TSUKUBA J. MATH.
Vol. 17 No. 1 (1993), 1–33

LOCAL INJECTIVITY OF PRYM MAPS FOR SOME FAMILIES OF COMPACT RIEMANN SURFACES

By

Katsuaki YOSHIDA

Introduction.

In this paper we consider some families of double coverings of compact Riemann surfaces (or complete irreducible non-singular algebraic curves over **C**) allowing ramifications, and we study the Prym varieties of these double coverings.

Let $\pi: \tilde{R}\to R$ be a double covering, where $\tilde{R}$ and R are compact Riemann surfaces of genera $\tilde{g}$ and g, and $J(\tilde{R})$ and $J(R)$ be Jacobians of $\tilde{R}$ and R, respectively. If π has $2n$ branch points, we have $\tilde{g}=2g+n-1$ by means of the Riemann-Hurwitz relation. We denote by ι the generator of the Galois group of $\tilde{R}/R$. Moreover we denote by the same ι the involution of $J(\tilde{R})$ induced by that of $\tilde{R}/R$. The norm map $Nm: J(\tilde{R})\to J(R)$ is defined by the induced map on divisor classes given by $D\to\pi(D)$ (D a divisor on $\tilde{R}$). The Prym variety $P=P(\tilde{R}/R)$ of $\tilde{R}/R$ (or $(\tilde{R}, \iota)$) is defined by the conneted component containing the origin of the kernel of Nm, and we have an isogeny $\iota_*: J(R)\times P\to J(\tilde{R})$ naturally (see Mumford [5], Fay [5], Sasaki [7]). The process taking Prym varieties defines the so-called Prym map $P: \tilde{R}/R\to P(\tilde{R}/R)$ from the family of $(\tilde{R}, \iota)$'s to the moduli space of polarized abelian varieties.

In case of unramified double coverings, Mumford [5] states some beautiful results concerning the relative dimension of the Prym map. For double coverings with ramification points, however, the contribution of those points to the Prym map might be unknown.

In this paper we will caluculate the relative dimension of the Prym map for some typical examples of $\tilde{R}/R$ with $2n$ ($n\geqq 1$) ramification points.

We consider the following three families of compact Riemann surfaces parametrized by t or t_i's:

(I) $\tilde{R}_t: y^4=(x-1-t)(x^2+x+1)$ genus 3

$R_t: y^2=(x-1-t)(x^2+x+1)$ genus 1

Received January 28, 1991, Revised March 16, 1992.

(II) $\tilde{R}_t:\ y^6=(x-1-t)\,(x^2+x+1)$ genus 4

$R_t:\ y^3=(x-1-t)\,(x^2+x+1)$ genus 1

(III) $\tilde{R}_{t_1,t_2,t_3}:\ y^4=(x-1-t_1)\,(x-\rho-\rho t_2)\,(x-\rho^2-\rho^2 t_3)$

$\times(x-\rho^3)\,(x-\rho^4)$ genus 6

$R_{t_1,t_1,t_3}:\ y^2=(x-1-t_1)\,(x-\rho-\rho t_2)\,(x-\rho^2-\rho^2 t_3)$

$\times(x-\rho^3)\,(x-\rho^4)$ genus 2,

where $\rho=\exp(2\pi i/5)$.

(I) is an example for $n=2$, and (II), (III) are for $n=3$.

Hereafter we write $\tilde{R}$ and R instead of $\tilde{R}_t$, $\tilde{R}_{t_1,t_2,t_3}$ and R_t, R_{t_1,t_2,t_3} for the sake of convenience occasionary.

In case (I) or (III), the surface $\tilde{R}$ has an automorphism T of order 4 defined by $T:(x, y)\to(x, iy)$ where $i=\sqrt{-1}$, and the involution $T^2:(x, y)\to(x, -y)$. In case (II), $\tilde{R}$ has an automorphism T of order 6 defined by $T:(x, y)\to(x, \sigma y)$ where $\sigma=\exp(2\pi i/6)$ and the involution $T^3:(x, y)\to(x, -y)$.

$\langle T\rangle$ will denote the group generated by T and as usual $\tilde{R}/\langle T\rangle$ will denote the surface obtained by identifying points on $\tilde{R}$ which are equivalent under the action of $\langle T\rangle$ on $\tilde{R}$. In case (I) or (III), $\tilde{R}/\langle T^2\rangle$ is canonically isomorphic to R, and in case (II), $\tilde{R}/\langle T^3\rangle$ is canonically isomorphic to R.

Under these notations, we can give our main results as follows:

THEOREM 1 (*Joint work with Sasaki*). *In case (I), the Prym variety P has a period matrix of the form*

$$\begin{pmatrix} I_2, & \begin{matrix} i & 0 \\ 0 & i \end{matrix} \end{pmatrix}.$$

That is to say, P is isomorphic to the product $J\times J$ where $J=\mathbf{C}/\langle 1, i\rangle$.

THEOREM 2. *In case (II) the Prym variety P has a period matrix of the form (I_3, Π) where Π is the following:*

$$\Pi=\frac{1}{2(1-\sigma)\{(2+\sigma)+z\}}\begin{pmatrix} (1+\sigma)+z & 1-\sigma^2 & 2 \\ 1-\sigma^2 & 3(\sigma-2) & 2(1-\sigma)(1+z) \\ 2 & 2(1-\sigma)(1+z) & 2\sigma+2\sigma(1-\sigma)z \end{pmatrix},$$

where $z=\left(\int_{B_3}\omega_1\right)\Big/\left(\int_{A_3}\omega_1\right)$ is the modulus of the surface $\hat{R}$, and

$$\hat{R}:\ y^2=(x-1-t)(x^2+x+1)\quad\textit{and}\quad \sigma=\exp(2\pi i/6).$$

THEOREM 3. *In case (III), the Prym variety P has a period matrix of the* *form* (I_4, Π) *where* Π *is the following*:

$$\Pi=\begin{pmatrix} -1/2+i/s & 1/2-ip/s & -iq/s & (1-i)/2-ir/s \\ 1/2-ip/s & i/2+ip^2/s & (1-i)/2+ipq/s & i+ipr/s \\ -iq/s & (1-i)/2+ipq/s & -(1-i)+iq^2/s & -i+iqr/s \\ (1-i)/2-ir/s & i+ipr/s & -i+iqr/s & i+ir^2/s \end{pmatrix}$$

where

$$p=\frac{\int_{A_3}\omega_1}{\int_{A_4}\omega_1}+(1-i)\frac{\int_{B_3}\omega_1}{\int_{A_4}\omega_1}+(1-i),\quad q=(i-1)\frac{\int_{B_3}\omega_1}{\int_{A_4}\omega_1}+(i-1),$$

$$r=(1-i)\frac{\int_{A_3}\omega_1}{\int_{A_4}\omega_1}-2i\frac{\int_{B_3}\omega_1}{\int_{A_4}\omega_1}-(1-i)\frac{\int_{B_4}\omega_1}{\int_{A_4}\omega_1}-(1+i),$$

$$s=-2p^2-q^2-2pq+4p-2r-2.$$

THEOREM 4. *In case* (I), *the Kodaira-Spencer map*

$$\kappa:\ T_{s,0}\longrightarrow H^1(\tilde{R}, \mathcal{T}_{\tilde{R}}) \quad \text{is given by}$$

$$\kappa((\partial/\partial t)_0)=(3x)^{-1}y^4\partial_x .$$

THEOREM 5. *In caie* (II), *the Kodaira-Spencer map* κ *is given by*

$$\kappa((\partial/\partial t)_0)=(1/3)x^2\partial_x .$$

THEOREM 6. *In case* (III), *the Kodaira-Spencer map* κ *is given by*

$$\kappa\begin{pmatrix}(\partial/\partial t_1)_0\\ (\partial/\partial t_2)_0\\ (\partial/\partial t_3)_0\end{pmatrix}=\frac{1}{5}\begin{pmatrix}1&1&1\\ \rho^2&\rho^3&\rho^4\\ \rho^4&\rho&\rho^3\end{pmatrix}\begin{pmatrix}x^4\partial_x\\ x^3\partial_x\\ x^2\partial_x\end{pmatrix},$$

where $\rho=\exp(2\pi i/5)$.

(The above notations in Theorem 4, 5, 6 are explained in § 2.)

OBSERVATION. In case (I), the parameter t is not reflected in the Prym variety as a variable, that is to say, the Prym is uniquely determined. In case (II), t is refrected as a variable, and in case (III), t_1, t_2 and t_3 are refrected as independent variables in the Prym.

In section 1, we find a canonical homology basis and we compute the period matrix of $J(\tilde{R})$ in each case. Moreover, by the operation of a second order transformation, we obtain the period matrix of the Prym variety. In section 2, we caluculate the Kodaira-Spencer map and we give not only to prove the independency of parameter t_i's but also to the explicit representation of influence of parameters on the Prym variety.

I would like to express my hearty thanks to Professor T. Sekiguchi for his advices and encouragement during the preparation of this paper. In paticular, he informed and teached me the notion and method of Kodaira-Spencer map and gave me some instructive suggestions. Also I would like to thank Professor R. Sasaki for his stimulative suggestions in our conversations which gave me the motivation of this paper.

§1. Period matrices and Prym varieties.

In this section, we shall construct the period matrices of surfaces in cases (I), (II) and (III), according to Farkas' method [1], [9].

CASE (I). $\tilde{R}: y^4=(x-1-t)(x^2+x+1)$

$R: y^2=(x-1-t)(x^2+x+1)$

where $1+t\neq\infty$, $\exp(2\pi i/3)$, $\exp(4\pi i/3)$.

