TSUKUBA J. MATH.
Vol. 4 (1980). 67–75

ON EXTENSIONS OVER ARTINIAN RINGS WITH SELF-DUALITIES

(Dedicated to Professor Goro Azumaya on his 60th birthday)

By

Kunio YAMAGATA

Let A be a left and right Artinian ring with an A-bimodule Q such that both ${}_AQ$ and Q_A are finitely generated, $\mathrm{soc}(Q_A)\cong\mathrm{top}(A_A)$ and $\mathrm{soc}({}_AQ)\cong\mathrm{top}({}_AA)$. Such a bimodule Q is called a *QF-module* [3]. Let T be a ring extension over A with kernel Q [2]. In this note we study when a given right T-module M has a waist MQ.

In §1 we will recall definitions and some properties of extensions from [3]. In §2 it will be given some criteria for a module M to have a waist MQ. In particular, for the trivial extension $T=A\ltimes Q$ it will be proved that an indecomposable projective right T-module P has a waist PQ if and only if every nonzero morphism from P/PQ to any indecomposable projective right A-module is monomorphic; if and only if the indecomposable projective left T-module $\mathrm{Hom}_T(P, T)$ has a waist $Q\,\mathrm{Hom}_T(P, T)$.

Throughout this paper, Artinian rings will be left and right Artinian and all modules will be finitely generated.

1. Preliminaries

Let A and T be Artinian rings such that there is a ring epimorphism $\rho: T\to A$. Then a given A-bimodule Q may be regarded as a T-bimodule, by setting

$$t_1qt_2=\rho(t_1)q\rho(t_2),\quad q\in Q,\quad t_1, t_2\in T.$$

Moreover, if $\mathrm{Ker}\,\rho$ is isomorphic to Q as T-bimodules, T is said to be an extension over A with kernel Q [2]. In this case, Q will be identified with the ideal $\mathrm{Ker}\,\rho$ in T, if there is no confusion. In the following we will recall from [3] some properties of the extensions. Let T be an extension over A with kernel $Q: 0\to Q\xrightarrow{\kappa} T\xrightarrow{\rho} A\to 0$. Then $Q^2=0$ in T, and hence every idempotent in A is lifted to T. Let e and $\boldsymbol{e}$ be idempotents in A and T, respectively, such

Received October 25, 1979.

that $\rho(e)=e$. Then for any A-module M, it holds that $me=me$ for $m\in M$, by regarding M as a T-module via ρ. If ρ is a splittable ring epimorphism, T is called a trivial extension of A by Q and it is isomorphic to the following extension

$$0 \longrightarrow Q \xrightarrow{\kappa_0} A\ltimes Q \xrightarrow{\rho_0} A \longrightarrow 0,$$

where $A\ltimes Q$ is a direct sum $A\oplus Q$ as additive groups with the multiplication

$$(a_1, q_1)(a_2, q_2)=(a_1a_2, a_1q_2+q_1a_2)$$

for $a_1, a_2\in A$ and $q_1, q_2\in Q$, and

$$\kappa_0(q)=(0, q), \quad \rho_0(a, q)=a \quad \text{for } a\in A,\ q\in Q.$$

LEMMA 1.1. [3, 1.2]. *Let A be an Artinian ring with a QF-module Q. Then every extension over A with kernel Q is quasi-Frobenius.*

By this lemma, for any extension T there is a duality between mod T and mod T°, where mod T denotes the category of finitely generated right T-modules and T° the opposite ring of T, more precisely $\mathrm{Hom}_T(\ \ , T)$ defines a duality between mod T and mod T°. We denote the functor $\mathrm{Hom}_T(\ \ , T)$ by $(\ \ )^*$. Let M be a right T-module and I a subset of T. Then $\boldsymbol{l}_M(I)$ denotes the (left) annihilator of I in M. Similarly, for a left T-module N, $\boldsymbol{r}_N(I)$ denotes the (right) annihilator of I in N. The following lemma is easily proved (cf. [3, 2.1]).

LEMMA 1.2. *Let A be an Artinian ring with a QF-module Q and T an extension over A with kernel Q. Then for a projective right T-module P, the following statements hold.*

(1) *PQ is injective in* mod A.

