TSUKUBA J. MATH.
Vol. 5 No. 1 (1981). 85—99

STABILITY FOR INFINITE-DIMENSIONAL FIBRE BUNDLDS

By

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Abstract. In this paper, we prove that any locally trivial fibre bundles $p: X\rightarrow B$ with fibre M a manifold modeled on an infinite-dimensional space E (e.g. the Hilbert space l_2 or the Hilbert cube Q) is bundle isomorphic to the bundle $p\circ\text{proj}: X\times E\rightarrow B$. Further, we can obtain a strong version of this Bundle Stability Theorem. From Bundle Stability Theorem, we can introduce the notion of deficiency in bundles. We show that a finite union of locally deficient sets is deficient and we prove a bundle version of Mapping Replacement Theorem.

§ 0. Introduction.

A Hilbert (*Hilbert cube*) *manifold,* briefly *l_2-manifold* (*Q-manifold*), is a paracompact space M admitting an open cover by sets homeomorphic ($\cong$) to open subsets of the Hilbert space l_2 (the Hilbert cube Q). These manifolds are topologically stable, that is, $M\cong M\times l_2$ ($M\cong M\times Q$). This Stability Theorem due to R. D. Anderson and R. M. Schori [A-S] is most fundamental in the theory of infinite-dimensional manifolds.

In this paper, we establish the stability theorem for locally trivial fibre bundles with fibre an l_2-manifold or a Q-manifold. We will call these bundles *l_2-manifold bundles* or *Q-manifold bundles,* respectively.

BUNDLE STABILITY THEOREM. (*Assume B is metrizable.*)

(A) *An l_2-manifold bundle is bundle isomorphic to* $p\circ\text{proj}: X\times l_2\rightarrow B$.

(B) *A Q-manifold bundle is bundle isomorphic to* $p\circ\text{proj}: X\times Q\rightarrow B$.

Here a bundle $p: X\rightarrow B$ is *bundle isomorphic* to a bundle $p': X'\rightarrow B$ if there exists homeomorphism $h: X\rightarrow X'$ such that $p'h=p$ (such a homeomorphism is called a *bundle homeomorphism*).

In this theorem, there is a bundle homeomorphism $h: X\times l_2\rightarrow X$ ($h: X\times Q - X$) is homotopic to the projection $\text{proj}: X\times l_2\rightarrow X$ ($\text{proj}: X\times Q\rightarrow X$) by a small bundle

Received October 8, 1980.

homotopy. In practice, we prove a more strong result, i.e. Theorem 4-3 (and Remark 5-1), under the more general situation including (A) and (B).

A subset K of M is *l_2-deficient* (*Q-deficient*) if there is a homeomorphism $h: M \to M \times l_2$ ($h: M \to M \times Q$) such that $h(K) \subset M \times \{0\}$. A closed subset K of M is a *Z-set* if there is a continuous map $f: M \to M \backslash K$ arbitrarily near to the identity (or equivalently, if for each non-empty homotopically trivial open set U in M, $U \backslash K$ is also non-empty and homotopically trivial). It is well-known that these two notion are identical for closed sets in l_2-manifolds or Q-manifolds. And these notion are very useful and very important in the theory of infinite-dimensional manifolds.

From Bundle Stability Theorem we can introduce the notion l_2-deficiency (Q-deficiency) in l_2-manifold (Q-manifold) bundles. In this paper, we see several easy properties of these deficient sets in bundles. We show that a locally deficient set is deficient and that a finite union of deficient sets is also deficient. And we prove a bundle version of Mapping Replacement Theorem due to R. D. Anderson and J. D. McCharen [A-M] which is an important tool in the theory of infinite-dimensional manifolds. Further aspects shall be developed in sequels [$\mathrm{Sa}_{2,3}$].

R. Y. T. Wong and T. A. Chapman ([$\mathrm{Wo}_{1,2}$] and [C-W]) have developed an entirely satisfactory infinite-dimensional bundle theory *over finite complex*. And T. A. Chapman and S. Ferry ([C-F] and [Fe]) have proved several theorems for *product bundle* with a Q-manifold fibre. And H. Toruńczyk, is his dissertation, have proved several theorems of infinite-dimensional bundles.

§1. Semi-Reflective Isotopy Property.

The unit interval [0, 1] is denoted by $\boldsymbol{I}$. A pointed topological space $(L, 0)$ is said to have the *semi-reflective isotopy property*, briefly: SRIP, if there exists an ambient invertible isotopy $\sigma: L^2 \times \boldsymbol{I} \to L^2$ (called a *semi-reflective isotopy*) such that

$$\sigma_0 = \mathrm{id},$$

$$\sigma_1(x, y) = (y, e(x)) \text{ for each } (x, y) \in L^2 \text{ and}$$

$$\sigma_t(0, 0) = (0, 0) \text{ for each } t \in \boldsymbol{I}$$

where $e: L \to L$ is a homeomorphism (called a *swerving homeomorphism*). If $e = \mathrm{id}$, we call the *reflective isotopy property* (*RIP*). (See [B-P] p. 289) It is easy to see that *if* $e^n = \mathrm{id}$, *then* $(L^n, 0)$ *has RIP.*

1-1 EXAMPLE: Any closed (or open) interval with a base point in its interior and any linear topological space with 0 a base point have *SRIP* and those semi-reflective isotopies have idenpotent swerving homeomorphisms (i. e. $e^2=\mathrm{id}$). If each $(L_\lambda, 0_\lambda)$ has *SRIP*, then $(\prod_{\lambda\in\Lambda} L_\lambda, 0)$ and $(\sum_{\lambda\in\Lambda} L_\lambda, 0)$ have *SRIP* where $\prod_{\lambda\in\Lambda} L_\lambda$ is the product space of L_λ $(\lambda\in\Lambda)$ and $\sum_{\lambda\in\Lambda} L_\lambda=\{x=(x_\lambda)\in\prod_{\lambda\in\Lambda} L \mid x_\lambda=0$ for almost all $\lambda\in\Lambda\}$ is a subspace of $\prod_{n\in N} L_\lambda$. We write $L^\omega=\sum_{n\in N} L_n$, $L_f^\omega=\sum L_n$ provided $L_n=L$ for each $n\in\boldsymbol{N}$. If $(L, 0)$ has a semi-reflective isotopy with an idenpotent swerving homeomorphism, then $(L^\omega, 0)$ and $(L_f^\omega, 0)$ has *RIP*. Then $Q=[-1, 1]^\omega$, $\boldsymbol{s}=(-1, 1)^\omega\cong\boldsymbol{R}^\omega\cong l_2$ and $\boldsymbol{R}_f^\omega$ have *RIP*.