Here we have an automorphis $T: (x, y)\to(x, iy)$ of $\tilde{R}$.

Then $\tilde{R}$ is regarded as a double covering of R with the involution T^2: $(x, y)\to(x, -y)$.

Since we can easily examine the divisors of differentials, we obtain a basis of the vector space of holomorphic differentials given by

$$\omega_1=y^{-3}dx, \qquad \omega_2=y^{-3}xdx, \qquad \omega_3=y^{-2}dx.$$

Then we see

$$(1)\quad \begin{cases} T\omega_j=i\omega_j, & T^2\omega_j=-\omega_j \qquad (j=1, 2) \\ T\omega_3=-\omega_3, & T^2\omega_3=\omega_3 \end{cases}$$

that is to say, ω_1 and ω_2 are the anti T^2-invariante differentials and ω_3 is T^2-invariante differential. So, ω_1 and ω_2 are the differentials which correspond to the differentials of Prym variety and ω_3 is considered as a differential of R by the natural projection $\tilde{R}\to R=\tilde{R}/\langle T^2\rangle$.

Here we construct a canonical homology basis on $\tilde{R}$ (cf. [6], [3]). We consider the following closed curves α, β, γ and δ on x-sphere as illustrated in Fig. 1.

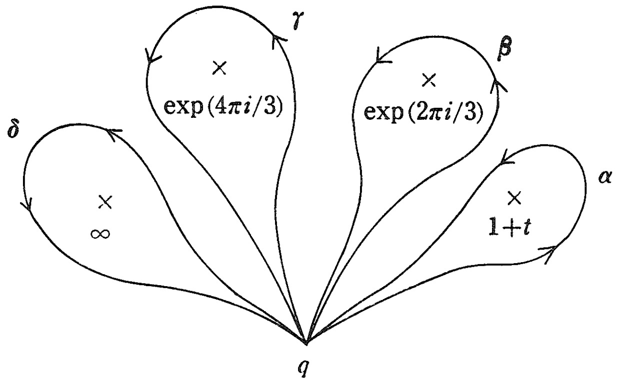


Fig. 1

y has four branches y_1, $y_2(=iy_1)$, $y_3(=-y_1)$, $y_4(=-iy_1)$. Let α_j be the lifting of α which is a path from y_j-branch to y_{j+1}-branch ($j=1, 2, 3$) and α_4 be the lifting of α from y_4-branch to y_1-branch. In the same way, let β_j, γ_j and δ_j be the liftings of β, γ and δ, respectively.

Then we get the following relations

$$(2)\quad \begin{cases} \alpha_1+\alpha_2+\alpha_3+\alpha_4\sim 0 & \beta_1+\beta_2+\beta_3+\beta_4\sim 0 \\ \gamma_1+\gamma_2+\gamma_3+\gamma_4\sim 0 & \delta_1+\delta_2+\delta_3+\delta_4\sim 0 \\ \alpha_1+\beta_2+\gamma_3+\delta_4\sim 0 & \alpha_2+\beta_3+\gamma_4+\delta_1\sim 0 \\ \alpha_3+\beta_4+\gamma_1+\delta_2\sim 0 & \alpha_4+\beta_1+\gamma_2+\delta_3\sim 0 \end{cases}$$

where $\alpha+\beta$ denotes the composition of α and β by joining the final point of α and the initial point of β, and $\sim$ means homotopic equivalence.

Then a canonical homology basis is represented as follows;

$$(3)\quad \begin{cases} A_1=\alpha_1-\delta_1-\delta_4+\alpha_4 & B_1=-\alpha_2+\delta_2+\gamma_3-\beta_3 \\ A_2=\alpha_1-\beta_1 & B_2=-\alpha_1+\gamma_1 \\ A_3=\alpha_3-\beta_3 & B_3=-\alpha_3+\gamma_3\,. \end{cases}$$

Here A_2 and B_2 are considered as a canonical homology basis of R by the natural projection $\tilde{R}\to R$. We illustrate this homology basis in Appendix.

Then, from (2), we see the following

$$(4)\quad \begin{cases} T^2A_1\approx -A_1 & T^2A_2\approx A_3 \\ T^2B_1\approx -B_1 & T^2B_2\approx B_3 \end{cases}$$

$$(5)\quad \begin{cases} A_1 \approx -A_3 - B_2 - TA_2 - TB_2 \\ B_1 \approx -TA_2 + B_3 - A_2 - B_2 \end{cases}$$

where $\approx$ means homological equivalence.

We put $\int_{A_j} \omega_i = a_{ij}$, $\int_{B_j} \omega_i = b_{ij}$.

From (1) and (4), we see that

$$a_{13} = -a_{12}, \quad a_{23} = -a_{22}, \quad a_{31} = 0, \quad a_{32} = a_{33},$$

$$b_{13} = -b_{12}, \quad b_{23} = -b_{22}, \quad b_{31} = 0, \quad b_{32} = b_{33}.$$

We put

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \qquad \phi_3 = \omega_3 / a_{32}$$

$$\begin{pmatrix} b'_{12} & b'_{12} \\ b'_{21} & b'_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \qquad \tau = b_{32} / a_{32}.$$

Then

$$\left(\int_{A_j} \phi_i \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \left(\int_{B_j} \phi_i \right) = \begin{pmatrix} b'_{11} & b'_{12} & -b'_{12} \\ b'_{21} & b'_{22} & -b'_{22} \\ 0 & \tau & \tau \end{pmatrix}.$$

From (1) and (5), we see that

$$1 = \int_{A_1} \phi_1 = -\int_{A_3} \phi_1 - \int_{B_2} \phi_1 - \int_{TA_2} \phi_1 - \int_{TB_2} \phi_1 = -b'_{12} + i b'_{12}$$

or $b'_{12} = -(1+i)/2$.

By the same way, we see that

$$b'_{11} = 1+i, \qquad b'_{21} = -(1+i), \qquad b'_{22} = i.$$

We put

$$\varphi_1 = \phi_1, \qquad \varphi_2 = (\phi_2 + \phi_3)/2, \qquad \varphi_3 = (\phi_3 - \phi_2)/2.$$

Then we see that

PROPOSITION 1. *$\tilde{R}$ has a period matrix of the form*

$$\left(\int_{A_j} \varphi_i, \ \int_{B_j} \varphi_i \right) = \left(I_3, \ \begin{matrix} 1+i & -(1+i)/2 & (1+i)/2 \\ -(1+i)/2 & (\tau+i)/2 & (\tau-i)/2 \\ (1+i)/2 & (\tau-i)/2 & (\tau+i)/2 \end{matrix} \right)$$

where I_3 is the 3×3 identity matrix and τ is the modulus of R with respect to *the homology basis on $\tilde{R}$ determined by the natural projection $\tilde{R}\to R$.*

Recall that in general there are $2g\times 2g$ integral matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ which act on Siegel space $\mathfrak{S}_g$ in the following fashion $\mathfrak{S}_g \ni \pi \to (A\pi+B)(C\pi+D)^{-1}\in\mathfrak{S}_g$ where A, B, C, D are $g\times g$ integral matrices, A^tB and C^tD are symmetric and $A^tD-B^tC=mI$. If $m=1$, the action is so called the element of the Siegel modular group or linear transformation, while the more general type of transformation is called an m-th order transformation ([1]).

PROPOSITION 2. *There is a second order transformation of $\mathfrak{S}_3$ which maps the period matrix of $\tilde{R}$ to the matrix*

$$\begin{pmatrix} \Pi & 0 \\ 0 & \tau \end{pmatrix} \quad \text{where} \quad \Pi=\begin{pmatrix} (1+i)/2 & -(1+i)/2 \\ -(1+i)/2 & i \end{pmatrix}\in\mathfrak{S}_2 .$$

PROOF. The proof is by computation. We takke $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where

$$A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad D=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad B=C=0.$$

Then a simple computation gives above result.

PROPOSITION 3. *There is a linear transformation of $\mathfrak{S}_2$ which maps Π to the matrix $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\in\mathfrak{S}_2$.*

PROOF. We take $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in Sp(2, \boldsymbol{Z})$ where

$$A=\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B=\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad C=\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \quad D=0.$$

Above matrix Π or $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ is the period matrix of the Prym variety. As a consequence of this, we have

THEOREM 1. *In case (I), the Prym variety P has a period matrix of the form $\left(I_2, \begin{matrix} i & 0 \\ 0 & i \end{matrix}\right)$. That is to say, P is isomorphic to the product $J\times J$ where $J=\mathbf{C}/\langle 1, i\rangle$.*

CASE (II). $\tilde{R}$: $y^6=(x-1-t)(x^2+x+1)$

R: $y^3=(x-1-t)(x^2+x+1)$

where $1+t\neq\infty$, $\exp(2\pi i/3)$, $\exp(4\pi i/3)$.

Here $T:(x, y)\rightarrow(x, \sigma y)$ where $\sigma=\exp(2\pi i/6)$. Then $\tilde{R}$ is regarded as a double covering of R with the involution

$$T^3:(x, y)\longrightarrow(x, -y).$$

Since we can easily examine the divisors of differentials, we obtain a basis of the vector space of holomorphic differentials given by

$$\omega_1=y^{-3}dx,\quad \omega_2=y^{-5}dx,\quad \omega_3=y^{-5}xdx,\quad \omega_4=y^{-4}dx.$$

Then we see

$$(6)\quad \begin{cases} T\omega_1=-\omega_1, & T\omega_i=\sigma\omega_i, & T\omega_4=\sigma^2\omega_4 \\ T^3\omega_1=-\omega_1{}' & T^3\omega_i=-\omega_i, & T^3\omega_4=-\omega_4 \end{cases} \quad (i=2, 3).$$

So, ω_1, ω_2 and ω_3 are the anti T^3-invariant differentials and ω_4 is T^3-invariant differential. Hence ω_1, ω_2 and ω_3 are the differentials of the Prym variety and ω_4 is considered as the differential of R by the natural projection $\tilde{R}\rightarrow R$.

