}m_{j,k} \tag{1.2}$$

$$m'_{i,0}=m'_{0,i}=1, \quad m'_{i,j}=m'_{j,i}, \quad m'_{i,j}m'_{i+j,k}=m'_{i,j+k}m'_{j,k} \tag{1.2'}$$

for all i, j, k in $N=\{0, 1, 2, \cdots\}$,

$$m_{i,j}m'_{k,l}=\sum m_{p,r}m_{q,s}m'_{p,q}m'_{r,s} \tag{1.3}$$

for any i, j, k, l in N such that $i+j=k+l$, where the sum runs over all p, q, r, s in N such that $p+q=i$, $r+s=j$, $p+r=k$, $q+s=l$.

We put $g(m, m')=(m_{i,j}, m'_{i,j})_{i,j\in N}$. Every pair of systems $(m_{i,j}, m'_{i,j})_{i,j\in N}$ satisfying (1.2), (1.2′), (1.3) is called a *polynomial type Hopf K-pair*, or equivalently an *h-pair* over K. It is easy to verify the following

LEMMA 1.1. *The map $(m, m')\mapsto g(m, m')$ establishes an one-to-one correspondence between elements of $h(K)$ and h-pairs over K. Moreover, if $\varphi: K\rightarrow K'$ is a morphism in $\mathfrak{A}$ and $g(m, m')=(m_{i,j}, m'_{i,j})_{i,j\in N}$ then $g(m_{\varphi}, m'_{\varphi})=(\varphi(m_{i,j}), \varphi(m'_{i,j}))_{i,j\in N}$.*

For the proof we remark only that conditions (1.2) are equivalent to the fact that $(K\langle t\rangle, m, e)$ is a graded commutative and associative K-algebra with unity, conditions (1.2′) correspond to the fact that $(K\langle t\rangle, m', e')$ is a graded, cocommutative and coassociative K-coalgebra with counity and condition (1.3) is equivalent to the fact that m' is a graded K-algebra morphism i.e. that the following diagram is commutative

$$\begin{array}{ccc} K\langle t\rangle\otimes K\langle t\rangle & \xrightarrow{m'\otimes m'} & K\langle t\rangle\otimes K\langle t\rangle\otimes K\langle t\rangle\otimes K\langle t\rangle \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle (m\otimes m)(1\otimes\tau\otimes 1)} \\ K\langle t\rangle & \xrightarrow{m'} & K\langle t\rangle\otimes K\langle t\rangle \end{array}$$

where $\tau: K\langle t\rangle\otimes K\langle t\rangle\to K\langle t\rangle\otimes K\langle t\rangle$ is the twisting morphism.

By the above lemma we may identify the set $h(K)$ with the set of h-pairs over an arbitrary commutative ring K.

For any i, j in N we denote by (i, j) the binomial coefficient $\frac{(i+j)!}{i!\,j!}$. Note that the pair of systems $(m_{i,j}=1, m'_{i,j}=(i, j))_{i,j\in N}$ is a polynomial type Hopf K-pair for an arbitrary ring K. The corresponding polynomial type Hopf algebra over K is the K-algebra of polynomials $K[t]$.

We have the following

LEMMA 1.2. *If $(m_{i,j}, m'_{i,j})_{i,j\in N}$ is a polynomial type Hopf K-pair then*

$$m_{i,j}m'_{i,j}=(i, j) \tag{1.4}$$

for all i, j in N.

PROOF. Remark that the condition (1.3) with $i=k$, $j=l$ in the case of the K-algebra of polynomials $K[t]$ has the form

$$(i, j)=\sum_{\substack{p+q=i\\ q+s=j}}(p, q)(q, s)\,. \tag{1.5}$$

It is clear that (1.4) holds in the case $i=0$ or $j=0$. Let $n>1$ and assume that (1.4) holds for all i, j in N satisfying $i+j<n$. Now, let i, j in N satisfy $i+j=n$ and $i, j>0$. By (1.3) we have

$$m_{i,j}m'_{i,j}=\sum_{\substack{p+q=i\\ q+s=j}}m_{p,q}m_{q,s}m'_{p,q}m'_{q,s}$$

Hence, using the inductive assumption and (1.5) we see that

$$m_{i,j}m'_{i,j}=\sum_{\substack{p+q=i\\ q+s=j}}(p, q)(q, s)=(i, j)\,.$$

Then the lemma follows by induction.