Throughout this paper, let $(E, 0)$ *denote a paracompact, perfectly normal pointed space which has SRIP and is homeomorphic to* $(E^\omega, 0)$ *or* $(E_f^\omega, 0)$.

A *manifold modeled on* E, briefly *E-manifold*, is a paracompact space M admitting an open cover by sets homeomorphic to open subsets of E. If $E=Q$, then M is a Hilbert cube manifold, and if $E=l_2$, then M is a Hilbert manifold. An *E-manifold bundle* is a locally trivial fibre bundle with an E-manifold fibre. An E-manifold bundle with fibre M is briefly called an *M-bundle*. Then an E-bundle is a locally trivial fibre bundle with fibre E.

The Stability Theorem for E-manifold has been established by R. M. Schori [Sch] and its strong version has been done by R. Geoghegan and D. W. Henderson [G-H] (cf. K. Sakai [Sa$_1$]. The stability theorem for product bundles is easily proved (cf. Theorem 4.6 in [Fe]). We present the Stability Theorem and its strong version for E-manifold bundles is Section 4. And in Section 5, we introduce deficiency in E-manifold bundles and we see several easy properties. The bundle version of Mapping Replacement Theorem is proved in Section 6.

§ 2. Reduced Cartesian Products.

Let X and Y be topological spaces and A a closed subset of X. The *product of X and Y reduced over A*, denoted by $(X\times Y)_A$, is defined to be the set $(X\backslash A)\times Y\cup A$ with the topology gererated by the basis consisting of all sets $(U\backslash A)\times V$ and $(U\backslash A)\times Y\cup(U\cap A)$ where U is open in X and V is open in Y. (See [B-P] p. 25). Note that $(X\times Y)_\phi=X\times Y$ and $(X\times Y)_X=X$.

Let $\pi_X=\pi_X^{X\times Y}: X\times Y\to X$, $\pi_Y=\pi_Y^{X\times Y}: X\times Y\to Y$ be the natural projections, that is, $\pi_X(x, y)=x$ and $\pi_Y(x, y)=y$ for each $(x, y)\in X\times Y$. The natural map $\tau^A=\tau_X^{(X\times Y)_A}: (X\times Y)_A\to X$ is defined by $\tau^A|A=\mathrm{id}$ and $\tau^A|(X\backslash A)\times Y=\pi_X$ $(=\pi_{X\backslash Y})$, and the natural map $\tau_A=\tau_{(X\times Y)_A}^{X\times Y}: X\times Y\to(X\times Y)_A$ is defined by $\tau_A|A\times Y=\pi_X$ $(=\pi_A)$ and $\tau_A|(X\backslash A)\times Y=\mathrm{id}$. Then τ^A and τ_A are continuous. Note that $\pi_X=\tau^A\tau_A$, $\tau^\phi=\tau_X=\pi_X$, $\tau^X=\mathrm{id}_X$ and $\tau_\phi=\mathrm{id}_{X\times Y}$.

Obviously if $(X, A)\cong(X', A')$ and $Y\cong Y'$, then $(X\times Y)_A\cong(X'\times Y')_{A'}$. Observe that

$$(X\times(Y\times Z)_B)_A=((X\times Y)_A\times Z)_{A\cup(X\setminus A)\times B},$$

so $X\times(Y\times Z)_B=((X\times Y)\times Z)_{X\times B}$ and $(X\times(Y\times Z))_A=((X\times Y)_A\times Z)_A$.

We shall define the *cone* $\boldsymbol{C}(X)$ and the *open cone* $\boldsymbol{C}^\circ(X)$ of topological space X as reduced products:

$$\boldsymbol{C}(X)=(\boldsymbol{I}\times X)_{\{0\}}\,;\ \boldsymbol{C}^\circ(X)=([0,1)\times X)_{\{0\}}.$$

Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of X. We say that $\mathcal{U}$ is a *refinement* of $\mathcal{V}$ or $\mathcal{U}$ *refines* $\mathcal{V}$, denote $\mathcal{U}<\mathcal{V}$, provided each $U\in\mathcal{U}$ is contained in some $V\in\mathcal{V}$. For $A\subset X$, define $\mathrm{st}(A;\mathcal{U})=\cup\{U\in\mathcal{U}\mid A\cap U\neq\emptyset\}$ and $\mathrm{st}(\mathcal{U})=\{\mathrm{st}(U;\mathcal{U})\mid U\in\mathcal{U}\}$. If $\mathrm{st}(\mathcal{U})<\mathcal{V}$, then $\mathcal{U}$ is called a *star-refinement* of $\mathcal{V}$. We say that a map $f: Y\to X$ is *$\mathcal{U}$-near* to a map $g: Y\to X$ or f and g are *$\mathcal{U}$-near* if for each $y\in Y$, there is some $U\in\mathcal{U}$ containing both $f(y)$ and $g(y)$. And a homotopy (an isotopy) $h: Y\times\boldsymbol{I}\to X$ is a *$\mathcal{U}$-homotopy* (a *$\mathcal{U}$-isotopy*) if for each $y\in Y$, $h(\{y\}\times\boldsymbol{I})$ is contained in some $U\in\mathcal{U}$.

A map $f: B\times X\to B\times Y$ (or $f: X\times B\to Y\times B$) is said to be *$B$-preserving* if $\pi_B f=\pi_B$. When $f: B\times X\to B\times Y$ (or $f: X\times B\to Y\times B$) is B-preserving, for each $b\in B$, define $f_b: X\to Y$ by $f_b(x)=f(b,x)$ (or $=f(x,b)$). Let $p: X\to B$ and $q: Y\to B$ be maps. A map $f: X\to Y$ is *B-preserving* if $qf=p$. A map $g: X\times Z\to Y\times Z'$ is *B-preserving* if $q\pi_Y g=p\pi_X$. And a homotopy $h: X\times\boldsymbol{I}\to Y$ is *B-preserving* if $qh_t=p$ for $t\in\boldsymbol{I}$. If $p: X\to B$ and $q: Y\to B$ are bundles, then a B-preserving continuous map (embedding, homeomorphism, etc.) $f: X\to Y$ is called a *bundle map* (a *bundle embedding*, a *bundle homeomorphism*, etc.) and a B-preserving homotopy (isotopy) $h: X\times\boldsymbol{I}\to Y$ is a called a *bundle homotopy* (a *bundle isotopy*).