Here we construct a canonical homology basis on $\tilde{R}$. We consider the following closed curves α, β, γ and δ on x-sphere as in Fig. 1. y has six branches y_1, $y_2(=\sigma y_1)$, $y_3(=\sigma^2 y_1)$, $y_4(=-y_1)$, $y_5(=-\sigma y)$ and $y_6(=-\sigma^2 y_1)$. Let α_i be the lifting of α which a path from y_i-branch to y_{i+1}-branch ($i=1, \cdots, 5$) and α_6 be the lifting from y_6-branch to y_1-branch. In the same way, let β_i and γ_i be the liftings of β and γ, respectively. We must pay attention to the lifting of δ, for the branch points over infinity are of order 1. Let δ_i be the lifting of δ from y_i-brach to y_{i+3}-branch ($i=1, 2, 3$) and δ_j be the lifting from y_j-branch to y_{j-3}-branch ($j=4, 5, 6$). Here we get the following relations

$$(7)\quad \begin{cases} \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6\sim 0, & \beta_1+\beta_2+\beta_3+\beta_4+\beta_5+\beta_6\sim 0 \\ \gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_6\sim 0, & \delta_1+\delta_4\sim 0, \quad \delta_2+\delta_5\sim 0, \quad \delta_3+\delta_6\sim 0 \\ \alpha_1+\beta_2+\gamma_3+\delta_4\sim 0, & \alpha_2+\beta_3+\gamma_4+\delta_5\sim 0, \quad \alpha_3+\beta_4+\gamma_5+\delta_6\sim 0 \\ \alpha_4+\beta_5+\gamma_6+\delta_1\sim 0, & \alpha_5+\beta_6+\gamma_1+\delta_2\sim 0, \quad \alpha_6+\beta_1+\gamma_2+\delta_3\sim 0 \end{cases}$$

Then a canonical homology basis is represented as follows;

$$(8)\quad \begin{cases} A_1=\gamma_1+\gamma_2+\gamma_3-\delta_1 & B_1=\delta_3-\gamma_5-\gamma_4-\gamma_3 \\ A_2=\alpha_2+\alpha_3+\alpha_4-\delta_2 & B_2=\beta_3+\beta_4+\beta_5-\alpha_5-\alpha_4-\alpha_3 \\ A_3=\alpha_1-\beta_1 & B_3=\gamma_1-\alpha_1 \\ A_4=\alpha_4-\beta_4 & B_4=\gamma_4-\alpha_4. \end{cases}$$

Here, A_3 and B_3 are considered as a canonical homology basis of R by the natural projection $\tilde{R}\rightarrow R$. We illustrate this homology basis in Appendix.

Then, from (6), (7) and (8), we get the following

$$(9)\quad \begin{cases} T^3A_1\approx -A_1 & T^3A_2\approx -A_2 & T^3A_3\approx A_4 \\ T^3B_1\approx -B_2 & T^3B_2\approx -B_2 & T^3B_3\approx B_4 \end{cases}$$

$$(10)\quad \begin{cases} A_1\approx -TA_4-B_4-TB_4 & B_1\approx +A_3+B_3+T^2B_4 \\ A_2\approx -T^2A_4+B_3 & B_2\approx -T^2A_3-A_4-TA_4. \end{cases}$$

We put

$$\int_{A_j}\omega_i=a_{ij}, \qquad \int_{B_j}\omega_i=b_{ij}.$$

From (6) and (9), we see thaat

$$a_{41}=a_{42}=b_{41}=b_{42}=0, \qquad a_{43}=a_{44}, \qquad b_{43}=b_{44}$$

$$a_{i4}=-a_{i3}, \qquad b_{i4}=-b_{i3}\ (i=1,\ 2,\ 3).$$

From (6) and (10), we see that

$$a_{11}=\int_{A_1}\omega_1=-\int_{TA_4}\omega_1-\int_{B_4}\omega_1-\int_{TB_4}\omega_1=\int_{A_4}\omega_1=a_{14}=-a_{13}$$

$$a_{12}=\int_{A_2}\omega_1=-\int_{T^2A_4}\omega_1+\int_{B_3}\omega_1=-\int_{A_4}\omega_1+\int_{B_3}\omega_1=a_{13}+b_{13}$$

$$b_{11}=\int_{B_1}\omega_1=\int_{A_3}\omega_1+\int_{B_3}\omega_1+\int_{T^2B_4}\omega_1=a_{13}+b_{13}-b_{13}=a_{13}$$

$$b_{12}=\int_{B_2}\omega_1=-\int_{T^2A_3}\omega_1-\int_{A_4}\omega_1-\int_{TA_4}\omega_1=-a_{13}.$$

By the same way

$$a_{21}=-\sigma^2a_{23}+(1-\sigma^2)b_{23}, \qquad a_{22}=-\sigma a_{23}+b_{23}$$

$$a_{31}=-\sigma^2a_{33}+(1-\sigma^2)b_{33}, \qquad a_{32}=-\sigma a_{33}+b_{33}$$

$$b_{21}=a_{23}+(1+\sigma)b_{23}, \qquad b_{22}=2a_{23}$$

$$b_{31}=a_{33}+(1+\sigma)b_{33}, \qquad b_{32}=2a_{33}.$$

We put $a_{13}=k,\ \ b_{13}=l,\ \ b_{43}/a_{43}=\tau.$

$$(a_{ij})^{-1}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}=\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}, \qquad \omega_4/a_{43}=\phi_4.$$

Then

$$\left(\int_{A_j}\phi_i\right)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-1\\0&0&1&1\end{pmatrix},\quad \left(\int_{B_j}\phi_i\right)=\begin{pmatrix}2b'_{11}&2b'_{12}&b'_{13}&-b'_{13}\\2b'_{21}&2b'_{22}&b'_{23}&-b'_{23}\\2b'_{31}&2b'_{32}&b'_{33}&-b'_{33}\\0&0&\tau&\tau\end{pmatrix}$$

where

$$(11)\quad \begin{cases} 2b'_{11}=\dfrac{(1+\sigma)k+l}{(1-\sigma)\{(2+\sigma)k+l\}} & 2b'_{21}=2b'_{12}=\dfrac{(1+\sigma)k}{(2+\sigma)k+l} \\ b'_{31}=b'_{13}=\dfrac{k}{(1-\sigma)\{(2+\sigma)k+l\}} & b'_{32}=b'_{23}=\dfrac{k+l}{(2+\sigma)k+l} \\ 2b'_{22}=\dfrac{-3k}{(2+\sigma)k+l} & b'_{33}=\dfrac{\sigma\{k+(1-\sigma)l\}}{(1-\sigma)\{(2+\sigma)k+l\}}\,. \end{cases}$$

We put $\varphi_i=\phi_i$ $(i=1, 2)$, $\varphi_3=(\phi_3+\phi_4)/2$, $\varphi_4=(\phi_4-\phi_3)/2$. Then we see that

PROPOSITION 4. *$\tilde{R}$ has a period matrix of the form*

$$\left(\int_{A_j}\varphi_i,\ \int_{B_j}\varphi_i\right)=\left(I_4,\ \begin{matrix}2b'_{11}&2b'_{12}&b_{1}^{\prime 3}&-b'_{13}\\2b'_{12}&2b'_{22}&b'_{23}&-b'_{23}\\b'_{13}&b'_{23}&(b'_{33}+\tau)/2&-(b'_{33}-\tau)/2\\-b'_{13}&-b'_{23}&-(b'_{33}-\tau)/2&(b'_{33}+\tau)/2\end{matrix}\right)$$

where b'_{ij} is that in (11) *and τ is the modulus of R with respect to the homology basis on R determined by the natural projection from $\tilde{R}\to R$.*

PROPOSITION 5. *The modulus τ is ω^2* $(\omega=\exp(2\pi i/3))$.

PROOF. The proof is a direct calculation.

PROPOSITION 6. *There is a second order transformation of $\mathfrak{S}_4$ which maps the period matrix of $\tilde{R}$ to the matrix $\begin{pmatrix}\Pi&0\\0&\omega^2\end{pmatrix}$ where*

$$\Pi=\begin{pmatrix}b'_{11}&b'_{12}&b'_{13}\\b'_{12}&b'_{22}&b'_{23}\\b'_{13}&b'_{23}&b'_{33}\end{pmatrix}\in\mathfrak{S}_3\,.$$

PROOF. We take a second order transformation $M=\begin{pmatrix}A&B\\C&D\end{pmatrix}$ of $\mathfrak{S}_4$ where

$$A=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-1\\0&0&1&1\end{pmatrix},\qquad B=C=0,\qquad D=\begin{pmatrix}2&0&0&0\\0&2&0&0\\0&0&1&-1\\0&0&1&1\end{pmatrix},$$

Summalizing above results, we have

THEOREM 2. *In case* (II), *the Prym variety* P *has a period matrix of the form* (I_3, Π) *where* Π *is that in Proposition* 6, *that is*

$$\Pi=\frac{1}{2(1-\sigma)\{(2+\sigma)+z\}}\begin{pmatrix}(1+\sigma)+z & 1-\sigma^2 & 2\\ 1-\sigma^2 & -3(1-\sigma) & 2(1-\sigma)(1+z)\\ 2 & 2(1-\sigma)(1+z) & 2\sigma+2\sigma(1-\sigma)z\end{pmatrix}$$

where $z=l/k=\left(\int_{B_3}\omega_1\right)\Big/\left(\int_{A_3}\omega_1\right)$.