(2) *$PQ=\boldsymbol{l}_P(Q)$.*

(3) *P/PQ is projective in* mod A.

2. Modules M with waists MQ.

To begin with, we will prove that a QF-module is just the module which defines a duality between finitely generated left A-modules and finitely generated right A-modules.

PROPOSITION 2.1. *Let A be an Artinian ring with a QF-module Q. Then* $\mathrm{Hom}_A(\ \ , Q)$ *defines a duality between* mod A *and* mod A°. *Moreover, for every extension T over A with kernel Q, the functor* $\mathrm{Hom}_T(\ \ , T)$ *is equivalent to* $\mathrm{Hom}_A(\ \ , Q)$ *on* mod A *and* mod A°.

PROOF. Let T be an extension over A with kernel Q (for the existence, consider the trivial extension $T=A\ltimes Q$). Then it suffices to show that $\mathrm{Hom}_T(\ ,T)$ and $\mathrm{Hom}_A(\ ,Q)$ are coincident on $\mathrm{mod}\,A$, because T induces a duality between $\mathrm{mod}\,T$ and $\mathrm{mod}\,T^\circ$ by Lemma 1.1 and, for finitely generated A-module X, $\mathrm{Hom}_T(X,T)$ is also finitely generated. Let M be in $\mathrm{mod}\,A$ and $f\in\mathrm{Hom}_T(M,T)$. Then $f(M)Q=0$ because $MQ=0$ in $\mathrm{mod}\,T$. Hence it follows from Lemma 1.2 that $f(M)\subset Q$. Therefore $\mathrm{Hom}_T(M,T)=\mathrm{Hom}_T(M,Q)$. Moreover it is clear that $\mathrm{Hom}_T(M,Q)=\mathrm{Hom}_A(M,Q)$.

In the following, A means an Artinian ring with a QF-module Q and T an extension over A with kernel Q:

$$0\longrightarrow Q\xrightarrow{\kappa}T\xrightarrow{\rho}A\longrightarrow 0.$$

LEMMA 2.2. *For a finitely generated right T-module M, let $0\to MQ\xrightarrow{u}M\xrightarrow{v}M/MQ\to 0$ be the canonically exact sequence. Then the following assertions hold.*

(1) *$MQ=0$ if and only if $QM^*=0$.*

(2) *$(M/MQ)^*\cong \boldsymbol{r}_{M^*}(Q)$ by v^*, $M^*/\boldsymbol{r}_{M^*}(Q)\cong(MQ)^*$ by u^*.*

(3) *If $f: M\to N$ is a morphism in $\mathrm{mod}\,T$, then $f(M)=(f^*(N^*))^*$.*

PROOF. (1) Since $(\ )^*$ defines a duality between $\mathrm{mod}\,T$ and $\mathrm{mod}\,T^\circ$, it suffices to show that $MQ=0$ implies $QM^*=0$. Let $f\in M^*$. Then $f(M)Q=f(MQ)=0$. Hence $f(M)\subset \boldsymbol{l}_T(Q)$. It then follows from Lemma 1.2 that $f(M)\subset Q$. Hence $Qf(M)=0$, because $Q^2=0$ in T. This means that $Qf=0$.

(2) Applying the duality $(\ )^*$ to the sequence $0\to MQ\xrightarrow{u}M\xrightarrow{v}M/MQ\to 0$, we have the canonically exact sequence

$$0\longrightarrow(M/MQ)^*\xrightarrow{v^*}M^*\xrightarrow{u^*}(MQ)^*\longrightarrow 0.$$

We may regard v^* as an inclusion. It then follows from (1) that $(M/MQ)^*\subset \boldsymbol{r}_{M^*}(Q)$. Conversely, let $f\in \boldsymbol{r}_{M^*}(Q)$. Since $Qf=0$, $Qf(M)=0$. Namely, $f(M)\subset \boldsymbol{r}_T(Q)=Q$ by Lemma 1.2. On the other hand, $u^*(f)(MQ)=f(MQ)=f(M)Q$. Hence $u^*(f)(MQ)\subset Q^2=0$, i.e. $u^*(f)=0$. This shows that $f\in(M/MQ)^*$. Thus we have shown that $(M/MQ)^*=\boldsymbol{r}_{M^*}(Q)$. The rest of assertions is an easy consequence of the above sequence.