§ 3. Main Lemma.

In this section, we will prove the following lemma.

3-1 LEMMA. *Let X be a space such that $X\times E$ is perfectly normal and W an open subspace of $X\times E$. Then for any closed sets A, C and D in W such that $C\cap D=\emptyset$, there exists a homeomorphism $h: (W\times E)_A\to(W\times E)_{A\cup D}$ such that*

i) $\pi_X\tau^{A\cup D}h=\pi_X\tau^A$

ii) $h\mid(C\setminus A)\times E\cup A=\mathrm{id}$

PROOF: According as $(E,0)\cong(E^\omega,0)$ or $(E^\omega_f,0)$, E^* denotes E^ω or E^ω_f. We may assume that W is an open set in $X\times E^*$. We will write $x=(x_0;x_1,x_2,\cdots)\in X\times E^*$. For each $n\in\boldsymbol{N}$, let $\pi_n: X\times E^*\to X\times E^n$ be the natural projection, i.e.

$\pi_n(x)=(x_0; x_1, \cdots, x_n)$. By an *n-basic subset* of $X\times E^*$, we will mean the inverse image of a subset of $X\times E^n$ by π_n, that is, $B\subset X\times E^*$ is n-basic if and only if $\pi_n^{-1}\pi_n(B)=B$. Note that if B is n-basic, then $\pi_n(\operatorname{int} B)=\operatorname{int}\pi_n(B)$, $\pi_n(\operatorname{cl} B)=\operatorname{cl}\pi_n(B)$ and $\pi_n(\operatorname{bd} B)=\operatorname{bd}\pi_n(B)$. Each m-basic set is n-basic for $n\geqq m$. A *basic set* is an n-basic for some n. (See [Sch] p. 89).

Since $(E, 0)$ has *SRIP*, there is a semi-reflective isotopy $\sigma: E^2\times \boldsymbol{I}\to E^2$ with a swerving homeomorphism $e: E\to E$. Define an $\boldsymbol{I}$-preserving continuous map $\theta: (X\times E^*)\times E\times \boldsymbol{I}\to (X\times E^*)\times \boldsymbol{I}$ by

$$\begin{cases} \theta(x, y, 0)=(x, 0) \quad \text{and} \\ \theta(x, y, t)=(x_0; x_1, \cdots, x_{n-1}, \\ \qquad \sigma(x_n, y, 2^n t-1), e(x_{n+1}), e(x_{n+2}), \cdots; t) \\ \qquad \text{if } 2^{-n}\leqq t\leqq 2^{-n+1}. \end{cases}$$

Note that $\theta|(X\times E^*)\times E\times(0, 1]$ is a homeomorphism and that if $t\leqq 2^{-n}$, then $\pi_n\theta_t(x, y)=\pi_n(x)$.

Using normality, construct collections $\mathscr{B}$ and $\mathscr{B}'$ of basic open sets in $X\times E^*$ such that $\bigcup\mathscr{B}=W\backslash(A\cup D)$, $C\cap\operatorname{cl}\bigcup\mathscr{B}'=\emptyset$ and $\bigcup(\mathscr{B}\cup\mathscr{B}')=W\backslash A$. Let $\mathscr{B}_n$ and $\mathscr{B}'_n$ denote the subcollections of $\mathscr{B}$ and $\mathscr{B}'$ consisting of all n-basic sets, respectively. By Lemma 5.2 of [Sch], take collections $\{K_n | n\in\boldsymbol{N}\}$ and $\{K'_n | n\in\boldsymbol{N}\}$ of closed sets in $X\times E^*$ such that $\bigcup_{n\in\boldsymbol{N}} K_n=W\backslash(A\cup D)=\bigcup\mathscr{B}$, $\bigcup_{n\in\boldsymbol{N}} K'_n=\bigcup\mathscr{B}'$ and each K_n and K'_n are n-basic and contained $\operatorname{int} K_{n+1}\cap\bigcup\mathscr{B}_n$ and $\operatorname{int} K'_{n+1}\cap\bigcup\mathscr{B}'_n$ respectively. Then $\bigcup_{n\in\boldsymbol{N}}(K_n\cup K'_n)=W\backslash A$ and each $K_n\cup K'_n$ is n-basic and contained $\operatorname{int}(K_{n+1}\cup K'_{n+1})\cap\bigcup(\mathscr{B}_n\cup\mathscr{B}'_n)$.

From Tietze Extension Theorem, there are continuous maps $f_n: \pi_n(K_n\backslash\operatorname{int} K_{n-1})\to[2^{-n-1}, 2^{-n}]$ and $f'_n: \pi_n(K_n\cup K'_n\backslash\operatorname{int}(K_{n-1}\cup K'_{n-1}))\to[2^{-n-1}, 2^{-n}]$ such that

$$f_n(\operatorname{bd}\pi_n(K_n))=f'_n(\operatorname{bd}\pi_n(K_n\cup K'_n))=2^{-n-1} \quad \text{and}$$
$$f_n(\operatorname{bd}\pi_n(K_{n-1}))=f'_n(\operatorname{bd}\pi_n(K_{n-1}\cup K'_{n-1}))=2^{-n}$$

where $K_0=K'_0=\emptyset$. Put $n(x)=\min\{n\in\boldsymbol{N} | x\in K_n\}$ for each $x\in W\backslash(A\cup D)$ and $m(x)=\min\{n\in\boldsymbol{N} | x\in K_n\cup K'_n\}$ for each $x\in W\backslash A$, and define continuous maps $f: W\backslash(A\cup D)\to(0, 1]$ and $f': W\backslash A\to(0, 1]$ by

$$f(x)=f_{n(x)}\pi_{n(x)}(x) \quad \text{and} \quad f'(x)=f'_{m(x)}\pi_{m(x)}(x).$$

These are well-defined because each K_n and $K_n\cup K'_n$ are n-basic. Note that

$$f(x)=f(x_0; x_1, \cdots, x_{n(x)}, *, *, \cdots)\leqq 2^{-n(x)}$$

for each $x\in W\backslash(A\cup D)$, and

$$f'(x)=f'(x_0; x_1, \cdots, x_{m(x)}, *, *, \cdots)\leqq 2^{-m(x)}$$

for each $x\in W\setminus A$, and that $m(x)\leqq n(x)$ for each $x\in W\setminus(A\cup D)$, and moreover if $x\notin \bigcup\mathcal{B}'$, then $n(x)=m(x)$. Take a continuous map $k: W\to \boldsymbol{I}$ such that $k(C)=0$ and $k(\mathrm{cl}\bigcup\mathcal{B}')=1$. And define a continuous map $g: W\setminus A\to(0, 1]$ by

$$g(x)=\begin{cases} f'(x) & \text{if } x\in D \\ (1-k(x))f(x)+k(x)f'(x) & \text{if } x\notin D. \end{cases}$$

Then observe that $g|C\setminus A=f|C\setminus A$ and

$$g(x)=g(x_0: x_1, \cdots, x_{m(x)}, *, *, \cdots)\leqq 2^{-m(x)}$$

for each $x\in W\setminus A$.