PROPOSITION 7. *z is corresponding to the modulus of the surface* $\hat{R}$: $y^2=(x-1-t)(x^2+x+1)$.

PROOF. $\tilde{R}/\langle T^2\rangle$ is canonically isomorphic to $\hat{R}$ and ω_1 is the holomorphic differential which corresponds to theodifferential of $\hat{R}$ and A_3, B_3 are the homology basis on $\hat{R}$ determined by the natural projection from $\tilde{R}\to\hat{R}$.

CASE (III) $\tilde{R}$: $y^4=(x-1-t_1)(x-\rho(1+t_2))(x-\rho^2(1+t_3))(x-\rho^3)(x-\rho^4)$

R: $y^2=(x-1-t_1)(x-\rho(1+t_2))(x-\rho^2(1+t_3))(x-\rho^3)(x-\rho^4)$

where $1+t_1$, $\rho(1+t_2)$, $\rho^2(1+t_3)$, ρ^3, ρ^4 and ∞ are distinct.

$$T:(x, y)\longrightarrow(x, iy).$$

Then $\hat{R}$ is regarded as a double covering of R with the involution

$$T^2:(x, y)\longrightarrow(x, -y).$$

Since we can easily examine the divisors of differentials, we obtain a basis of the vector space of holomorphic differentials given by

$$\omega_1=y^{-1}dx,\quad \omega_2=y^{-3}dx,\quad \omega_3=y^{-3}xdx,\quad \omega_4=y^{-3}x^2dx,$$

$$\omega_5=y^{-2}dx,\quad \omega_6=y^{-2}xdx,$$

Then we see

$$(12)\quad \begin{cases} T\omega_1=-i\omega_1,\ T\omega_i=i\omega_i,\ T\omega_j=-\omega_j \\ T^2\omega_1=-\omega_1,\ T^2\omega_i=-\omega_i,\ T^2\omega_j=\omega_j \end{cases} \quad (i=2,3,4,\ j=5,6)$$

namely, ω_1, ω_2, ω_3 and ω_4 are anti T^2-invariant differentials and ω_5 and ω_6 are T^2-invariant differentials. So ω_1, ω_2, ω_3 and ω_4 are differentials of Prym variety, and ω_5 and ω_6 are considered as differentials of R by the natural projection $\tilde{R}\to R=\tilde{R}/\langle T^2\rangle$.

Here we construct a canonical homology basis on $\tilde{R}$. We consider the following closed curves α, β, γ, δ, ε and ζ on x-sphere in Fig. 2.

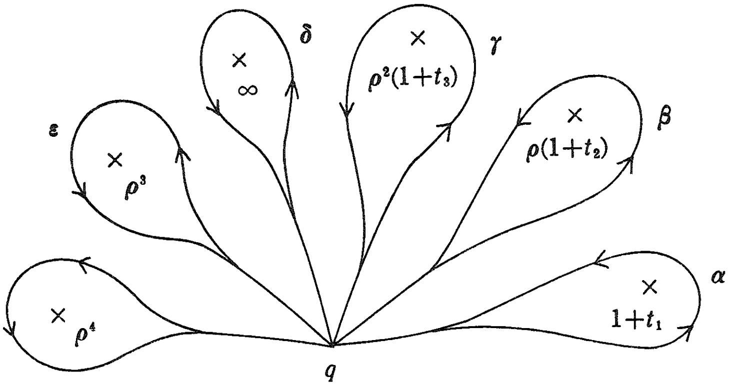


Fig. 2.

y has four branches y_1, $y_2(=iy_1)$, $y_3(=-y_1)$ and $y_4(=-iy_1)$. Let α_i be the lifting of α which is a path from y_i-branch to y_{i+1}-branch ($i=1,2,3$) and α_4 be the lifting of α from y_4-branch to y_1-branch. In the same way, let β_i, γ_i, δ_i, ε_i and ζ_i be the liftings of β, γ, δ, ε and ζ, respectively. Here we get the following relations

$$(13)\quad \begin{cases} \alpha_1+\alpha_2+\alpha_3+\alpha_4\sim 0, & \beta_1+\beta_2+\beta_3+\beta_4\sim 0, \quad \gamma_1+\gamma_2+\gamma_3+\gamma_4\sim 0 \\ \delta_4+\delta_3+\delta_2+\delta_1\sim 0, & \varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4\sim 0, \quad \zeta_1+\zeta_2+\zeta_3+\zeta_4\sim 0 \\ \alpha_1+\beta_2+\gamma_3+\delta_4+\varepsilon_3+\zeta_4\sim 0, & \alpha_2+\beta_3+\gamma_4+\delta_1+\varepsilon_4+\zeta_1\sim 0 \\ \alpha_3+\beta_4+\gamma_1+\delta_2+\varepsilon_1+\zeta_2\sim 0, & \alpha_4+\beta_1+\gamma_2+\delta_3+\varepsilon_2+\zeta_3\sim 0 \end{cases}$$

Then a canonical homology basis is represented as follows

$$(14)\quad \begin{cases} A_1=\zeta_4+\zeta_1-\gamma_1-\gamma_4, \quad A_2=\beta_3+\beta_4+\varepsilon_1+\varepsilon_2, \quad A_3=\alpha_1-\beta_1 \\ A_4=\gamma_1+\delta_2, \quad A_5=\alpha_3-\beta_3, \quad A_6=\gamma_3+\delta_4 \\ B_1=-\gamma_3+\delta_3+\varepsilon_2+\beta_3-\alpha_3+\zeta_3, \quad B_2=\beta_3-\alpha_3-\zeta_2-\gamma_1+\delta_1-\varepsilon_3 \\ B_3=\zeta_1-\alpha_1, \quad B_4=\varepsilon_1-\gamma_1, \quad B_5=\zeta_3-\alpha_3, \quad B_6=\varepsilon_3-\gamma_3 \end{cases}$$

Here, A_3, B_3, A_4 and B_4 are considered as a canonical homology basis of R by the natural projection $\hat{R}\to R$. We illustrate this homology basis in Appendix.

Then, from (12), (13) and (14), we get the following

$$(15)\quad \begin{cases} T^2A_1\approx -A_1 & T^2A_2\approx -A_2 \quad T^2A_3\approx A_5 \quad T^2A_4\approx A_6 \\ T^2B_1\approx -B_1 & T^2B_2\approx -B_2 \quad T^2B_3\approx B_5 \quad T^2B_4\approx B_6 \end{cases}$$

$$(16)\quad \begin{cases} A_1\approx A_3+TA_3-TA_4-A_6-TB_3-TB_4-2B_5-TB_5-B_6 \\ A_2\approx 2A_3+TA_3-A_4-TA_4+TA_5-TB_3-B_5 \\ B_1\approx -TA_3+TA_4-A_5+A_6+TB_4+B_5+TB_5+B_6 \\ B_2\approx A_3-TA_4-A_5+TA_6-TB_3-TB_4-B_5-B_6 \end{cases}$$

We put

$$\int_{A_j}\omega_i=a_{ij}, \qquad \int_{B_j}\omega_i=b_{ij}.$$

From (15), we see that

$$a_{51}=a_{52}=a_{61}=a_{62}=0, \quad a_{53}=a_{55}, \quad a_{54}=a_{56}, \quad a_{63}=a_{65},$$

$$a_{64}=a_{66}, \quad a_{13}=-a_{15}, \quad a_{14}=-a_{16}, \quad a_{23}=-a_{25}, \quad a_{24}=-a_{26},$$

$$a_{33}=-a_{35}, \quad a_{34}=-a_{36}, \quad a_{43}=-a_{45}, \quad a_{44}=-a_{46}.$$

There are same relations in $\{b_{ij}\}$.

We put

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}^{-1} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix},$$

$$\begin{pmatrix} a_{53} & a_{54} \\ a_{63} & a_{64} \end{pmatrix}^{-1} \begin{pmatrix} \omega_5 \\ \omega_6 \end{pmatrix} = \begin{pmatrix} \phi_5 \\ \phi_6 \end{pmatrix}, \quad \begin{pmatrix} a_{53} & a_{54} \\ a_{63} & a_{64} \end{pmatrix}^{-1} \begin{pmatrix} b_{53} & b_{54} \\ b_{63} & b_{64} \end{pmatrix} = (\pi_{ij}) = \pi$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}^{-1} \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix} = \begin{pmatrix} 2b'_{11} & 2b'_{12} & b'_{13} & b'_{14} \\ 2b'_{21} & 2b'_{22} & b'_{23} & b'_{24} \\ 2b'_{31} & 2b'_{32} & b'_{33} & b'_{34} \\ 2b'_{41} & 2b'_{42} & b'_{43} & b'_{44} \end{pmatrix}$$

where, from Riemann's equation, $b'_{ij}=b'_{ji}$, $\pi_{12}=\pi_{21}$. Then

$$\left(\int_{A_j}\phi_i\right)=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&-1&0\\0&0&0&0&0&-1\\0&0&1&0&1&0\\0&0&0&1&0&1\end{pmatrix}$$

$$\left(\int_{B_j}\phi_i\right)=\begin{pmatrix}2b'_{11}&2b'_{12}&b'_{13}&b'_{14}&-b'_{13}&-b'_{14}\\2b'_{12}&2b'_{22}&b'_{23}&b'_{24}&-b'_{23}&-b'_{24}\\2b'_{13}&2b'_{23}&b'_{33}&b'_{34}&-b'_{33}&-b'_{34}\\2b'_{14}&2b'_{24}&b'_{34}&b'_{44}&-b'_{34}&-b'_{44}\\0&0&\pi_{11}&\pi_{12}&\pi_{11}&\pi_{12}\\0&0&\pi_{12}&\pi_{22}&\pi_{12}&\pi_{22}\end{pmatrix}.$$