For any sequence $(m_n)_{n\in N^+}$, of elements in a commutative ring K, we define $m_0!=1$, $m_{n+1}!=m_n!\,m_{n+1}$. By a *multiplication* over K we mean any system $(m_{i,j})_{i,j\in N}$, of elements in K, which satisfies (1.2). We have

LEMMA 1.3. *Let $(m_{i,j})_{i,j\in N}$ be a multiplication over K and suppose that all $m_{i,j}$, $i, j\in N$ are non-zero-divisors in K. Then*

$$m_{i,j}=\frac{m_{i+j}!}{m_i!\,m_j!}$$

for all i, j *in* N, *where* $m_n=m_{n-1,1}$, $n\in N^+$.

The proof is left to the reader. Compare Proposition 1.5 in [5].

For a real number r we denote by $[r]$ the integral part of r. If $i_1, \cdots, i_s\in N^+$, $d\in N^+$, then we put

$$[i_1, \cdots, i_s: d]=\left[\frac{i_1+\cdots i_s}{d}\right]-\left[\frac{i_1}{d}\right]-\cdots-\left[\frac{i_s}{d}\right]$$

Remark that for any $i, j, k\in N$, $d\in N^+$, we have

$$[i, j: 1]=0$$

$$[i, 0: d]=0$$

$$[i, j: d]=[j, i: d]$$

$$[i, j: d]+[i+j, k: d]=[i, j+k: d]+[j, k: d].$$

For any sequence $\boldsymbol{x}=(x_d)_{d\in N^+}$, of elements in an arbitrary commutative ring K, we define a system $m(\boldsymbol{x})=(m_{i,j})_{i,j\in N}$, by formulas

$$m_{i,j}=\prod_{d\in N^+} x_d^{[i,j:d]}, \qquad i, j\in N.$$

One easily checks the following

LEMMA 1.4. *The system* $m(\boldsymbol{x})$ *is a multiplication over* K.

We define a sequence $\boldsymbol{v}=(v_d)_{d\in N^+}$, by

$$(1.6)\qquad v_d=\begin{cases} 1, & \text{if } d \text{ is not a power of a prime number or } d=1,\\ p, & \text{if } d=p^r \text{ where } p \text{ is a prime number and } r\in N^+.\end{cases}$$

Since $n!=\prod_p p^{\sum_r\left[\frac{n}{p^r}\right]}$ (see for example [3]) then the following lemma immediately follows from definitions of $\boldsymbol{v}$ and $[i, j: d]$.

LEMMA 1.5. $m(\boldsymbol{v})=(i, j)_{i,j\in N}$.

Let us denote by H the Z-algebra $Z[X_d, X_d': d\in N^+]$ with relations $X_1=X_1'=1$, $X_d\cdot X_d'=v_d$, $d\in N^+$. We shall need the following result.

LEMMA 1.6. *The ring* H *is torsionless as an abelian group.*

PROOF. Note that H is generated as an abelian group by elements of the form

(1.7) $$X_{d_1}^{\alpha_1}\cdot \cdots \cdot X_{d_r}^{\alpha_r}\cdot X'^{\beta_1}_{d'_1}\cdot \cdots \cdot X'^{\beta_s}_{d'_s}$$

where $1<d_1<\cdots<d_r$, $1<d'_1<\cdots<d'_s$, $d_i\neq d'_j$, $\alpha_1,\cdots,\alpha_r$, $\beta_1,\cdots,\beta_s\in N^+$. Let $Q(T_d: d\geqq 2)$ be the field of rational functions in variables T_d, $d\geqq 2$. We define the morphism in $\mathfrak{A}$

$$\phi: H\longrightarrow Q(T_d: d\geqq 2)$$

by formulas $\phi(X_d)=T_d$, $\phi(X'_d)=\dfrac{v_d}{T_d}$, $d\geqq 2$. The images of the elements (1.7) are monomials

$$v_{d'_1}^{\beta_1}\cdots v_{d'_s}^{\beta_s}\frac{T_{d_1}^{\alpha_1}\cdots T_{d_r}^{\alpha_r}}{T_{d'_1}^{\beta_1}\cdots T_{d'_s}^{\beta_s}}$$

that are linearly independent over the field of rational numbers Q. It follows that ϕ is an injection. Consequently elements (1.7) form a free abelian group basis of H and the lemma is proved.