Now define $h_f: (W\times E)_{A\cup D}\to W$ and $h_g: (W\times E)_A\to W$ by

$$\begin{cases} h_f|A\cup D=\mathrm{id} \\ h_f(x, y)=\theta_{f(x)} & \text{for each } (x, y)\in(W\setminus(A\cup D))\times E \end{cases}$$

and

$$\begin{cases} h_g|A=\mathrm{id} \\ h_g(x, y)=\theta_{g(x)}(x, y) & \text{for each } (x, y)\in(W\setminus A)\times E. \end{cases}$$

Then $h_f|(C\setminus A)\times E\cup A=h_g|(C\setminus A)\times E\cup A$.

Now, we will show that h_f and h_g are homeomorphisms. Then $h_f^{-1}h_g: (W\times E)_A\to(W\times E)_{A\cup D}$ is clearly a desired homeomorphism. From similarity, we may check up h_f alone.

Continuity of h_f: Since $h_f|(W\setminus(A\cup D))\times E$ is continuous, we have to examine that h_f is continuous at $x\in A\cup D$. Let V be an n-basic neighbourhood of x in W. Since $K_n\cap(A\cup D)=\emptyset$, $V\setminus K_n$ is a neighbourhood of x in W, so

$$U=((V\setminus K_n)\setminus(A\cup D))\times E\cup((V\setminus K_n)\cap(A\cup D))$$

is a neighbourhood of x in $(W\times E)_{A\cup D}$. For $(x', y')\in((V\setminus K_n)\setminus(A\cup D))\times E$, $x'\notin K_n$ implies $n(x')>n$ therefore $f(x')\leqq 2^{-n(x')}<2^{-n}$. Then

$$\pi_n h_f(x', y')=\pi_n\theta_{f(x')}(x', y')=\pi_n(x')\in\pi_n(V)$$

so $h_f(x', y')\in\pi_n^{-1}\pi_n(V)=V$. Hence $h_f(U)\subset V$.

Inverse of h_f: Define $h'_f: W\to(W\times E)_{A\cup D}^{-}$ by

$$h'_f(x)=\begin{cases} x & \text{if } x\in A\cup D \\ \theta_{f(x)}^{-1}(x) & \text{if } x\notin A\cup D. \end{cases}$$

For each $x\in W\setminus(A\cup D)$ put $(x', y')=\theta_{f(x)}^{-1}(x)\in(W\setminus(A\cup D))\times E$. Since $x=\theta_{f(x)}(x', y')$ and $f(x)\leqq 2^{-n(x)}$,

$$\pi_{n(x)}(x)=\pi_{n(x)}\theta_{f(x)}(x', y')=\pi_{n(x)}(x')$$

therefore $f(x)=f(x')$. Hence

$$\begin{aligned} h_f(h'_f(x)) &= h_f(x', y') \\ &= \theta_{f(x')}(x', y') \\ &= \theta_{f(x)}(\theta_{f(x)}^{-1}(x)) \\ &= x . \end{aligned}$$

For each $(x, y)=(W\backslash(A\cup D))\times E$, put $x'=\theta_{f(x)}(x, y)\in W\backslash(A\cup D)$. Similarly as above, $f(x)=f(x')$. Hence

$$\begin{aligned} h'_f(h_f(x, y)) &= h'_f(x') \\ &= \theta_{f(x')}^{-1}(x') \\ &= \theta_{f(x)}^{-1}(\theta_{f(x)}(x, y)) \\ &= (x, y) . \end{aligned}$$

Therefore $h'_f=h_f^{-1}$.

Continuity of $h_f^{-1}=h'_f$: Since $h'_f|W\backslash(A\cup D)$ is continuous, we have to examine that h'_f is continuous at $x\in A\cup D$. Let V be an n-basic neighbourhood of x in W. Note that $V\backslash K_n$ is a neighbourhood of x in W. For $x'\in(V\backslash K_n)\backslash(A\cup D)$, put $h'_f(x')=(x'', y'')$. Then $\pi_{n(x')}(x')=\pi_{n(x')}(x'')$, so $\pi_n(x')=\pi_n(x'')$ because $n<n(x')$. Since V is n-basic, $x''\in V$ that is $h'_f(x')=(x'', y'')\in(V\backslash(A\cup D))\times E$. Hence

$$h'_f(V\backslash K_n)\subset(V\backslash(A\cup D))\times E\cup(V\cap(A\cup D)) . \qquad \square$$

3-2 Remark: In the above proof, note that

$$\theta((x, 0), 0, t)=(x, 0, t)$$

for each $((x, 0), 0, t)\in X\times E^*\times E\times \boldsymbol{I}$, then

$$h_f^{-1}h_g((x, 0), 0)=\begin{cases} ((x, 0), 0) & \text{if } (x, 0)\in(W\backslash(A\cup D))\cap X\times\{0\} \\ (x, 0) & \text{if } (x, 0)\in(D\backslash A)\cap X\times\{0\} . \end{cases}$$

Hence *we can require a homeomorphism h in Lemma* 3-1 *to satisfy*

iii) $h|((W\backslash A)\cap X\times\{0\})\times\{0\}=\tau_{A\cup D}$.