We put

$$\phi_i=C_i\omega_1+\tilde{\phi}_i\ (i=1,\ 2,\ 3,\ 4)$$

where $\tilde{\phi}_i$ are linear combinations of ω_2, ω_3 and ω_4. Since C_i is the i-th entry of the first column of $(a_{ij})^{-1}{}_{1\leq i,j\leq 4}$,

$$a_{11}C_1+a_{12}C_2+a_{13}C_3+a_{14}C_4=1.$$

We put

$$\int_{A_3}\omega_1=a_{13}=k \qquad \int_{A_4}\omega_1=a_{14}=l$$

$$\int_{B_4}\omega_1=b_{14}=m \qquad \int_{B_4}\omega_1=b_{14}=n.$$

Then

$$\int_{TA_3}\phi_i=\int_{TA_3}(C_i\omega_1+\tilde{\phi}_i)=2iC_ik-i\int_{A_3}\phi_i, \qquad \int_{TA_4}\phi_i=2iC_il-i\int_{A_4}\phi_i$$

$$\int_{TB_3}\phi_i=2iC_im-i\int_{B_3}\phi_i, \qquad \int_{TB_4}\phi_i=2iC_in-i\int_{B_4}\phi_i.$$

From (16), we see that

$$1=\int_{A_1}\phi_1=\int_{A_3}\phi_1+\int_{TA_3}\phi_1-\int_{TA_4}\phi_1-\int_{A_6}\phi_1-\int_{TB_3}\phi_1-\int_{TB_4}\phi_1$$
$$-2\int_{B_5}\phi_1-\int_{TB_5}\phi_1-\int_{B_6}\phi_1$$

or

$$1=2iC_1(k-l-n)+2b'_{13}+(1+i)b'_{14}.$$

We have the following relations similarly,

$$0=-2iC_1(l+m)+(1+i)b'_{12}$$

$$2b'_{11}=2iC_1(-k+l-m+n)-(1-i)b'_{13}-(1+i)b'_{14}$$

$$2b'_{12}=-2iC_1(2l+m+n)+(1+i)(b'_{13}+b'_{14})$$

$$0=2iC_2(k-l-n)+2b'_{23}+(1+i)b'_{24}$$

$$1=-2iC_2(l+m)+(1+i)b'_{23}$$

$$2b'_{12}=2iC_2(-k+l-m+n)-(1-i)b'_{23}-(1+i)b'_{24}$$

$$2b'_{22}=-2iC_2(2l+m+n)+(1+i)(b'_{23}+b'_{24})$$

$$0=(1-i)+2iC_3(k-l-n)+2b'_{33}+(1+i)b'_{34}$$

$$0=2-2iC_3(l+m)+(1+i)b'_{33}$$

$$2b'_{13}=(1+i)+2iC_3(-k+l-m+n)-(1-i)b'_{33}-(1+i)b'_{34}$$

$$2b'_{23}=2-2iC_3(2l+m+n)+(1+i)(b'_{33}+b'_{34})$$

$$0=(1+i)+2iC_4(k-l-n)+2b'_{34}+(1+i)b'_{44}$$

$$0=-(1-i)-2iC_4(l+m)+(1+i)b'_{34}$$

$$2b'_{14}=-(1+i)+2iC_4(-k+l-m+n)-(1-i)b'_{34}-(1+i)b'_{44}$$

$$2b'_{24}=2i-2iC_4(2l+m+n)+(1+i)(b'_{34}+b'_{44})$$

From the above relations, we have

$$(17)\quad \begin{cases} 2b'_{11}=-1+2i/s, \quad 2b'_{12}=1-2ip/s, \quad b'_{13}=-iq/s \\ b'_{14}=(1-i)/2-ir/s, \quad 2b'_{22}=i+2ip^2/s, \quad b'_{23}=(1-i)/2+ipq/s \\ b'_{24}=i+ipr/s, \quad b'_{33}=-(1-i)+iq^2/s, \quad b'_{34}=-i+iqr/s \\ b'_{44}=i+ir^2/s \\ \text{where} \quad p=k/l+(1-i)m/l+(1-i) \\ \qquad q=-(1-i)m/l-(1-i) \\ \qquad r=(1-i)k/l-2im/l-(1-i)n/l-(1+i) \\ \qquad s=-2p^2-q^2-2pq+4p-2r-2 \end{cases}$$

REMARK 1. In (17), p, q and r are represented as other forms. Indeed, we put

$$\begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix}^{-1} \begin{pmatrix} a_{21} \\ a_{31} \\ a_{41} \end{pmatrix} = \begin{pmatrix} p' \\ q' \\ r' \end{pmatrix}$$

then, we can find $p=p'$, $q=q'$ and $r=r'$ by means of concrete caluculation. Furthermore, we put

$$\Delta(\hat{A}_3, \hat{A}_4, \hat{B}_3)=\det\begin{pmatrix} a_{23} & a_{24} & b_{23} \\ a_{33} & a_{34} & b_{33} \\ a_{43} & a_{44} & b_{43} \end{pmatrix}, \quad \Delta(\hat{A}_3, \hat{A}_4, \hat{B}_4)=\det\begin{pmatrix} a_{23} & a_{24} & b_{24} \\ a_{33} & a_{34} & b_{34} \\ a_{43} & a_{44} & b_{44} \end{pmatrix}$$

$$\Delta(\hat{A}_3, \hat{B}_3, \hat{B}_4)=\det\begin{pmatrix} a_{23} & b_{23} & b_{24} \\ a_{33} & b_{33} & b_{34} \\ a_{43} & b_{43} & b_{44} \end{pmatrix}, \quad \Delta(\hat{A}_4, \hat{B}_3, \hat{B}_4)=\det\begin{pmatrix} a_{24} & b_{23} & b_{24} \\ a_{34} & b_{33} & b_{34} \\ a_{44} & b_{43} & b_{44} \end{pmatrix}$$

Then, we see, by means of caluculation,

$$(18) \quad \begin{cases} p=(1-i)+\dfrac{\Delta(\hat{A}_3, \hat{A}_4, \hat{B}_4)}{\Delta(\hat{A}_3, \hat{A}_4, \hat{B}_3)} \\ q=-(1-i)-2\dfrac{\Delta(\hat{A}_3, \hat{A}_4, \hat{B}_3)}{\Delta(\hat{A}_3, \hat{A}_4, \hat{B}_4)}+(1+i)\dfrac{\Delta(\hat{A}_4, \hat{B}_3, \hat{B}_4)}{\Delta(\hat{A}_3, \hat{A}_4, \hat{B}_3)} \\ r=(1-i)+(1-i)\dfrac{\Delta(\hat{A}_3, \hat{A}_4, \hat{B}_4)}{\Delta(\hat{A}_3, \hat{A}_4, \hat{B}_3)}-(1+i)\dfrac{\Delta(\hat{A}_4, \hat{B}_3, \hat{B}_4)}{\Delta(\hat{A}_3, \hat{A}_4, \hat{B}_3)} \end{cases}$$

or from (17)

$$(19) \quad \begin{cases} \dfrac{\Delta(\hat{A}_3, \hat{A}_4, \hat{B}_4)}{\Delta(\hat{A}_3, \hat{A}_4, \hat{B}_3)}=k/l+(1-i)m/l \\ \dfrac{\Delta(\hat{A}_4, \hat{B}_3, \hat{B}_4)}{\Delta(\hat{A}_3, \hat{A}_4, \hat{B}_3)}=(1-i)k/l-im/l \\ \dfrac{\Delta(\hat{A}_3, \hat{B}_3, \hat{B}_4)}{\Delta(\hat{A}_3, \hat{A}_4, \hat{B}_3)}=-in/l+(1-i) \end{cases}$$

We put

$$\varphi_i=\phi_i, \quad \varphi_{i+2}=(\phi_{i+2}+\phi_{i+4})/2, \quad \varphi_{i+4}=(\phi_{i+4}-\phi_{i+2})/2 \qquad (i=1, 2).$$

Then we see that

PROPOSITION 8. *$\tilde{R}$ has a period matrix of the form*

$$\left(\int_{A_j}\varphi_i, \int_{B_j}\varphi_i\right)=\begin{pmatrix} & 2L & M & -M \\ I_6, & M & (\pi+S)/2 & (\pi-S)/2 \\ & -M & (\pi-S)/2 & (\pi+S)/2 \end{pmatrix}$$

where

$$L=\begin{pmatrix} -1/2+i/s & 1/2-ip/s \\ 1/2-ip/s & i/2+ip^2/s \end{pmatrix}, \quad M=\begin{pmatrix} -iq/s & (1-i)/2-ir/s \\ (1-i)/2+ipq/s & i+ipr/s \end{pmatrix}$$

$$S=\begin{pmatrix} -(1-i)+iq^2/s & -i+iqr/s \\ -i+iqr/s & i+ir^2/s \end{pmatrix}, \quad \pi=\begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{12} & \pi_{22} \end{pmatrix}.$$

Furthermore, π is the period matrix for R with respect to the homology basis on R determined by the natural projection $\tilde{R}\to R$.