2. The main theorems.

Let K be a commutative ring. A *reduced h-pair* over K is a pair of sequences $(\boldsymbol{x}=(x_d)_{d\in N^+}, \boldsymbol{x}'=(x'_d)_{d\in N^+})$ of elements in K, satisfying the following conditions

(2.1) $$x_1=x'_1=1\,,\quad x_d x'_d=v_d\,,\quad d\in N^+.$$

THEOREM 2.1. *Let $(\boldsymbol{x}=(x_d)_{d\in N^+}, \boldsymbol{x}'=(x'_d)_{d\in N^+})$ be a reduced h-pair over K. Then $(m(\boldsymbol{x}), m(\boldsymbol{x}'))$ is a polynomial type Hopf structure over K.*

PROOF. Let $m(\boldsymbol{x})=(m_{i,j})_{i,j\in N}$ and $m(\boldsymbol{x}')=(m'_{i,j})_{i,j\in N}$. By Lemma 1.4 the pair of systems $(m_{i,j}, m'_{i,j})_{i,j\in N}$ satisfies (1.2) and (1.2′). Now by (1.6), (2.1) and Lemma 1.5 we have

(2.2) $$m_{i,j}m'_{i,j}=(i,\ j)$$

for all i, j in N. First we suppose that K is torsionless as an abelian group. Then (2.2) implies that all elements $m_{i,j}$, $m'_{i,j}$ are non-zero-divisors in K. Hence, using Lemma 1.3, we see that

(2.3) $$m_{i,j}=\frac{m_{i+j}!}{m_i!\,m_j!},\qquad m'_{i,j}=\frac{m'_{i+j}!}{m'_i!\,m'_j!}$$

for all i, j in N, where $m_n=m_{n-1,1}$, $m'_n=m'_{n-1,1}$ for all n in N^+.

Now, let i, j, k, l in N satisfy $i+j=k+l$. The equality (1.3) in the special case of the K-algebra of polynomials $K[t]$ has a form

$$(k,\ l)=\sum(p,\ q)(r,\ s)$$

where the sum runs as in (1.3). Combining this with (2.2) we have

$$m_{k,l}m'_{k,l}=\sum m_{p,q}m_{r,s}m'_{p,q}m'_{r,s}.$$

Multiplying the above equality by $\frac{m_k!\, m_l!}{m_i!\, m_j!}$ (in the classical ring of quotients of the ring K) and using (2.3) we get (1.3). Hence, by Lemma 1.1, the pair $(m(\boldsymbol{x}), m(\boldsymbol{x}'))$ is a polynomial type Hopf structure over K.

By the definition of the ring H we know that $(\mathcal{X}=(X_d)_{d\in N^+}, \mathcal{X}'=(X'_d)_{d\in N^+})$ is a reduced h-pair over H. Since H is torsionless as an abelian group the pair $(m(\mathcal{X}), m(\mathcal{X}'))$ belongs to $h(H)$. Now, if K is an arbitrary ring, we consider the ring homomorphism $\varphi: H\to K$ defined by $\varphi(X_d)=x_d$, $\varphi(X'_d)=x'_d$, $d\in N^+$. It is clear that $(m(\boldsymbol{x}), m(\boldsymbol{x}'))=h(\varphi)(m(\mathcal{X}), m(\mathcal{X}'))$. Thus $(m(\boldsymbol{x}), m(\boldsymbol{x}'))$ belongs to $h(K)$. This completes the proof of the theorem.

THEOREM 2.2. *Let (m, m') be an h-structure over K. Then there exists a unique reduced h-pair $(\boldsymbol{x}, \boldsymbol{x}')$ over K such that $(m, m')=(m(\boldsymbol{x}), m(\boldsymbol{x}'))$.*

PROOF. Let $m=(m_{i,j})_{i,j\in N}$ and $m'=(m'_{i,j})_{i,j\in N}$. We have to construct a unique pair of sequences $(\boldsymbol{x}=(x_d)_{d\in N^+}, \boldsymbol{x}'=(x'_d)_{d\in N^+})$ of elements in K such that

$$\begin{gathered} x_1=x'_1=1, \quad x_d x'_d=v_d, \quad d\in N^+, \\ m_{i,j}=\prod_{d\in N^+} x_d^{[i,j:d]}, \quad m'_{i,j}=\prod_{d\in N^+} x_d'^{[i,j:d]}, \quad i, j\in N. \end{gathered} \tag{2.4}$$

Let $n>1$ and assume that there exists a unique pair of sequences $((x_1, \cdots x_{n-1}), (x'_1, \cdots, x'_{n-1}))$ of elements in K satisfying (2.4) for $d\leqq n-1$ and $i+j\leqq n-1$.