(3) Let $g: M\to f(M)$, $h: f(M)\to N$ be the canonical epimorphism and monomorphism such that $f=hg$. Then $f^*=g^*h^*$ and $g^*: f(M)^*\to M^*$ is monomophic and $h^*: N^*\to f(M)^*$ is epimorphic. Hence $f^*(N^*)=h^*(N^*)=f(M)^*$. Thus we have that $f(M)=(f^*(N^*))^*$.

COROLLARY 2.3. *Let P be an indecomposable projective right T-module. Then the following diagram is commutative.*

$$\begin{tikzcd} 0 \arrow[r] & (P/PQ)^* \arrow[r] \arrow[d, equal, "\wr"] & P^* \arrow[r] \arrow[d, equal] & (PQ)^* \arrow[r] \arrow[d, equal, "\wr"] & 0 \\ 0 \arrow[r] & QP^* \arrow[r] & P^* \arrow[r] & P^*/QP^* \arrow[r] & 0 \end{tikzcd},$$

where all morphisms are canonical.

PROOF. This is immediately obtained from Lemma 1.2 and Proposition 2.1.

Let M be a module and W a nonzero proper submodule. Then W is said to be a *waist* in M if $X\subset W$ or $W\subset X$ for every submodule X of M [1], or we say that M has a waist W.

PROPOSITION 2.4. *Let M be a finitely generated right T-module such that $MQ\neq 0$. Then MQ is a waist in M if and only if $\boldsymbol{r}_{M^*}(Q)$ is a waist in M^*. Particularly, for a primitive idempotent $\boldsymbol{e}$ in T, $\boldsymbol{e}Q$ is a waist in $\boldsymbol{e}T$ if and only if $Q\boldsymbol{e}$ is a waist in $T\boldsymbol{e}$.*

PROOF. This is easily proved by Lemma 2.2 and [1, Proposition 4].

PROPOSITION 2.5. *For any finitely generated indecomposable right T-module M with $MQ\neq 0$, the following statements are equivalent.*

(1) *MQ is a waist in M.*

(2) *For any morphism $f: M\to N$ in* mod T *such that $f(MQ)\neq 0$,*

$$f(M)/f(MQ)\cong M/MQ.$$

(3) *For any morphism $g: L\to M^*$ in* mod T° *such that $g(QL)\neq 0$,*

$$\boldsymbol{r}_{g(L)}(Q)=\boldsymbol{r}_{M^*}(Q).$$

Moreover, if $MQ=\boldsymbol{l}_M(Q)$, the above statements are equivalent to

(4) *For any morphism $f: N\to M$ in* mod T *such that $f(NQ)\neq 0$,*

$$\boldsymbol{l}_{f(N)}(Q)=MQ.$$

PROOF. (1)$\Rightarrow$(2): Let $f: M\to N$ be a morphism in mod T with $f(MQ)\neq 0$. Then $MQ\not\subset \operatorname{Ker} f$ and so $\operatorname{Ker} f\subset MQ$, because MQ is a waist in M by assumption. Hence $f^{-1}(f(MQ))=MQ+\operatorname{Ker} f=MQ$. Thus we have that $M/MQ\cong f(M)/f(MQ)$.

(2)$\Rightarrow$(3): Let $g: L\to M^*$ in mod T° with $g(QL)\neq 0$. By applying the functor $(\ \ )^*$, we have a morphism $g^*: M\to L^*$ in mod T such that $g^*(MQ)\neq 0$.

For, $g(L)=(g^*(M))^*$ by (2.2). Since $Qg(L)\neq 0$, $(g^*(M)Q)^*\neq 0$. Hence $g^*(M)Q\neq 0$. Now then, it follows from the assumption that

$$g^*(M)/g^*(MQ)\cong M/MQ\,.$$

Again, applying $(\ \ )^*$, we have from Lemma 2.2 that

$$\boldsymbol{r}_{g^*(M)^*}(Q)=\boldsymbol{r}_{M^*}(Q)\,.$$

On the other hand, it follows from Lemma 2.2 that $g^*(M)^*=g(L)$. As a consequence, we have $\boldsymbol{r}_{g(L)}(Q)=\boldsymbol{r}_{M^*}(Q)$