PROPOSITION 9. *There is a second order transformation of $\mathfrak{S}_6$ which maps the period matrix of $\tilde{R}$ to the matrix*

$$\begin{pmatrix} \Pi & 0 \\ 0 & \pi \end{pmatrix} \quad \textit{where} \quad \Pi=\begin{pmatrix} L & M \\ M & S \end{pmatrix}\in\mathfrak{S}_4.$$

PROOF. We take the second transformation $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $\mathfrak{S}_6$ where

$$A=\begin{pmatrix} I_2 & 0 & 0 \\ 0 & I_2 & -I_2 \\ 0 & I_2 & I_2 \end{pmatrix}, \quad B=C=0, \quad D=\begin{pmatrix} 2I_2 & 0 & 0 \\ 0 & I_2 & -I_2 \\ 0 & I_2 & I_2 \end{pmatrix}.$$

THEOREM 3. *In case* (III), *the Prym variety P has a period matrix of the form* (I_4, Π) *where Π is that in proposition* 9, *that is,*

$$\Pi=\begin{pmatrix} -1/2+i/s & 1/2-ip/s & -iq/s & (1-i)2-ir/s \\ 1/2-ip/s & i/2+ip^2/s & (1-i)/2+ipq/s & i+ipr/s \\ -iq/s & (1-i)/2+ipq/s & -(1-i)+iq^2/s & -i+iqr/s \\ (1-i)/2-ir/s & i+ipr/s & -i+iqr/s & i+ir^2/s \end{pmatrix}.$$

REMARK 2. Π has three parameters p, q and r. These parameters are independednt, but the strict proof will be given at § 2.

§ 2. Kodaira-Spencer map and local injectivity.

First we recall the Kodaira-Spencer map associated with a deformation according to Sekiguchi's paper ([8], [4]).

Let S be a k-scheme, $o\in S$ a k-rational point and $\pi: X\to S$ be a smooth and proper morphism. Denote X_0 the fibre over $o\in S$.

We have a short exact sequence of sheaves on X;

$$0 \longrightarrow \mathcal{T}_{X/S} \longrightarrow \mathcal{T}_X \longrightarrow \pi^*\mathcal{T}_S \longrightarrow 0$$

defining the relative tangent sheaf $\mathcal{T}_{X/S}$. By restriction this to X_0, we obtain an exact sequence

$$0 \longrightarrow \mathcal{T}_{X_0} \longrightarrow \mathcal{T}_X \otimes \mathcal{O}_{X_0} \longrightarrow \mathcal{T}_{S_0} \otimes_k \mathcal{O}_{X_0} \longrightarrow 0 .$$

From the exact seqence of cohomology, we obtain the so-called Kodaira-Spencer map

$$\kappa : T_{S,0} \longrightarrow H^1(X_0, \mathcal{T}_{X_0}) .$$

From now on, we will compute this map exactly in our cases by Sekiguchi's method.

CASE (I). In the above general theory, $k = \mathbf{C}$

$$X = \tilde{R}_t : y^4 = (x-1-t)(x^2+x+1)$$

$$X_0 = \tilde{R}_0 : y^4 = x^3 - 1$$

$$S = \mathrm{Spec}(\mathbf{C}[t]/(t^2)) .$$

A smooth model of $\tilde{R}_t$ is given by

$$\tilde{R}_t = \mathcal{U}_t \cup \mathcal{V}_t$$

with

$$\mathcal{U}_t = \mathrm{Spec}(\mathbf{C}[t][X, Y]/(Y^4 - (X^3-1) - t(X^2+X+1)))$$

$$= \mathrm{Spec}(\mathbf{C}[t][x, y])$$

$$\mathcal{V}_t = \mathrm{Spec}(\mathbf{C}[U, V]/(V^4 + U^4 - U + t(U^4 + U^3 + U^2)))$$

$$= \mathrm{Spec}(\mathbf{C}[t][u, v])$$

and

$$\mathcal{U}_t \cap \mathcal{V}_t = \mathrm{Spec}(\mathbf{C}[t][x, y, x^{-1}]) = \mathrm{Spec}(\mathbf{C}[t][u, v, u^{-1}])$$

where $\begin{cases} x = u^{-1} \\ y = u^{-1}v \end{cases}$ or $\begin{cases} X = U^{-1} \\ Y = U^{-1}V \end{cases}$

By using the Čeck cohomology over the covering $\{\mathcal{U}_t, \mathcal{V}_t\}$ of $\tilde{R}_t$, we can compute the cohomology groups $H^i(\tilde{R}_t, \mathcal{T})$ for a coherent sheaf $\mathcal{T}$ on $\tilde{R}_t$. In particular, we obtain the following:

LEMMA 1.
(i) *A basis of* $H^0(\tilde{R}_t, \Omega_{\tilde{R}_t})$ *is given by*

$$\omega_1 = y^{-3}dx, \qquad \omega_2 = y^{-3}x\,dx, \qquad \omega_3 = y^{-2}dx .$$

(ii) *A basis of* $H^0(\tilde{R}_t, \Omega^{\otimes 2})$ *is given by*

$$\Omega_1=\omega_1^2=y^{-6}(dx)^2, \quad \Omega_2=\omega_1\omega_2=y^{-6}x(dx)^2, \quad \Omega_3=\omega_2^2=y^{-6}x^2(dx)^2,$$

$$\Omega_4=\omega_3^2=y^{-4}(dx)^2, \quad \Omega_4=\omega_1\omega_3=y^{-5}(dx)^2, \quad \Omega_6=\omega_2\omega_3=y^{-5}x(dx)^2.$$

(iii) *A basis of* $H^1(\tilde{R}_t, \check{\Omega}_{\tilde{R}_t})$ *is given by*

$$\theta_1=x^{-1}y^6\partial_x, \quad \theta_2=x^{-2}y^6\partial_x, \quad \theta_3=x^{-3}y^6\partial_x,$$

$$\theta_4=x^{-1}y^4\partial_x, \quad \theta_5=x^{-1}y^5\partial_x, \quad \theta_6=x^{-2}y^5\partial_x,$$

where ∂_x *is the derivation on* $\mathbf{C}[x, y]$ *defined by*

$$\partial_x(x)=1, \quad \partial_x(y)=dy/dx.$$

PROOF. (i) is mentioned in §1. (ii) is asserted by Max Noeter's theorem. (iii) First, we have to compute $H^1(\tilde{R}_t, \Omega_{\tilde{R}_t})$. We see

$$\Gamma(\mathcal{U}_t, \Omega_{\tilde{R}_t})=\mathbf{C}[x, y]y^{-3}dx \quad \text{and} \quad \Gamma(\mathcal{V}_t, \Omega_{\tilde{R}_t})=\mathbf{C}[u, v]v^{-3}du$$

$$\Gamma(\mathcal{U}_t, \Omega_{\tilde{R}_t})\cap\Gamma(\mathcal{V}_t, \Omega_{\tilde{R}_t})=\mathbf{C}[x^{-1}, x^{-1}y]y^{-3}xdx.$$

Therefore $x^{-1}dx$ is a basis of $H^1(\tilde{R}_t, \Omega_{\tilde{R}_t})$. From Serre duality theorem, corresponing to a basis of $H^0(\tilde{R}_t, \Omega^{\otimes 2})$, we obtain

$$H^1(\tilde{R}_t, \check{\Omega}_{\tilde{R}_t})=\langle x^{-1}y^6\partial_x,\ x^{-2}y^6\partial_x,\ x^{-3}y^6\partial_x,\ x^{-1}y^5\partial_x,\ x^{-2}y^5\partial_x,\ x^{-1}y^4\partial_x\rangle$$

where $\check{\Omega}_{\tilde{R}_t}$ coincides the tangent sheaf $\mathcal{T}_{\tilde{R}_t}$ of $\tilde{R}_t$.

We look for derivations over $\mathbf{C}$:

$$\mathcal{D}: \mathbf{C}[t][x, y] \longrightarrow \mathbf{C}[t][x, y] \quad \text{and} \quad \mathcal{D}: \mathbf{C}[t][u, v] \longrightarrow \mathbf{C}[t][u, v]$$

such that

$$\mathcal{D}(t)=1 \quad \text{and} \quad \tilde{\mathcal{D}}(t)=1.$$

If $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are such derivations, then we get the following; Since

$$y^4=x^3-1-t(x^2+x+1) \quad \text{and} \quad v^4=-u(u^3-1)-tu^2(u^2+u+1),$$

(1) $4y^3\mathcal{D}(y)=\{3x^2-t(2x+1)\}\mathcal{D}(x)-(x^2+x+1),$

(1′) $4v^3\tilde{\mathcal{D}}(v)=\{-4u^3+1-tu(4u^2+3u+2)\}\tilde{\mathcal{D}}(u)-u^2(u^2+u+1).$

Put

$$(2)\quad \begin{cases} \mathcal{D}(x)=(A+By+Cy^2+Dy^3)+(G+Hy+Jy^2+Ky^3)t \\ \mathcal{D}(y)=(L+My+Ny^2+Py^3)+(Q+Ry+Sy^2+Ty^3)t \end{cases}$$

$$(2)'\quad \begin{cases} \tilde{\mathcal{D}}(u)=(\tilde{A}+\tilde{B}v+\tilde{C}v^2+\tilde{D}v^3)+(\tilde{G}+\tilde{H}v+\tilde{J}v^2+\tilde{K}v^3)t \\ \tilde{\mathcal{D}}(v)=(\tilde{L}+\tilde{M}v+\tilde{N}v^2+\tilde{P}v^3)+(\tilde{Q}+\tilde{R}v+\tilde{S}v^2+\tilde{T}v^3)t. \end{cases}$$