We define

$$\bar{m}_{i,j}=\prod_{d=1}^{n-1} x_d^{[i,j:d]}, \quad \bar{m}'_{i,j}=\prod_{d=1}^{n-1} x_d'^{[i,j:d]}$$

for all i, j in N^+, $i+j=n$. It is easy to see that

$$\bar{m}_{i,j}=\bar{m}_{j,i}, \quad \bar{m}'_{i,j}=\bar{m}'_{j,i}, \quad i, j\in N^+, \quad i+j=n \tag{2.5}$$

$$m_{i,j}\bar{m}_{i+j,k}=m_{j,k}\bar{m}_{i,j+k}, \ m'_{i,j}\bar{m}'_{i+j,k}=m'_{j,k}\bar{m}'_{i,j+k} \tag{2.6}$$

for i, j, k in N^+, $i+j+k=n$,

$$\bar{m}_{i,j}\bar{m}'_{i,j}=c_{i,j} \tag{2.7}$$

for all i, j in N^+, $i+j=n$, where $c_{i,j}=\frac{(i, j)}{v_n}$.

Consider K-modules

$$L_n=(\bigoplus_{\substack{i+j=n\\ i,j>0}} K\cdot X_{i,j})/R_n, \quad L'_n=(\bigoplus_{\substack{i+j=n\\ i,j>0}} K\cdot X'_{i,j})/R'_n,$$

where $X_{i,j}$, $X'_{i,j}$ are free generators and R_n, R'_n are K-submodules generated respectively by elements

$$X_{i,j}-X_{j,i}, \quad X'_{i,j}-X'_{j,i}, \quad \text{with} \quad i, j \in N^+, \quad i+j=n$$

$$m_{i,j}X_{i+j,k}-m_{j,k}X_{i,j+k},$$

$$m'_{i,j}X'_{i+j,k}-m'_{j,k}X'_{i,j+k}, \quad \text{with} \quad i, j, k \in N^+, \quad i+j+k=n$$

By conditions (2.5) and (2.6) there exist K-linear maps

$$\bar{l}_n : L_n \longrightarrow K, \quad \bar{l}'_n : L'_n \longrightarrow K$$

given by

$$\bar{l}_n(\bar{X}_{i,j})=\bar{m}_{i,j}, \quad \bar{l}'_n(\bar{X}'_{i,j})=\bar{m}'_{i,j} \tag{2.8}$$

for i, j in N^+, $i+j=n$, where $\bar{X}_{i,j}$, $\bar{X}'_{i,j}$ are cosets of $X_{i,j}$, $X'_{i,j}$ in L_n, L'_n, respectively. We will show that $\bar{l}_n$ and $\bar{l}'_n$ are isomorphisms.

It is well known that there exist rational integers $d_{i,j}$, $i, j \in N^+$, $i+j=n$, such that

$$\sum_{i+j=n} d_{i,j}c_{i,j}=1 \tag{2.9}$$

(see Section 4.1 in [4]).

If we put

$$\bar{X}_n=\sum d_{i,j}\bar{m}'_{i,j}\bar{X}_{i,j}$$

$$\bar{X}'_n=\sum d_{i,j}\bar{m}_{i,j}\bar{X}'_{i,j}$$

then from (2.7), (2.8) and (2.9) it follows that $\bar{l}_n(\bar{X}_n)=1$ and $\bar{l}'_n(\bar{X}'_n)=1$. Hence $\bar{l}_n$ and $\bar{l}'_n$ are epimorphisms. In the next section we prove that $\bar{l}_n$ and $\bar{l}'_n$ are monomorphisms. Thus the elements $\bar{X}_n$ and $\bar{X}'_n$ are free generators of the cyclic K-modules L_n and L'_n, respectively, and the following equalities hold

$$\bar{X}_{i,j}=\bar{m}_{i,j}\bar{X}_n, \quad \bar{X}'_{i,j}=\bar{m}'_{i,j}\bar{X}'_n \tag{2.10}$$

for all i, j in N^+ such that $i+j=n$.

