TSUKUBA J. MATH.
Vol. 12 No. 2 (1988). 521–524

A DIRECT PROOF THAT EACH PEANO CONTINUUM WITH A FREE ARC ADMITS NO EXPANSIVE HOMEOMORPHISMS

By

Kazuhiro KAWAMURA

A homeomorphism $f: X\rightarrow X$ of a compact metric space X is said to be *expansive* if there exists a constant $c>0$ (called *expansive constant*) such that

(*) for each pair x, y of distinct points of X, there exists an integer n such that $d(f^{n}(x), f^{n}(y))>c$, where d is a metric for X. Expansiveness does not depend on the choice of metrics for compact metric spaces.

A compact connected metric space is called a *continuum*. A *Peano continuum* means a locally connected continuum. An arc A in a continuum X with end points $\{a, b\}$ is denoted by $[a, b]$. $bd\,A$ means $\{a, b\}$ and $\mathrm{int}\,A=A-bd\,A$. An arc A in X is called a *free arc* if $\mathrm{int}\,A$ is open in X. Let (X, d) be a continuum. For a point $x\in X$ and $\varepsilon>0$, $U(x, \varepsilon)$ denotes the ε-neighbourhood of x. The Hausdorff metric is denoted by d_{H}.

In this paper, we give a direct proof of the following theorem, which is a consequence of Proposition C in Hiraide [2].

THEOREM. *Let X be a Peano continuum with a free arc. Then there does not exist expansive homeomorphisms of X.*

The author benefits from reading Proposition C in [2] and wishes to thank to Professor K. Sakai for his helpful suggestions.

First we list known results which are necessary for the proof of Theorem.

LEMMA 1 ([3] p. 257, theorem 4). *Let (X, d) be a Peano continuum. For each $\varepsilon>0$, there exists a $\delta>0$ such that each pair of points $x, y\in X$ with $d(x, y)<\delta$ can be joined by an arc whose diameter is less than ε.*

LEMMA 2 ([3] p. 179, theorem 1). *A continuum X is homeomorphic to an arc if and only if there exist two points a and b of X such that*

1) *$X-a$ and $X-b$ are connected and*
2) *for each $x\in X$ with $a\neq x\neq b$, $X-x$ is not connected.*

Received October 14, 1987, Revised January 28, 1988.

LEMMA 3 ([1] p. 63-68). *Let $f: X \to X$ be an expansive homeomorphism of a compoct metric space X.*

1) *For each integer k, f^k is also expansive.*

2) *Suppose a closed subset A of X satisfies $f(A)=A$. Then $f|A$ is also expansive.*

3) *There exist no expansive homeomorphisms of arcs and simple closed curves.*

To prove Theorem, we first show the following.

(A) Let (L_n) be an increasing sequence of free arcs in X and $M=\mathrm{Lim}\,L_n$ (Lim means the limit by the Hausdorff metric).

Then M is either a free arc or

a simple closed curve such that $M \cap cl(X-M)$ is a point.

Let $L_n=[p_n, q_n]$. It is easy to see that $M=cl(\cup L_n)$. Without loss of generality, we may assume that there exist two points p and q of M such that $p=\lim p_n$ and $q=\lim q_n$. We consider two cases.

Case a) $p \neq q$. In this case, M is a free arc. To see this, we show

1) $M=cl(\cup L_n)=\cup L_n \cup \{p, q\}$.

Suppose that there exists a point $u \in cl(\cup L_n)-\cup L_n \cup \{p, q\}$. We can choose a sequence u_n's of points in L_n's which converges to u. Since $p \neq u \neq q$, we may assume that $u_n \in \mathrm{int}\,L_n$. By Lemma 1, there exists a sequence A_n's of arcs joing u and u_n and $\mathrm{diam}\,A_n \to 0$ as $n \to \infty$. On the other hand, $u_n \in \mathrm{int}\,L_n$ and $u \notin L_n$, and so $A_n \cap bd\,L_n \neq \emptyset$. Therefore there exists an integer $N>0$ such that for each $n>N$, $\mathrm{diam}\,A_n > \min\{d(u, p), d(u, q)\}/2>0$, which is a contradiction. Hence $cl(\cup L_n) \subset \cup L_n \cup \{p, q\}$. Clearly $cl(\cup L_n) \supset \cup L_n \cup \{p, q\}$, and therefore $M=\cup L_n \cup \{p, q\}$. It is easy to see that $M-p$ and $M-q$ are connected and $M-x$ is not connected for each $x \in M-\{p, q\}$. By Lemma 2, M is an arc. $M-\{p, q\}$ is open in X, so M is a free are.

Case b) $p=q$. In this case, M is a simple closed curve and $M \cap cl(X-M)$ is a point. To prove this, take $c \in \mathrm{int}\,L_1$ and let $A_n=[p_n, c]$ and $B_n=[q_n, c]$. Since L_n's are free arcs, $p=q \neq c$. Applying the argument of Case a), we see that $A=\mathrm{Lim}\,A_n$ and $B=\mathrm{Lim}\,B_n$ are free arcs with end points $\{p, c\}$ and $\{q, c\}$ respectively. Clearly $M=A \cup B$ and since $A_n \cap B_n=\{c\}$, $A=\cup A_n \cup \{p\}$ and $B=\cup B_n \cup \{q\}$, we have $A \cap B=\{c, p=q\}$. Therefore M is a simple closed curve. Since M is a limit of free arcs, $M \cap cl(X-M)$ is a point.