From (1), (2), (1′) and (2′), we get

$$(3)\quad \begin{cases} 4M(x^3-1)=3Ax^2-(x^2+x+1) \\ 4N(x^3-1)=3Bx^2 \\ 4P(x^3-1)=3Cx^2 \\ 4L=3Dx^2 \\ -4M(x^2+x+1)+4R(x^3-1)=-A(2x+1)+3Gx^2 \\ -4N(x^2+x+1)+4S(x^3-1)=-B(2x+1)+3Hx^2 \\ -4P(x^2+x+1)+4T(x^3-1)=-C(2x+1)+3Jx^2 \\ 4Q=D(2x+1)+3Kx^2 \end{cases}$$

$$(3)'\quad \begin{cases} 4\tilde{M}(-u^4+u)=\tilde{A}(-4u^3+1)-(u^4+u^3+u^2) \\ 4\tilde{N}(-u^4+u)=\tilde{B}(-4u^3+1) \\ 4\tilde{P}(-u^4+u)=\tilde{C}(-4u^3+1) \\ 4\tilde{L}(-u^4+u)=\tilde{D}(-4u^3+1) \\ -4\tilde{M}(u^4+u^3+u^2)+4\tilde{R}(-u^4+u)=-\tilde{A}(4u^3+3u^2+2u)+\tilde{G}(-4u^3+1) \\ -4\tilde{N}(u^4+u^3+u^2)+4\tilde{S}(-u^4+u)=-\tilde{B}(4u^3+3u^2+2u)-\tilde{H}(-4u^3+1) \\ -4\tilde{P}(u^4+u^3+u^2)+4\tilde{T}(-u^4+u)=-\tilde{C}(4u^4+3u^2+2u)+\tilde{J}(-4u^3+1) \\ 4\tilde{Q}=-\tilde{D}(4u^3+3u^2+2u)+\tilde{K}(-4u^3+1) \end{cases}$$

In these equalities, we may put

$$B=C=D=H=J=K=N=P=L=S=T=Q=0$$

and

$$\tilde{B}=\tilde{C}=\tilde{D}=\tilde{H}=\tilde{J}=\tilde{K}=\tilde{N}=\tilde{P}=\tilde{L}=\tilde{S}=\tilde{T}=\tilde{Q}=0.$$

Then, we get

$$(4)\quad \begin{cases} 4M(x^3-1)=3Ax^2-(x^2+x+1) \\ -4M(x^2+x+1)+4R(x^3-1)=-A(2x+1)+3Gx^2 \end{cases}$$

$$(4')\quad \begin{cases} 4\tilde{M}(-u^4+u)=\tilde{A}(-4u^3+1)-(u^4+u^3+u^2) \\ -4\tilde{M}(u^4+u^3+u^2)+4\tilde{R}(-u^4+u)=-\tilde{A}(4u^3+3u^2+2u)+\tilde{G}(-4u^3+1). \end{cases}$$

Here, we can put

$$(5)\quad \begin{cases} A=(x^2+x+1)/3, \qquad M=(x+1)/4 \\ G=-(x^2+x+1)/3, \qquad R=-(3x+2)/12, \end{cases}$$

$$(5') \quad \begin{cases} \tilde{A}=-u(u^2+u+1)/3, & \tilde{M}=-(4u^2+4u+1)/12 \\ \tilde{G}=u(u-1)(u^2+u+1)/3, & \tilde{R}=(4u^3+u-1)/12. \end{cases}$$

Therefore, by (2), (2′), (5) and (5′), we get

$$(6) \quad \begin{cases} \mathscr{D}(x)=(x^2+x+1)/3-(x^2+x+1)t/3 \\ \mathscr{D}(y)=(x+1)y/4-(3x+2)yt/12 \end{cases}$$

$$(6') \quad \begin{cases} \tilde{\mathscr{D}}(u)=-u(u^2+u+1)/3+u(u-1)(u^2+u+1)t/3 \\ \tilde{\mathscr{D}}(v)=-u(4u^2+4u+1)v/12+(4u^3+u-1)vt/12. \end{cases}$$

Now we put

$$\bar{\mathscr{D}}=\mathscr{D} \pmod t \qquad \bar{\tilde{\mathscr{D}}}=\tilde{\mathscr{D}} \pmod t$$

Then we get

$$\bar{\mathscr{D}}(x)=\bar{\mathscr{D}}(u^{-1})=-u^{-2}\bar{\mathscr{D}}(u)=(u+1+u^{-1})/3=(x+1+x^{-1})/3$$

$$\bar{\tilde{\mathscr{D}}}(y)=\bar{\tilde{\mathscr{D}}}(u^{-1}v)=u^{-2}\{u\bar{\tilde{\mathscr{D}}}(v)-v\bar{\tilde{\mathscr{D}}}(u)\}=(4u)^{-1}v=y/4.$$

Hence on $\mathcal{U}_0\cap\mathcal{V}_0$, we get

$$(7) \quad \begin{cases} (\bar{\mathscr{D}}-\bar{\tilde{\mathscr{D}}})(x)=(x^2+x+1)/3-(x+1+x^{-1})/3=(x^2-x^{-1})/3=y^4/3x \\ (\bar{\mathscr{D}}-\bar{\tilde{\mathscr{D}}})(y)=(x+1)y/4-y/4=xy/4=(y^4/3x)(dy/dx) \end{cases}$$

Here, we notice that

$$\kappa((\partial/\partial_t)_0)=(\bar{\mathscr{D}}-\bar{\tilde{\mathscr{D}}})\in H^1(\tilde{R}_0, \mathcal{T}_{R_0})$$

and from (7),

$$\bar{\mathscr{D}}-\bar{\tilde{\mathscr{D}}}=(3x)^{-1}y^4\partial_x.$$

Therefore, from lemma 1 (iii), we could prove the following

THEOREM 4. *The Kodaira-Spencer map* $\kappa: T_{S,0}\to H^1(\tilde{R}_0, \mathcal{T}_{\tilde{R}_0})$ *is given by*

$$(8) \qquad \kappa((\partial/\partial_t)_0)=(3x)^{-1}y^4\partial_x=\theta_4/3.$$

REMARK 3. Theorem 4 is an alternative proof of the fact that there is no contribution of the parameter t to Prym variety (This is a direct result of Theorem 1). Indeed, from Serre duality, θ_4 corresponds to Ω_4 which is a quadratic differential for R, while $\Omega_4=\omega_3^2$ does not corresponds to Prym variety since ω_3 is not a Prym differential.

CASE (II) $X=\tilde{R}_t:\ y^6=(x-1-t)(x^2+x+1)$

$X_0=\tilde{R}_0:\ y^6=x^3-1$

$S=\mathrm{Spec}\,(\mathbf{C}[t]/(t^2))$

A smooth model of $\tilde{R}_t$ is given by

$$\tilde{R}_t=\mathcal{U}_t\cup\mathcal{V}_t$$

with

$$\mathcal{U}_t=\mathrm{Spec}\,(\mathbf{C}[t][X, Y]/(Y^6-(X^3-1)+t(X^2+X+1)))$$
$$=\mathrm{Spec}\,(\mathbf{C}[t][x, y])$$
$$\mathcal{V}_t=\mathrm{Spec}\,(\mathbf{C}[t][U, V]/(1-(V^3-U^6)+t(V^2U^2+VU^4+U^6)))$$
$$=\mathrm{Spec}\,(\mathbf{C}[t][u, v])$$

and

$$\mathcal{U}_t\cap\mathcal{V}_t=\mathrm{Spec}\,(\mathbf{C}[t][x, y, y^{-1}])=\mathrm{Spec}\,(\mathbf{C}[t][u, v, u^{-1}])$$

where

$$x=u^{-2}v \qquad y=u^{-1}.$$

LEMMA 2.

(i) *A basis of* $H^0(\tilde{R}_t, \Omega_{\tilde{R}_t})$ *is given by*

$$\omega_1=y^{-3}dx, \qquad \omega_2=y^{-5}dx, \qquad \omega_3=y^{-5}xdx, \qquad \omega_4=y^{-4}dx$$

(iii) *A basis of* $H^0(\tilde{R}_t, \Omega^{\otimes 2})$ *is given by*

$$\Omega_1=\omega_1^2=y^{-6}(dx)^2, \qquad \Omega_2=\omega_1\omega_2=y^{-8}(dx)^2, \qquad \Omega_3=\omega_1\omega_3=y^{-8}x(dx)^2$$
$$\Omega_4=\omega_2^2=y^{-10}(dx)^2, \qquad \Omega_5=\omega_2\omega_3=y^{-10}x(dx)^2, \qquad \Omega_6=\omega_3^2=y^{-10}x^2(dx)^2$$
$$\Omega_7=\omega_1\omega_4=y^{-7}(dx)^2, \qquad \Omega_8=\omega_2\omega_4=y^{-9}(dx)^2, \qquad \Omega_9=\omega_3\omega_4=y^{-9}x(dx)^2$$

(iii) *A basis of* $H^1(\tilde{R}_t, \check{\Omega}_{\tilde{R}_t})$ *is given by*

$$\theta_1=x^2\partial_x, \qquad \theta_2=x^2y^2\partial_x, \qquad \theta_3=xy^2\partial_x$$
$$\theta_4=x^2y^4\partial_x, \qquad \theta_5=xy^4\partial_x, \qquad \theta^9=y^4\partial_x$$
$$\theta_7=x^2y\partial_x, \qquad \theta_8=x^2y^3\partial_x, \qquad \theta_9=xy^3\partial_x$$

PROOF. The proof of the above is the same as the lemma 1 and is therefore omitted.