Now we consider K-linear maps

$$l_n : L_n \longrightarrow K, \quad l'_n : L'_n \longrightarrow K$$

defined as follows

$$l_n(\bar{X}_{i,j})=m_{i,j}, \quad l'_n(\bar{X}'_{n,j})=m'_{i,j}, \quad i, j \in N^+, \quad i+j=n.$$

Let $x_n=l_n(\bar{X}_n)$ and $x'_n=l'_n(\bar{X}'_n)$. Then (2.10) implies that $m_{i,j}=\bar{m}_{i,j}x_n$ and $m'_{i,j}=\bar{m}'_{i,j}x'_n$. Moreover

$$\begin{aligned} x_nx'_n &= (\sum d_{i,j}c_{i,j})x_nx'_n \\ &= \sum d_{i,j}\bar{m}_{i,j}x_n\bar{m}'_{i,j}x'_n \\ &= \sum d_{i,j}m_{i,j}m'_{i,j} \\ &= \sum d_{i,j}(i, j) \end{aligned}$$

$$=(\sum d_{i,j}c_{i,j})v_n$$

$$=v_n .$$

Finally, if y_n, y'_n are elements in K such that $y_n y'_n=v_n$ and $m_{i,j}=\bar{m}_{i,j}y_n$, $m'_{i,j}=\bar{m}'_{i,j}y'_n$, then

$$\begin{aligned} y_n &= \sum d_{i,j}c_{i,j}y_n \\ &= \sum d_{i,j}\bar{m}_{i,j}\bar{m}'_{i,j}y_n \\ &= \sum d_{i,j}m_{i,j}\bar{m}'_{i,j} \\ &= \sum d_{i,j}\bar{m}_{i,j}\bar{m}'_{i,j}x_n \\ &= x_n . \end{aligned}$$

Similarly one can prove that $y'_n=x'_n$.

Hence the pair of sequences $(x_1, \cdots, x_n)$ and $(x'_1, \cdots, x'_n)$ is a unique pair that satisfies (2.4) for $d\leqq n$ and $i+j=n$. Consequently, by induction, the theorem is proved.

Theorems 2.1 and 2.2 yield

COROLLARY 2.3. *The functor* $h:\mathfrak{A}\to\mathfrak{S}$ *is representable by the ring* H *in the sense that* $h(-)=\mathfrak{A}(H, -)$.

COROLLARY 2.4. *Let* $(u_n)_{n\in N}$, $u_0=1$, *be sequence of invertible elements in* K. *Then there exists a unique sequence* $(\alpha_d)_{d\in N^+}$, *of inverible elements in* K *such that*

$$\frac{u_{i+j}}{u_j u_j}=\prod_{d\in N^+}\alpha_d^{[i,j:d]}$$

for any i, j *in* N.

PROOF. Since the pair $\left(m_{i,j}=\frac{u_{i+j}}{u_i u_j},\ m'_{i,j}=\frac{(i,\ j)}{m_{i,j}}\right)_{i,j\in N}$ is an h-structure over K, then the corollary follows by Theorem 2.2.

3. A generalized Lazard's Lemma.

We keep in this section the notation of the preceding sections, in particular, the notation that has been introduced in the proof of Theorem 2.2. It is easy to see that the following theorem is a generalization and a slight modification of Lazard's Lemma (see [2, 4, 6]).

THEOREM 3.1. *Let* $(m_{i,j}, m'_{i,j})_{i,j\in N}$ *be a polynomial type Hopf structure over an arbitrary commutative ring* K. *Then for any* $n\geqq 2$ *the* K*-linear maps*

$\bar{l}_n: L_n \to K$ and $\bar{l}'_n: L'_n \to K$ are isomorphisms.