(3)$\Rightarrow$(1): From Proposition it suffices to show that $\boldsymbol{r}_{M^*}(Q)$ is a waist in M^*. Let X be a submodule of M^* such that $X\not\subset\boldsymbol{r}_{M^*}(Q)$. We have only to show that $\boldsymbol{r}_{M^*}(Q)\subset X$. Let $p: P\to X$ be a projective cover of M. Since $X\not\subset\boldsymbol{r}_{M^*}(Q)$, there is then an indecomposable summand P' of P such that $p(P_{,})\not\subset\boldsymbol{r}_{M^*}(Q)$. Let $f: P' \xrightarrow{p'} X\subset M'$, where p' is the restriction of p on P'. Then $f(QP')\neq 0$. Because, if $f(QP')=0$, then $f(P')\subset\boldsymbol{r}_X(Q)\subset\boldsymbol{r}_{M^*}(Q)$, and hence $p(P')=p'(P')\subset\boldsymbol{r}_{M^*}(Q)$, which contradicts the cohice of P'. Thus we can apply the assumption (3) to f, so that we have that $\boldsymbol{r}_{f(P')}(Q)=\boldsymbol{r}_{M^*}(Q)$. Therefore $\boldsymbol{r}_{M^*}(Q)\subset f(X)$, and hence $\boldsymbol{r}_{M^*}(Q)\subset X$.

For the condition (4), suppose that $MQ=\boldsymbol{l}_M(Q)$. It then follows from Proposition 2.4 that the (1) is equivalent to that QM^* is a waist in M^*. Hence we know that the (4) is equivalent to the (1) in view of the equivalence of (1) and (3).

Let A be of finite representation type and T an extension which is also of finite representation type. Then the number of isomorphism classes of indecomposable right T-modules are twice as many the number of isomophism classes of indecomposable right A-modules if and only if $\boldsymbol{l}_M(Q)$ is injective in $\mathrm{mod}\,A$ for any indecomposable right T-module M with $MQ\neq 0$ [3, 2.12]. For such an extension T we have the following.

COROLLARY 2.6. *Let T be an extension over A with kernel Q. Suppose that for any indecomposable right T-module M with $MQ\neq 0$, $\boldsymbol{l}_M(Q)$ is injective in* $\mathrm{mod}\,A$. *Then every indecomposable projective right T-module P has a waist PQ. In particular, if A is hereditary, every indecomposable projective right T-module P has a waist PQ.*

PROOF. Let P be an indecomposable projective right T-module. Let $f: M\to P$ be a morphism in $\mathrm{mod}\,T$ such that $f(MQ)\neq 0$. Then $f(M)$ is indecomposable, because $\mathrm{soc}\,(f(M))=\mathrm{soc}\,(P)$ which is simple. Hence, by assumption, $\boldsymbol{l}_{f(M)}(Q)$ is injective in $\mathrm{mod}\,A$. Since PQ is indecomposable injective in $\mathrm{mod}\,A$ by Proposition 2.1, it therefore follows that $\boldsymbol{l}_{f(M)}(Q)=PQ$. This shows that PQ is a waist in P by Proposition 2.5, because $PQ=\boldsymbol{l}_P(Q)$ by Proposition 2.1.

LEMMA 2.7. *Let R be a semi-perfect ring and e a primitive idempotent in R. Then the following statements are equivalent.*

(1) *For any primitive idempotent e' in R, every nonzero morphism from eR to $e'R$ is monomorphic.*

(2) *For any primitive idempotent e' in R, every nonzero morphism from Re to Re' is monomorphic.*

PROOF. It is sufficient to show that (1)$\Rightarrow$(2) by symmetry. Assume (1) and let $f: Re \to Re'$ be a nonzero morphism. Since the ring R is semi-perfect, every finitely generated left module has a projective cover which is isomorphic to a direct sum of primitive left ideals. Hence, to show that f is monomorphic it suffices to show that for a nonzero morphism $g: Re'' \to Re$, where e'' is a primitive idempotent, $fg: Re'' \to Re'$ is nonzero. Applying $\mathrm{Hom}_R(\ ,R)$, we have the commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_R(Re', R) \arrow[r, "\mathrm{Hom}_R(f,R)"] \arrow[d, equal, "\wr"] & \mathrm{Hom}_R(Re, R) \arrow[r, "\mathrm{Hom}_R(g,R)"] \arrow[d, equal, "\wr"] & \mathrm{Hom}_R(Re'', R) \arrow[d, equal, "\wr"] \\
e'R \arrow[r, "\hat{f}"'] & eR \arrow[r, "\hat{g}"'] & e''R,
\end{tikzcd}$$