In the above proof, put $A=D=\emptyset$, construct $\mathcal{B}$ so fine that $\mathrm{st}(\mathcal{B})<\mathcal{U}$ for an open cover $\mathcal{U}$ of W and define $\Theta^{\mathcal{U}}:(X\times E^*)\times E\times\boldsymbol{I}\to(X\times E^*)\times\boldsymbol{I}$ by

$$\Theta^{\mathcal{U}}(x, y, t)=\begin{cases} (\theta_{tf(x)}(x, y), t) & \text{if } x\in W \\ (x, t) & \text{if } x\notin W . \end{cases}$$

Then note that $\Theta^{\mathcal{U}}$ is X-preserving because θ is so. From the proof of Lemma

2-1 of [Sa_1], we have the following lemma:

3-3 LEMMA: *Let X be a space such that $X \times E$ is perfectly normal and W an open subspace of $X \times E$. Then for each open $\mathcal{U}$ of W, there exists an X- and $\boldsymbol{I}$-preserving continuous map $\Theta^{\mathcal{U}}: X \times E \times E \times \boldsymbol{I} \to X \times E \times \boldsymbol{I}$ such that*

i) $\Theta^{\mathcal{U}}(x, 0, 0, t) = (x, 0, t)$ *for each* $(x, 0, 0, t) \in X \times E \times E \times \boldsymbol{I}$,

ii) $\Theta^{\mathcal{U}}_0 = \pi_{X \times E}$,

iii) $\Theta^{\mathcal{U}}_t | ((X \times E) \setminus W) \times E = \pi_{X \times E}$ *for each* $t \in \boldsymbol{I}$,

iv) $\Theta^{\mathcal{U}} | W \times E \times (0, 1]: W \times E \times (0, 1] \to W \times (0, 1]$ *is a homeomorphism,*

v) $\Theta^{\mathcal{U}} | W \times \{0\} \times \boldsymbol{I}: W \times \{0\} \times \boldsymbol{I} \to W \times \boldsymbol{I}$ *is a closed embedding, and*

vi) *for each* $(x, y) \in W$, *there is some* $U \in \mathcal{U}$ *such that* $\Theta^{\mathcal{U}}(\{(x, y)\} \times E \times \boldsymbol{I}) \subset U \times \boldsymbol{I}$.

§4. Stability Theorem for Infinite-Dimensional Bundles.

In [Mi], E. Micheal established a useful criterion for a topological property $\mathcal{P}$ in order that the implication "if a topological space X has $\mathcal{P}$ locally, then X has $\mathcal{P}$" hold. In the proof of his theorem, he actually proved the following:

4-1 THEOREM (Micheal): *Let X be a paracompact* (i.e. *fully normal*) *space and $\mathcal{G}$ an open cover of X which satisfies the following conditions:*

a) *U is open in X and* $U \subset V \in \mathcal{G} \Rightarrow U \in \mathcal{G}$.

b) $U, V \in \mathcal{G} \Rightarrow U \cup V \in \mathcal{G}$.

c) *For any discrete subcollection* $\{\mathcal{B}_\lambda | \lambda \in \Lambda\}$ *of* $\mathcal{G}$, $\bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{G}$.

Then $X \in \mathcal{G}$.

Using this theorem, we establish the stability theorem for a locally trivial fibre bundle with fibre M a manifold modeled on $E \cong E^\omega$ or E^ω_f which has *SRIP*. It is a bundle version of Schori Stability Theorem (Theorem 5.10 in [Sch]).

4-2 BUNDLE STABILIEY THEOREM: *Let $p: X \to B$ be an E-manifold bundle such that $X \times E$ and $B \times E$ are paracompact, perfectly normal. Then $p\pi_X: X \times E \to B$ is bundle isomorphic to $p: X \to B$.*

PROOF: Let $\mathcal{G}$ is the collection of all open sets in X whose each open subset W satisfies the following condition:

(*) For any closed sets A, C and D in W such that $C\cap D=\emptyset$, there exists a homeomorphism $h:(W\times E)_A\to(W\times E)_{A\cup D}$ such that $h|(C\backslash A)\times E\cup A=\mathrm{id}$, $p\tau^{A\cup D}h=p\tau^A$.

Then $\mathcal{G}$ is an open cover of X, that is, each $x\in X$ has a neighbourhood which is a member of $\mathcal{G}$. In fact, there are an open neighbourhood U of $p(x)$ in B and a homeomorphism $f: p^{-1}(U)\to U\times M$ such that $\pi_U f=p$, where M is an E-manifold which is the fibre of $p: X\to B$. And there are an open neighbourhood V of $\pi_M f(x)$ in M homeomorphic to an open set in E. From Lemma 3-1, it is easily shown that each open subset of $f^{-1}(U\times V)$ satisfies the condition (*).

Now we will see that $\mathcal{G}$ satisfies the conditions a), b) and c) in Theorem 4-1. Then it follows $X\in\mathcal{G}$, therefore there exists a homeomorphism $h:(X\times E)_\emptyset=X\times E\to(X\times E)_X=X$ such that $ph=p\pi_X$.

Obviously, conditions a) and c) are satisfied. To see condition b), let $W=W'\cup W''$ where W' and W'' satisfy (*) and A, C and D closed sets in W so that $C\cup D=\emptyset$. Since W is normal, there are open sets V' and V'' in W such that $\mathrm{cl}_W V'\cap\mathrm{cl}_W V''=\emptyset$, $W\backslash W''\subset V'$ and $W\backslash W^*\subset V''$.

Let V be an open set in W so that $W\backslash W'\subset V\subset\mathrm{cl}_W V\subset V''$. Put $A'=A\cap W'$, $C'=(C\cup\mathrm{cl}_W V)\cap W'$ and $D'=D\backslash V''$. Since W' satisfies (*), there exists a homeomorphism $h':(W'\times E)_{A'}\to(W'\times E)_{A'\cup D'}$ such that $h'|(C'\backslash A')\times E\cup A'=\mathrm{id}$ and $p\tau^{A'\cup D'}h'=p\tau^{A'}$. Define a homeomorphism $h_1:(W\times E)_A\to(W\times E)_{A\cup D'}$ by $h_1|(W'\times E)_{A'}=h'$ and $h_1|(W\times E)_A\backslash(W'\times E)_{A'}=\mathrm{id}$. Then $h_1|(C\backslash A)\times E\cup A=\mathrm{id}$ and $p\tau^{A\cup D'}h_1=p\tau^A$.

Put $A''=(A\cup D')\cap W''$, $C''=(C\cup\mathrm{cl}_W V')\cap W''$ and $D''=D\cap\mathrm{cl}_W V''$, then using above argument, we obtain a homeomorphism $h_2:(W\times E)_{A\cup D'}\to(W\times E)_{A\cup D}$ such that $h_2|(C\backslash(A\cup D'))\times E\cup(A\cup D')=\mathrm{id}$ and $p\tau^{A\cup D}h_2=p\tau^{A\cup D'}$.