PROOF. We already know that $\bar{l}_n$ and $\bar{l}'_n$ are epimorphisms. Thus $\bar{l}_n$ and $\bar{l}'_n$ are monomorphisms too if and only if L_n and L'_n are cyclic K-modules. Moreover, we may assume that K is a $Z_{(p)}$-algebra for some prime number p (see for example, Theorem 1 in [1, Chapter II, §3]). Here $Z_{(p)}$ denotes the localization of rational integers Z with respect to the prime ideal pZ.

Let p be a prime number and let K be a $Z_{(p)}$-algebra. Let t, r be natural numbers such that

$$n=p^t+r \quad \text{and} \quad p^t<n\leqq p^{t+1}.$$

We will show that $\bar{X}=\bar{X}_{p^t,r}$ is a generator of L_n and $\bar{X}'=\bar{X}'_{p^t,r}$ is a generator of L'_n. The case of L_n is proved only and its proof proceeds in several steps.

1° Let j, $s\in N$ and $0\leqq j<p^{s+1}-p^s$. Then $m_{p^s,j}$ is invertible in K.

2° Let j, $s\in N$ and $0\leqq j<p^s$. Then $m_{p^{s+1}-p^s,j}$ is invertible in K.

For the proof remark that by Lemma 1.2 we have $m_{p^s,j}m'_{p^s,j}=(p^s, j)$ and $m_{p^{s+1}-p^s,j}m'_{p^{s+1}-p^s,j}=(p^{s+1}-p^s, j)$. Since K is a $Z_{(p)}$-algebra then it is sufficient to prove that p-adic valuation of (p^s, j) and $(p^{s+1}-p^s, j)$ is zero. But this is easy.

As a consequence of 1° and the equality $m_{p^s,j}\bar{X}_{p^s+j,k}=m_{j,k}\bar{X}_{p^s,j+k}$ in L_n one gets

3° $\bar{X}_{p^s+j,k}$ belongs to the submodule $K\bar{X}_{p^s,j+k}$ for any j, k, s in N, $p^s+j+k=n$, $0<j<p^{s+1}-p^s$, $0<k$.

Now by 3° we have

4° $\bar{X}_{p^t+1,r-1}, \cdots, \bar{X}_{n-1,1}$ belong to $K\bar{X}$.

5° $\bar{X}_{p^{t-1},n-p^{t-1}}$ belongs to $K\bar{X}$ (if $t>0$).

We consider two cases: (i) $0<r<p^{t-1}$ and (ii) $p^{t-1}\leqq r\leqq p^{t+1}-p^t$. In the case (i) the element $\bar{X}_{p^{t-1},n-p^{t-1}}$ belongs to $K\bar{X}$ because of the equality $m_{p^{t-1},p^t-p^{t-1}}\bar{X}_{p^t,r}=m_{p^t-p^{t-1},r}\bar{X}_{p^{t-1},n-p^{t-1}}$ and 2°. In the case (ii) the equalities $\bar{X}_{p^{t-1},n-p^{t-1}}=\bar{X}_{n-p^{t-1},p^{t-1}}$ and $n-p^{t-1}=p^t+(r-p^{t-1})$ imply that $\bar{X}_{p^{t-1},n-p^{t-1}}$ belongs to $K\bar{X}$ by 4°.

By 3° and 5° we get

6° $\bar{X}_{p^{t-1}+1,n-p^{t-1}-1}, \cdots, \bar{X}_{p^{t-1},n-p^t+1}$ belong to $K\bar{X}$.

The last statement follows from the inequality $n-i>p^{t-1}$ for $i<p^{t-1}$ and the equality $\bar{X}_{i,n-i}=\bar{X}_{n-i,i}$ and 4° or 5° or 6°

7° $\bar{X}_{1,n-1}, \cdots, \bar{X}_{p^{t-1}-1,n-p^{t-1}+1}$ belong to $K\bar{X}$ (if $t>1$).

This completes the proof of the theorem.

4. Isomorphisms.

We say that h-structures (m, m') and (n, n') over K are isomorphic if and only if the polynomial type Hopf algebras $(K\langle t\rangle, m, m', e, e')$ and $(K\langle t\rangle, n, n, e, e')$ are isomorphic. An isomorphism $\phi: K\langle t\rangle\to K\langle t\rangle$ is called a *strict isomorphism* if $\phi(t_1)=t_1$. Then the corresponding polynomial type Hopf algebras and h-structures over K are called *strictly isomorphic*. It is easy to see that if $\phi: K(t)\to K\langle t\rangle$ is an isomorphism and $\phi(t_n)=u_n t_n$, $n\in N$, then all u_n are invertible elements in K, $u_0=1$, and for any invertible element u in K the K-linear map $\bar{\phi}: K\langle t\rangle\to K\langle t\rangle$ defined by $\bar{\phi}(t_n)=u^n u_n t_n$, $n\in N$, is an isomorphism, too. Hence we may restrict our considerations to the strict isomorphisms only.