where $\hat{f}$ and $\hat{g}$ are canonical morphisms which make the diagram commutative. By assumption $\hat{g}$ is a monomorphism, and so $\hat{g}\hat{f}$ is nonzero. Hence $\mathrm{Hom}_R(g, R)\mathrm{Hom}_R(f, R) \neq 0$, which clearly means that $fg \neq 0$.

THEOREM 2.8. *Let A be an Artinian ring with a QF-module Q, and $T = A \ltimes Q$ the trivial extension with the canonical epimorphism $\rho: T \to A$. Let $\boldsymbol{e}$ be a primitive idempotent in T and $e = \rho(\boldsymbol{e})$. Then the following statements are equivalent.*

(1) *eQ is a waist in $\boldsymbol{e}T$.*

(2) *If $f: eA \to e'A$ is a nonzero morphism for a primitive idempotent e', then f is a monomorphism.*

(3) *If $f: Ae \to Ae'$ is a nonzero morphism for a primitive idempotent e', then f is a monomorphism.*

(4) *$eQ = aQ$ for any nonzero element $a \in eA$.*

PROOF. (1)$\Rightarrow$(2): Let $f: eA \to e'A$ be a nonzero morphism, where e' is a primitive idempotent in A, and let $\boldsymbol{e}'$ be an idempotent in T such that $e' = \rho(\boldsymbol{e}')$. Let $g: \boldsymbol{e}T \to \boldsymbol{e}'T$ be an extension of f. Then $g(eQ) \neq 0$. Because, if $g(eQ) = 0$ to the contrary, then $g(\boldsymbol{e}T) \subset e'Q$ by Lemma 1.2. Hence $f = 0$, a contradiction. It therefore follows from Proposition 2.5 that

$$eT/eQ \cong g(eT)/g(eQ).$$

This means that $g(eT)/g(eQ)$ is projective in mod A. Hence

$$g(eT)/(e'Q \cap g(eT)) \cong g(eT)/g(eQ).$$

On the other hand, $f(eA) \cong g(eT)/(e'Q \cap g(eT))$. Thus we know that $f(eA)$ is projective in mod A. As a consequence, f is monomorphic because eA in indecomposable.

(2)$\Rightarrow$(3): This is proved in Lemma 2.7.

(3)$\Rightarrow$(4): It is sufficient to show that $eQ=eae'Q$ for any $0 \neq eae' \in A$, where e' is a primitive idempotent in A. Let $f: e'Q \to eQ$ be a morphism defined by $f(e'x)=eae'x$ for $x \in Q$. Applying the duality $\mathrm{Hom}_A(\ \ , Q)$ (cf. Proposition 2.1), we have

$$\begin{tikzcd}
\mathrm{Hom}_A(eQ, Q) \arrow[rr, "\mathrm{Hom}_A(f,\, Q)"] \arrow[d, equal, "\wr"] & & \mathrm{Hom}_A(e'Q, Q) \arrow[d, equal, "\wr"] \\
Ae \arrow[rr, "\hat{f}"'] & & Ae'
\end{tikzcd},$$

where $\hat{f}: Ae \to Ae'$ is a morphism which makes the diagram commutative. Since $\hat{f}$ is nonzero, it follows from (3) that $\hat{f}$ is monomorphic, i.e. f is epimorphic. Hence we have $eQ=eae'Q$.

(4)$\Rightarrow$(1): Let $t \in eT/eQ$. Since T is the trivial extension $A \ltimes Q$, T is a direct sum $A \oplus Q$ in mod A (cf. §1). Let $t=(a, q)$, where $a \in eA$ and $q \in eQ$. Then $a \neq 0$ by the choice of t. Then clearly $aQ \subset tT$. By the assumption (4) $aQ=eQ$. Hence $tT \supset eQ$, which shows that eQ is a waist in eT.