Then $h=h_2h_1:(W\times E)_A\to(W\times E)_{A\cup D}$ is a desired homeomorphism. □

From 3-3 and 4-2, we can easily obtain the following strong version of 4-2 which is a bundle version of Geoghegan-Henderson Strong Stability Theorem [G-H] and Theorem 2-2 in [Sa_1].

4-3 Strong Bundle Stability Theorem: *Let $p: X\to B$ be an E-manifold bundle such that $X\times E$ and $B\times E$ paracompact, perfectly normal and let W be an open set in X. Then for each open cover $\mathcal{U}$ of W, there exists an $\boldsymbol{I}$-preserving continuous map $\Delta^{\mathcal{U}}: X\times E\times\boldsymbol{I}\to X\times\boldsymbol{I}$ such that*

i) *$p\Delta_t^{\mathcal{U}}=p\pi_X$ for each $t\in\boldsymbol{I}$,*

ii) *$\Delta_0^{\mathcal{U}}=\pi_X$,*

iii) *$\Delta_t^{\mathcal{U}}|(X\backslash W)\times E=\pi_X$ for each $t\in\boldsymbol{I}$,*

iv) *$\Delta^{\mathcal{U}}|W\times E\times(0, 1]$ is a homeomorphism, and*

v) *for each* $x \in W$, *there is some* $U \in \mathcal{U}$ *such that* $\Delta^{\mathcal{U}}(\{x\} \times E \times \boldsymbol{I}) \subset U \times \boldsymbol{I}$.

PROOF: Let $h: X \to X \times E$ be a bundle homeomorphism. Then $\Delta^{\mathcal{U}} = (h^{-1} \times \mathrm{id}_I) \cdot \Theta^{h(\mathcal{U})}(h \times \mathrm{id}_{E \times I})$ fulfills our requirements. □

In particular, it follows from the above theorem that

I) *for each open cover* $\mathcal{U}$ *of* X, *there exists a bundle homeomorphism* $h: X \times E \to X$ *homotopic to the projection* $\pi_X: X \times E \to X$ *by a bundle* $\mathcal{U}$*-homotopy; and*

II) *for each open set* W, *there exists* B*-preserving homemorphisms* $g: W \times E \to W$ B*-preservingly homotopic to the projection* $\pi_W: W \times E \to W$.

§ 5. Deficiency in Bundles.

Let $p: X \to B$ be a map. A subset K of X is said to be *B-preservingly E-deficient in X (with respect to $p: X \to B$)* if there exists a homeomorphism $h: X \to X \times E$ such that $p\pi_X h = p$ and $\pi_E h(K) = 0$ (i.e. $h(K) \subset X \times \{0\}$). And if each $x \in K$ has a neighbourhood W in X such that $K \cap W$ is B-preservingly E-deficient in W with respect to $p|W: W \to B$, then K is said to be *locally B-preservingly E-deficient in X (with respect to $p: X \to B$)*.

From Bundle Stability Theorem 4-2 and its strong version 4-3, these notion of deficiency and local deficiency have the sense for E-manifold bundles.

Throughout the following, *let $p: X \to B$ denote an E-manifold bundle such that $X \times E$ and $B \times E$ are paracompact, perfectly normal.*

First, we remark the following:

5-1 REMARK: *In* 4-3, *let K be a B-preservingly E-deficient set in X.* In the proof, using a bundle homeomorphism $h: X \to X \times E$ such that $h(K) \subset X \times \{0\}$, *we can require* $\Delta^{\mathcal{U}}$ *to satisfy*

vi) $\Delta_t^{\mathcal{U}} | K \times \{0\} = \pi_X$ *for each* $t \in \boldsymbol{I}$.

This remark yields the following:

5-2 PROPOSITION: *If K is a B-preservingly E-deficient in X, then there exists a bundle homeomorphism $h: X \to X \times E$ such that $h(x) = (x, 0)$ for each $x \in K$.*

And moreover if W is an open subset of X, then $K \cap W$ is B-preservingly E-deficient in W.

Now, we will show that any locally B-preservingly E-deficient set is B-

preservingly E-deficient.

5-3 THEOREM: *If K is a locally B-preservingly E-deficient set in X, then K is B-preservingly E-deficient in X.*

PROOF: Let $\mathcal{G}_K$ be the collection of all open sets in X whose each open subset W satisfied the following condition:

$(*)_K$ For any closed sets A, C and D in W such that $C \cap D = \emptyset$, there exists a homeomorphism $h: (W \times E)_A \to (W \times E)_{A \cup D}$ such that $h|(C \setminus A) \times E \cup A = \mathrm{id}$, $p\tau^{A \cup D} h = p\tau^A$ and $h|(K \setminus A) \times \{0\} = \tau_{A \cup D}$.

Using Remark 3-2, it is same as 4-2 to see that $\mathcal{G}_K$ is an open cover of X and that $\mathcal{G}_K$ satisfies the condition b) in Theorem 4-1. It is clear that conditions a) and c) in 4-1 are satisfied. Then the result follows from Theorem 4-1. □

The following corollary is a direct consequence on 5-3.

5-4 COROLLARY: *A necessary and sufficient condition that K is B-preservingly E-deficient in X is that for each $x \in B$, there exist a neighbourhood U of x in B and a bundle homemorphism $h: p^{-1}(U) \to U \times M$ such that $\pi_M h(K \cap p^{-1}(U))$ is E-deficient in M, where M is an E-manifold which is the fibre of $p: X \to B$.*

In the following, we will show that a finite union of B-preservingly E-deficient sets in X is also B-preservingly E-deficient in X. *We must assume that $(\boldsymbol{C}(E), 0) \cong (E, 0)$. The Hilbert cube Q and any locally convex linear metric space F homeomorphic to F^ω or to F_f^ω satisfy this assumption.* It is well known that $\boldsymbol{C}(Q) \cong Q$ and Q is homogeneous (cf. [Ch₂]), then these imply $(\boldsymbol{C}(Q), 0) \cong (Q, 0)$. Since $F \cong \boldsymbol{C}^\circ(F)$ by Lemma 2 in [He] (with a remark in the proof of Theorem 3.1 in [Ch₁]) and $F \times (0, 1] \cong F$, $\boldsymbol{C}(F)$ is an F-manifold. From contractibility of $\boldsymbol{C}(F)$, $\boldsymbol{C}(F) \cong F$ by Classification Theorem in [He]. (Using Negligibility Theorem in [Cu₁], $\boldsymbol{C}(F) \cong \boldsymbol{C}(F) \setminus F \times \{1\} = \boldsymbol{C}^\circ(F) \cong F$ because $F \times \{1\}$ is F-deficient closed in $\boldsymbol{C}(F)$.) Our theorem (5-6) is valid for not closed sets, thus it is an extension of Proposition 5.3 in [Cu₂].