THEOREM 4.1. *Let (m, m') and (n, n') be h-structures over K and let $(\boldsymbol{x}=(x_d)_{d\in N^+}, \boldsymbol{x}'=(x'_d)_{d\in N^+})$ and $(\boldsymbol{y}=(y_d)_{d\in N^+}, \boldsymbol{y}'=(y'_d)_{d\in N^+})$ be reduced h-pairs over K such that $(m, m')=(m(\boldsymbol{x}), m(\boldsymbol{x}'))$ and $(n, n')=(m(\boldsymbol{y}), m(\boldsymbol{y}'))$. Then (m, m') and (n, n') are strictly isomorphic if and only if there exists a sequence $(\alpha_d)_{d\in N^+}$, of invertible elements in K such that*

$$y_d=\alpha_d x_d, \quad y'_d=\alpha_d^{-1}x'_d, \quad d\in N^+. \tag{4.1}$$

PROOF. Let $m=m(\boldsymbol{x})=(m_{i,j})_{i,j\in N}$, $m'=m(\boldsymbol{x}')=(m'_{i,j})_{i,j\in N}$, $n=m(\boldsymbol{y})=(n_{i,j})_{i,j\in N}$, $n'=m(\boldsymbol{y}')=(n'_{i,j})_{i,j\in N}$ and let $\phi: K\langle t\rangle\to K\langle t\rangle$ be a strict isomorphism of corresponding Hopf algebras. If $\phi(t_n)=u_n t_n$, $n\in N$, then $u_0=u_1=1$ and u_n are invertible elements in K. Since ϕ preserves multiplication and comultiplication then it is easy to see that

$$u_{i+j}m_{i,j}=u_i u_j n_{i,j}, \quad u_i u_j m'_{i,j}=u_{i+j}n'_{i,j}. \tag{4.2}$$

for all i, j in N. By Corollary 2.4 there exists a unique sequence $\boldsymbol{\alpha}=(\alpha_d)_{d\in N^+}$, $\alpha_1=1$, of invertible elements in K such that

$$\left(u_{i,j}=\frac{u_{i+j}}{u_i u_j}\right)_{i,j\in N}=m(\alpha). \tag{4.3}$$

By the definition of $m(\boldsymbol{x})$, (4.2) and (4.3) we have

$$n_{i,j}=\prod_{d\in N^+}(\alpha_d x_d)^{[i,j:d]}, \quad n'_{i,j}=\prod_{d\in N^+}(\alpha_d^{-1}x'_d)^{[i,j:d]}$$

for all i, j in N. Hence by the unique part of Theorem 2.2 we get (4.1).

Conversely, let (4.1) holds for some sequence of invertible elements $\boldsymbol{\alpha}=(\alpha_d)_{d\in N^+}$, $\alpha_1=1$, in K. If we put $m(\boldsymbol{\alpha})=(u_{i,j})_{i,j\in N}$ then by Lemma 1.3 we have

$$u_{i,j}=\frac{u_{i+j}!}{u_i!\,u_j!}$$

for all i, j in N, where $u_n=u_{n-1,1}$, $n\in N^+$. It is clear that

(4.4) $$u_{i+j}!\,m_{i,j}=u_i!\,u_j!\,n_{i,j},\quad u_i!\,u_j!\,m'_{i,j}=u_{i+i}!\,n'_{i,j}$$

for all i, j in N. Now let $\phi\colon K\langle t\rangle\to K\langle t\rangle$ be a K-linear map defined by $\phi(t_n)=u_n!\,t_n$. Then by (4.4) ϕ preserves multiplication and comultiplication of corresponding polynomial type Hopf algebras and therefore it is a strict isomorphism. This completes the proof of the theorem.

Acknowledgements

I would like to express my gratitude to Andrzej Tyc for his kind encouragement and many helpful conversations.

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