Let $\mathcal{F}$ be the collection defined by

$$\mathcal{F}=\left\{K \,\middle|\, \begin{array}{l} K \text{ is a subcontinuum of } X \text{ and there exists an increasing} \\ \text{sequence of free arcs which converges to } K \end{array}\right\}.$$

$\mathcal{F}$ is a partially ordered set by the usual inclusions. We show

(B) Each totally ordered subset of $\mathcal{F}$ has an upper bound.

Let $\mathcal{K}$ be a totally ordered subset of $\mathcal{F}$ and $K_0=cl(\cup\mathcal{K})$. We must find an increasing sequence of free arcs which converges to K_0. Notice that each $K\in\mathcal{F}$ is either a free arc or a simple closed curve by (A). We consider two cases.

Case a) Each $K\in\mathcal{F}$ is a free arc. Let $\{x_1, \cdots, x_n\}\subset K_0$ be a finite set such that $K_0\subset\bigcup_{i=1}^{n}U(x_i, 1/2)$. For each $i=1, \cdots, n$, there exist $K_{a_i}\in\mathcal{F}$ and a point $p_i\in K_{a_i}$ such that $d(p_i, x_i)<1/2$. Take a $K_1\in\mathcal{F}$ which contains all of $K_{a_1}, \cdots, K_{a_n}$. Then it is easy to see that $d_H(K_1, K_0)<1$.

Take a finite set $\{y_1, \cdots, y_m\}\subset K_0$ such that $K_0\subset\bigcup_{i=1}^{m}U(y_i, 1/4)$. For each $i=1, \cdots, m$, there exist K_{b_i} and a point $q_i\in K_{b_i}$ such that $d(q_i, y_i)<1/4$. Take a $K_2\in\mathcal{K}$ which contains all of $K_1, K_{b_1}, \cdots, K_{b_m}$. Then $d_H(K_2, K_0)<1/2 \cdots$. Continuing this processes, we can take an increasing sequence of free arcs which converges to K_0.

Case b) There exists an $L\in\mathcal{K}$ which is a simple closed curve. Each $N\in\mathcal{K}$ which contains L is a simple closed curve. Hence $K_0=L$ which is the limit of an increasing sequence of free arcs. Therefore K_0 is an upper bound of $\mathcal{K}$. This proves (B).

Using Zorn's lemma, we can find a maximal element M of $\mathcal{F}$.

Now suppose that $f: X\to X$ is an expansive homeomorphism with expansive constant $c>0$. If $f^n(M)=M$ for some integer $n\neq 0$, we have a contradiction by Lemma 3, 2) and 3). Thus we have $f^n(M)\neq M$ for each $n\neq 0$. Then the following holds.

(C) C-1) $\operatorname{diam} f^n(M)\to 0$ as $n\to\infty$ and

C-2) $\operatorname{diam} f^{-n}(M)\to 0$ as $n\to\infty$.

We prove C-1). Suppose that there exist an $\varepsilon>0$ and a subsequence (n_i) such that $\operatorname{diam} f^{n_i}(M)>\varepsilon$. Taking a subsequence if necessary, we may assume that $f^{n_i}(M)$ converges to a continuum M_0. Set $M_i=f^{n_i}(M)$. Again, we consider two cases.

Case a) M is a free arc. By the maximality of M, $M_i\cap M_j\subset bd\, M_i\cap bd\, M_j$ for each $i\neq j$. For each i, take a point $x_i\in M_i$ such that $d(x_i, bd\, M_i)\geqq\varepsilon/2$. Without loss of generality, we may assume that x_i's converge to a point

$x \in M_0$. By Lemma 1, there exists a sequence (A_i) of arcs joing x and x_i such that $\operatorname{diam} A_i \to 0$ as $i \to \infty$. If $x \notin M_i$ for each i, then $A_i \cap bd\, M_i \neq \emptyset$ for each i. If $x \in M_i$ for some i, then for each $j \neq i$, either $x \notin M_j$ or $x \in bd\, M_j \cap bd\, M_i$. Therefore $A_j \cap bd\, M_j \neq \emptyset$ for each j. In any case, $\operatorname{diam} A_k \geqq \varepsilon/2$ for each k, which is a contradiction.

Case b) M is a simple closed curve. Let $M \cap cl(X-M)=\{b\}$ and $b_i = f^{n_i}(b)$. In this case, $M_i \cap M_j = \emptyset$ or $\{b_i = b_j\} = M_i \cap M_j$ for each $i \neq j$. For each i, take a point $x_i \in M_i$ such that $d(x_i, b_i) \geqq \varepsilon/2$. Using the same argument as in Case a), we have a contradiction.

The proof of C-2) is similar, so we omit it.

Finally we take an integer m such that for each $n>m$, $\operatorname{diam} f^n(M)<c/2$ and $\operatorname{diam} f^{-n}(M)<c/2$. There exists a $\delta>0$ such that if $d(x, y)<\delta (x, y \in M)$, then $\max_{|i|=1,\cdots,m} d(f^i(x), f^i(y))<c$. Then, for distinct points x, y of M with $d(x, y)<\delta$, $d(f^i(x), f^i(y))<c$ for each integer i. This contradiction completes the proof.

References

[1] Aoki, N. and Shiraiwa, K., *Rikigaku-kei to entropy* (Dynamical systems and entropy), Kyoritsu Schuppan (1985), 1st ed., in Japanese.

[2] Hiraide, K., Expansive homeomorphisms on compact surfaces are pseudo-Anosov, preprint.

[3] Kuratowski, K., *Topology* II, Academic Press (1968).

Institute of Mathematics
University of Tsukuba
Ibaraki 305, Japan