5-5 LEMMA: *If $(\boldsymbol{C}(E), 0) \cong (E, 0)$, then there is a homeomorphism $f: \boldsymbol{I} \times E \to \boldsymbol{C}(E) = (\boldsymbol{I} \times E)_{\{0\}}$ such that $f|\boldsymbol{I} \times \{0\} = \tau_{\{0\}}$, that is, $f(0, 0) = 0$ and $f(t, 0) = (t, 0)$ for each $t \in (0, 1]$. So $(\boldsymbol{C}(E), 0) \cong (E, 0)$ implies $(E \times \boldsymbol{I}, (0, 0)) \cong (E, 0)$.*

PROOF: Let $h: E \to \boldsymbol{C}(E) = (\boldsymbol{I} \times E)_{\{0\}}$ be a homeomorphism such that $h(0) = 0$. Then h induces a homeomorphism $h^*: (\boldsymbol{I} \times E)_{\{0\}} \to (\boldsymbol{I} \times (\boldsymbol{I} \times E)_{\{0\}})_{\{0\}}$ defined by $h^*(0) = 0$ and $h^*|(0, 1] \times E = \mathrm{id}_{(0,1]} \times h$. Observe that

$$(\boldsymbol{I}\times(\boldsymbol{I}\times E)_{\{0\}})_{\{0\}}=((\boldsymbol{I}\times\boldsymbol{I})_{\{0\}}\times E)_{\{0\}\cup(0,1]\times\{0\}}$$

and that

$$\boldsymbol{I}\times(\boldsymbol{I}\times E)_{\{0\}}=((\boldsymbol{I}\times\boldsymbol{I})\times E)_{\boldsymbol{I}\times\{0\}}\,.$$

We can easily constract a homeomorphism $g:\boldsymbol{I}\times\boldsymbol{I}\rightarrow(\boldsymbol{I}\times\boldsymbol{I})_{\{0\}}$ so that $g|\boldsymbol{I}\times\{0\}=\tau_{\{0\}}$. This g induces a homeomorphism

$$g^*:((\boldsymbol{I}\times\boldsymbol{I})\times E)_{\boldsymbol{I}\times\{0\}}\rightarrow((\boldsymbol{I}\times\boldsymbol{I})_{\{0\}}\times E)_{\{0\}\cup(0,1]\times\{0\}}$$

defined by $g^*|\boldsymbol{I}\times\{0\}=g|\boldsymbol{I}\times\{0\}$ and $g^*|\boldsymbol{I}\times(0,1]\times E=(g|\boldsymbol{I}\times(0,1])\times\mathrm{id}_E$.

Now define $f=h^{*-1}g^*(\mathrm{id}_I\times h):\boldsymbol{I}\times E\rightarrow(\boldsymbol{I}\times E)_{\{0\}}$.

$$\begin{array}{c}
\boldsymbol{I}\times E\\
\Big\downarrow{\scriptstyle \mathrm{id}_I\times h}\\
\boldsymbol{I}\times(\boldsymbol{I}\times E)_{\{0\}}\\
\|\\
((\boldsymbol{I}\times\boldsymbol{I})\times E)_{\boldsymbol{I}\times\{0\}}\\
\Big\downarrow{\scriptstyle g^*}\\
((\boldsymbol{I}\times\boldsymbol{I})_{\{0\}}\times E)_{\{0\}\cup(0,1]\times E}\\
\|\\
(\boldsymbol{I}\times(\boldsymbol{I}\times E)_{\{0\}})_{\{0\}}\\
\Big\downarrow{\scriptstyle h^{*-1}}\\
(\boldsymbol{I}\times E)_{\{0\}}
\end{array}$$

For $t\in(0,1]$, $f(t,0)=h^{*-1}g^*(t,0)=h^{*-1}(t,0)=(t,0)$ and $f(0,0)=h^{*-1}g^*(0,0)=h^{*-1}(0)=0$. Hence f is a desired homeomorphism. □

5-6 THEOREM: *Assume* $(\boldsymbol{C}(E),0)\cong(E,0)$. *Then a finite union of B-preservingly E-deficient sets in X is also B-preservingly E-deficient.*

PROOF: Let K and L be B-preservingly E-deficient in X. We may show that $K\cup L$ is B-preservingly E-deficient in X. Since $(E\times\boldsymbol{I},(0,0))\cong(E,0)$, there is a bundle homeomorphism $g:X\rightarrow X\times\boldsymbol{I}$ such that $g(L)\subset X\times\{0\}$. Put $A=g^{-1}(X\times\{0\})$. Then g induces a B-preserving homeomorphism $g^*:(X\times E)_A\rightarrow((X\times\boldsymbol{I})\times E)_{X\times\{0\}}$ defined by $g^*|A=g|A$ and $g^*|(X\backslash A)\times E=(g|X\backslash A)\times\mathrm{id}_E$. By 5-5, there is a homeomorphism $f:\boldsymbol{I}\times E\rightarrow(\boldsymbol{I}\times E)_{\{0\}}$ such that $f|\boldsymbol{I}\times\{0\}=\tau_{\{0\}}$.

$$\begin{array}{ccc} X\times E & \xrightarrow{\quad g\times \mathrm{id}_E \quad} & X\times \boldsymbol{I}\times E \\ \Big\downarrow h' & & \Big\downarrow \mathrm{id}_X\times f \\ & & X\times(\boldsymbol{I}\times E)_{\{0\}} \\ & & \parallel \\ (X\times E)_A & \xleftarrow[\quad g^{*^{-1}} \quad]{} & ((X\times \boldsymbol{I})\times E)_{X\times\{0\}} \end{array}$$

Then $h'=g^{*-1}(\mathrm{id}_X\times f)(g\times \mathrm{id}_E): X\times E\to(X\times E)_A$ is a B-preserving homeomorphism such that $h'|X\times\{0\}=\tau_A|X\times\{0\}$.