Let e be a primitive idempotent in T such that eQ is a waist in eT and let $e=\rho(e)$. Then it is easy to see that there is some integer m such that $eQ = e\,\mathrm{rad}\,(T)^m$ (cf. [1, Proposition 1]). Hence it holds that

$$\begin{aligned}
e\,\mathrm{rad}\,(T)^i/e\,\mathrm{rad}\,(T)^{i+1} &\cong e\,\mathrm{rad}\,(A)^i/e\,\mathrm{rad}\,(A)^{i+1} && \text{for } i<m \\
&\cong eQ\,\mathrm{rad}\,(A)^{i-m}/eQ\,\mathrm{rad}\,(A)^{i+1-m} && \text{for } i \geqq m.
\end{aligned}$$

Thus Theorem 2.8 may be useful to calculate the trivial extensions of Artinian rings with self-dualities.

COROLLARY 2.9. *Let A be an Artinian ring with a QF-module Q and let $T=A \ltimes Q$ be the trivial extension of A by Q. Then the following statements are* *equivalent.*

(1) *Every indecomposable projective right T-module P has a waist PQ.*

(2) *Every nonzero morphism between indecomposable projective right A-modules is monomorphic.*

(3) *Every submodule of any projective right A-module is a sum of projective submodules.*

(4) *Every submodule of any indecomposable projective right A-module is a sum of projective submodules.*

(5) *Every nonzero morphism between indecomposable projective left A-modules is monomorphic.*

PROOF. (1)$\Rightarrow$(2): This is clear from Theorem 2.6.

(2)$\Rightarrow$(3): Let P be a projective right A-module and M a nonzero submodule of P. Let $P=\bigoplus_{i\in I}P_i$, where each P_i is indecomposable, and $p_i: P\to P_i$ be the projection. Since M has a projective cover, it suffices to show that every nonzero morphism $f: P'\to M$ is monomorphic, where P' is indecomposable projective in mod A. Since $f(P')\neq 0$, there is i such that $p_if(P')\neq 0$. Then $p_if: P'\to P_i$ is monomorphic by assumption, and so is f.

(3)$\Rightarrow$(4): This is trivial.

(4)$\Rightarrow$(1): Let P be indecomposable projective in mod T. Then P/PQ is clearly indecomposable projective in mod A. Now let $P/PQ\cong eA$ and $f: eA\to e'A$ a nonzero morphism. Then $f(eA)$ is a sum of projective submodules by assumption. Therefore $f(eA)$ itself is projective because it has the unique maximal submodule, so that f must be a monomorphism. Hence PQ is a waist in P by Theorem 2.8.

(2)$\Leftrightarrow$(5): This is an immediate consequence of Lemma 2.7.

EXAMPLE 2.10. In conclusion we will note that in Corollary 2.9 if T is not the trivial extension, the condition (1) does not necessarily imply the others.

Let $\boldsymbol{Z}$ be the set of integers and p a prime number ($p>1$). Let $A=\boldsymbol{Z}/p^2\boldsymbol{Z}$ and $T=\boldsymbol{Z}/p^4\boldsymbol{Z}$. Then A is a serial local Artinian ring, in particular, it is quasi-Frobenius. It is then easy to see that T is an extension over A with kernel Q, where $Q=\boldsymbol{Z}/p^2\boldsymbol{Z}$. T is however not trivial. Since T is serial local, it is clear that the condition (1) in Corollary 2.9 holds, but the others do not hold.

REMARK. At the Second International Conference on Representations of Algebras in Ottawa, the author heard from D. Simson that he had proved the equivalence of (2) and (5) in Corollary 2.9 for arbitrary Artinial rings, which is also obtained in our proof (cf. Lemma 2.7).

References

[1] Auslander, M., Green, E. L. and Reiten, I,, *Modules having waists.* "Representations of Algebras", Lecture Notes in Math. 488, Springer-Verlag, 20-28 (1975).

[2] Cartan, H. and Eilenberg, S., Homological algebra. Princeton, Now Jersey, Princeton University press 1956.

[3] Yamagata, K., *Extensions over hereditary Artinian rings with self-dualities* I, preprint.

Institute of Mathematics,
University of Tsukuba,
Sakuramura Niiharigun Ibaraki
305 Japan