From proof of 5-3, there exists a B-preserving homeomorphism $h'':(X\times E)_A \to X$ such that $h''|A\cup(K\setminus A)\times\{0\}=\tau^A$. Then $h=h''h': X\times E\to X$ is a bundle homeomorphism such that $h|(K\cup L)\times\{0\}=\pi_X$. Hence $K\cup L$ is B-preservingly E-deficient in X. □

§ 6. **Mapping Replacement.**

Recall our assumption that $p: X\to B$ *is an E-manifold bundle such that* $X\times E$ *and* $B\times E$ *are paracompact, perfectly normal.*

In this section, we will prove two theorems, using results in Section 3. The first theorem is a bundle version of Theorem 4.1 in [Ch_1] (Theorem 2-5 in [Sa_1]).

6-1 THEOREM: *Let K be a B-preservingly E-deficient subset of X. Then for each open cover* $\mathcal{U}$ *of X, there exists an invertible bundle* $\mathcal{U}$*-isotopy* $g_t: X\to X$ $(t\in\boldsymbol{I})$ *such that*

i) $g_0=\mathrm{id}$,

ii) $g_t|K=\mathrm{id}$ *for each* $t\in\boldsymbol{I}$, *and*

iii) $g_t(X)$ *is a B-preservingly E-deficient closed set in X for each* $t\in(0, 1]$.

PROOF: Since K is B-preservingly E-deficient in X, there is a bundle homeomorphism $h: X\to X\times E$ such that $h(K)\subset X\times\{0\}$. Define a closed embedding $i: X\times E\to X\times E\times E$ by $i(x, y)=(x, y, 0)$. Then $g=h^{-1}\pi_{X\times E}\Theta^{h(\mathcal{U})}(ih\times \mathrm{id}_I): X\times \boldsymbol{I}\to X$ is a desired isotopy, where $\Theta^{h(\mathcal{U})}$ is a map in Lemma 3-3. □

The second theorem is a bundle version of Mapping Replacement Theorem due to R. D. Anderson and J. D. McCharen [A-M] (Lemma 5.1 in [Ch_1]; Theorem 3-1 in [Sa_1]). In case of a product Q-manifold bundle, it has been proved (Proposition 4.9 in [Fe], Corollary 2.4 in [C-F]). In the following, we assume metrizability of B and E, hence metrizability of all spaces and that $(E\times \boldsymbol{I}, 0)\cong(E, 0)$. The Hilbert cube Q and any linear metric space F homeomorphic to F^{ω} or to F_f^{ω} have this property.

6-2 MAPPING REPLACEMENT THEOREM: *Assume that E and B are metrizable and that $(E\times \boldsymbol{I}, 0)\cong(E, 0)$. Let $Y\supset Z$ be closed subsets of $B\times E$. If $f: Y\to X$ is a B-preserving continuous map such that $f|Z$ is a closed embedding and $f(Z)$ is B-preservingly E-deficient in X, then for each open cover $\mathcal{U}$ of X, there is a B-preserving $\mathcal{U}$-homotopy $f^*: Y\times \boldsymbol{I}\to X$ such that*

i) $f_0^*=f$,

ii) $f_t^*|Z=f|Z$ *for each* $t\in \boldsymbol{I}$,

iii) $f_1^*: Y\to X$ *is a closed embedding, and*

iv) $f_1^*(Y)$ *is B-preservingly E-deficient in* X.

PROOF (cf. Proof of Theorem 3-1 in [Sa$_1$]): According as $E\cong E^{\omega}$ of $E\cong E_f^{\omega}$, E^* denotes E^{ω} or E_f^{ω}. Note that $(E, 0)\cong(E^*\times \boldsymbol{I}, 0)$. Let d and d^* be metrics on Y and $X\times E^*\times \boldsymbol{I}$, respectively, defined as follows

$$d(y, y')=d_Y(y, y')+d_X(f(y), f(y'))$$

and

$$d^*((x, z, t), (x', z', t'))=d_X(x, x')+\sum_{i=1}^{\infty}2^{-i}d_E(z_i, z_i')+2^{-1}|t-t'|$$

where d_X, d_Y and d_E are metrics bounded by 1/4 on X, Y and E respectively.

Let $\mathcal{V}$ be a star-refinement of $\mathcal{U}$. From Theorem 6-1, we have an invertible bundle $\mathcal{V}$-isotopy $g: X\times \boldsymbol{I}\to X$ such that $g_0=\mathrm{id}$, $g_t|f(Z)=\mathrm{id}$ for each $t\in \boldsymbol{I}$ and $g_1(X)$ is B-preservingly E-deficient closed in X. Let $h: X\to X\times E^*\times \boldsymbol{I}$ be a homeomorphism so that $p\pi_X h=p$ and $hg_1(x)=(g_1(x), 0, 0)$ for each $x\in X$. Using the above metrics, define a continuous map $k: Y\to[0, 1/2]$ by

$$k(y)=d(y, Z)=\inf\{d(y, y')\,|\,y'\in Z\}$$

and a continuous map $e: X\times E^*\times \boldsymbol{I}\to \boldsymbol{I}$ by

$$e(x, z, t)=\sup\{d^*((x, z, t), X\times E^*\times \boldsymbol{I}\setminus h(V))\,|\,V\in\mathcal{V}\}.$$

(Since $|e(x, z, t)-e(x', z', t')|<d^*((x, z, t), (x', z', t'))$, e is continuous. This map e is called a majorant for with respect to d^* in [Sa$_1$]; see [Cu] 2.)

Now, let $\theta: X\times E^*\times E\times \boldsymbol{I}\to X\times E^*\times \boldsymbol{I}$ be the X- and $\boldsymbol{I}$-preserving continuous map defined in the proof of Lemma 3-1 and define a homotopy $f': Y\times \boldsymbol{I}\to X$ by

$$f_t'(y)=h^{-1}\theta(g_1f(y), 0, \pi_E(y), tk(y)ehg_1f(y)),$$

Note that $(pg_1f(y), \pi_E(y))=(\pi_B(y), \pi_E(y))=y$ for each $y\in Y$. Then by the same arguments in the proof of Theorem 3-1 in [Sa_1], a homotopy $f^*: Y\times I\to X$ by

$$f_t^*(y)=\begin{cases} g_{2t}f(x) & \text{if } 0\leqq t\leqq 1/2 \\ f'_{2t-1}(x) & \text{if } 1/2\leqq t\leqq 1 \end{cases}$$

fulfills our requirements. □

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