We look for derivations over $\mathbf{C}$

$$\mathcal{D}: \mathbf{C}[t][x, y] \longrightarrow \mathbf{C}[t][x, y] \quad \text{and} \quad \tilde{\mathcal{D}}: \mathbf{C}[t][u, v] \longrightarrow \mathbf{C}[t][u, v]$$

Since

$$y^6=x^3-1-t(x^2+x+1) \quad \text{and} \quad v^3=u^6+1+t(u^2v^2+u^4v+u^6),$$

we get the following

(9) $6y^5\mathcal{D}(y)=\{3x^2-t(2x+1)\}\mathcal{D}(x)-(x^2+x+1)$

and

(9′) $\{3v^2-t(2u^2v+u^4)\}\tilde{\mathscr{D}}(v)=\{6u^5+t(2uv^2+4u^3v+6u^5)\}\tilde{\mathscr{D}}(u)+(u^2v^2+u^4v+u^6)$.

Put

$$(10)\quad\begin{cases}\mathscr{D}(x)=A+By+Cy^2+Dy^3+Ey^4+Fy^5+t(G+Hy+Iy^2+Jy^3+Ky^4+Ly^5)\\ \mathscr{D}(y)=M+Ny+Oy^2+Py^3+Qy^4+Ry^5+t(S+Ty+Uy^2+Vy^3+Wy^4+Zy^5)\end{cases}$$

$$(10')\quad\begin{cases}\tilde{\mathscr{D}}(u)=\tilde{A}+\tilde{B}v+\tilde{C}v^2+t(\tilde{D}+\tilde{E}v+\tilde{F}v^2)\\ \tilde{\mathscr{D}}(v)=\tilde{G}+\tilde{H}v+\tilde{I}v^2+t(\tilde{J}+\tilde{K}v+\tilde{L}v^2)\end{cases}$$

From (9), (10), (9′) and (10′), we get

$$(11)\quad\begin{cases}6N(x^3-1)=3Ax^2-(x^2+x+1)\\ 6O(x^3-1)=3Bx^2\\ 6P(x^3-1)=3Cx^2\\ 6Q(x^3-1)=3Dx^2\\ 6R(x^3-1)=3Ex^2\\ 6M=3Fx^2\\ -6N(x^2+x+1)+6T(x^3-1)=-A(2x+1)+3Gx^2\\ -6O(x^2+x+1)+6U(x^3-1)=-B(2x+1)+3Hx^2\\ -6P(x^2+x+1)+6V(x^3-1)=-C(2x+1)+3Ix^2\\ -6Q(x^2+x+1)+6W(x^3-1)=-D(2x+1)+3Jx^2\\ -6R(x^2+x+1)+6Z(x^3-1)=-E(2x+1)+3Kx^2\\ 6S=-F(2x+1)+3Lx^2\end{cases}$$

$$(11')\quad\begin{cases}3\tilde{H}(u^6+1)=6\tilde{A}u^5+u^6\\ 3\tilde{I}(u^6+1)=6\tilde{B}u^5+u^4\\ 3\tilde{G}=6\tilde{C}u^5+u^2\\ -\tilde{G}u^4+3\tilde{H}u^6+\tilde{I}u^2(u^6+1)+3\tilde{K}(u^6+1)\\ \qquad=6\tilde{A}u^5+2\tilde{B}u(u^6+1)+4\tilde{C}u^3(u^6+1)+6\tilde{D}u^5\\ -2\tilde{G}u^2+2\tilde{H}u^4+3\tilde{I}u^6+3\tilde{L}(u^6+1)=4\tilde{A}u^3+6\tilde{B}u^5+2\tilde{C}u+6\tilde{E}u^5\\ \tilde{H}u^2+2\tilde{I}u^4+3\tilde{J}=2\tilde{A}u+4\tilde{B}u^3+6\tilde{C}u^5+6\tilde{F}u^5\end{cases}$$

In these we may put

$$B=C=D=E=F=H=I=J=K=L=0$$

$$M=O=P=Q=R=S=U=V=W=Z=0$$

and

$$\tilde{C}=\tilde{F}=\tilde{J}=0.$$

Moreover we can put

$$(12)\quad \begin{cases} A=(x^2+x+1)/3, & G=-(x^2+x+1)/3 \\ N=(x+1)/6, & T=-(3x+2)/18 \end{cases}$$

$$(12')\quad \begin{cases} \tilde{A}=u^7/6, \quad \tilde{B}=u^5/6, \quad \tilde{D}=u^7/18, \quad \tilde{G}=u^2/3 \\ \tilde{H}=u^6/3, \quad \tilde{I}=u^4/3, \quad \tilde{K}=u^6/9, \quad \tilde{L}=2u^4/9. \end{cases}$$

Therefore, by (10), (10′), (12) and (12′), we get

$$(13)\quad \begin{cases} \mathcal{D}(x)=(x^2+x+1)/3-(x^2+x+1)t/3 \\ \mathcal{D}(y)=(x+1(y/6-(3x+2)yt/18 \end{cases}$$

$$(13')\quad \begin{cases} \tilde{\mathcal{D}}(u)=(u^7+u^5v)/6+(u^7+2u^5v)t/18 \\ \tilde{\mathcal{D}}(v)=(u^2+u^6v+u^4v^2)/3+(2u^4v+u^6v^2)t/9. \end{cases}$$

Now we put

$$\bar{\mathcal{D}}\equiv\mathcal{D}, \qquad \bar{\tilde{\mathcal{D}}}\equiv\tilde{\mathcal{D}} \pmod t.$$

Then, we get

$$\bar{\tilde{\mathcal{D}}}(x)=\bar{\tilde{\mathcal{D}}}(u^{-2}v)=u^{-3}\{u\bar{\tilde{\mathcal{D}}}(v)-2v\bar{\tilde{\mathcal{D}}}(u)\}=1/3$$

$$\bar{\tilde{\mathcal{D}}}(y)=-u^{-2}\bar{\tilde{\mathcal{D}}}(u)=-(x+1)y^{-5}/6$$

Hence on $\mathcal{U}_0\cap\mathcal{V}_0$, we get

$$(14)\quad \begin{cases} (\tilde{\mathcal{D}}-\bar{\tilde{\mathcal{D}}})(x)=(x^2+x)/3 \\ (\tilde{\mathcal{D}}-\bar{\tilde{\mathcal{D}}})(y)=(x^2/3)(dy/dx)+(x/3)(dy/dx). \end{cases}$$

Here $\kappa((\partial/\partial_t)_0)=\tilde{\mathcal{D}}-\bar{\tilde{\mathcal{D}}}$ is determined by the part $(x^2/3)\partial_x$ and $(x/3)\partial_x$ is a boundary component from lemma 2, (iii). Therefore, we could prove the following

THEOREM 5. *The Kodaira-Spencer map* κ: $T_{S,0}\rightarrow H^1(\tilde{R}_0, \mathcal{T}_{\tilde{R}_0})$ *is given by*

$$(15)\qquad \kappa((\partial/\partial_t)_0)=(x^2/3)\partial_x=\theta_1/3.$$

REMARK 4. Theorem 5 shows that there is the contribution of the paramenter t to Prym variety. Indeed, from Serre duality, θ_1 corresponds to Ω_1 which is the quadratic differential for $\hat{R}$, and $\Omega_1=\omega_1^2$ corresponds to Prym since ω_1 is a Prym differential.

CASE (III)

$X=\tilde{R}_{t_1,t_2,t_3}:\ y^4=(x-1-t_1)(x-\rho-\rho t_2)(x-\rho^2-\rho^2 t_3)(x-\rho^3)(x-\rho^4)$

$X_0=\tilde{R}_{0,0,0}:\ y^4=x^5-1$

$S=\mathrm{Spec}(\mathbf{C}[[t_1, t_2, t_3]])$

For our purpose, we may consider $\tilde{R}_{t_1,0,0}$, $\tilde{R}_{0,t_2,0}$ and $\tilde{R}_{0,0,t_3}$ separately. So we reset

$$X=\tilde{R}_{t_1,0,0}=\tilde{R}_{t_1}:\ y^4=(x-1-t_1)(x^4+x^3+x^2+x+1)$$

$$S=\mathrm{Spec}(\mathbf{C}[t_1]/(t_1^2))$$

A smooth model of $\tilde{R}_{t_1}$ is given by

$$\tilde{R}_{t_1}=\mathcal{U}_{t_1}\cup\mathcal{V}_{t_1}$$

with

$$\mathcal{U}_{t_1}=\mathrm{Spec}(\mathbf{C}[t_1][X, Y]/(Y^4-(X^5-1)+t_1(X^4+X^3+X^2+X+1)))$$
$$=\mathrm{Spec}(\mathbf{C}[t_1][x, y])$$
$$\mathcal{V}_{t_1}=\mathrm{Spec}(\mathbf{C}[t_1][U, V]/(U-V^5+U^5+t_1(UV^4+U^2V^3+U^3V^2+U^4V+U^5)))$$
$$=\mathrm{Spece}(\mathbf{C}[t_1][u, v])$$

and

$$\mathcal{U}_{t_1}\cap\mathcal{V}_{t_1}=\mathrm{Spec}(\mathbf{C}[t_1][x, y, y^{-1}])=\mathrm{Spec}(\mathbf{C}[t_1][u, v, u^{-1}])$$

where

$$x=u^{-1}v \qquad y=u^{-1}$$

LEMMA 3.

(i) *A basis of* $H^0(X, \Omega_X)$ *is given by*

$$\omega_1=y^{-1}dx, \quad \omega_2=y^{-3}dx, \quad \omega_3=y^{-3}xdx$$
$$\omega_4=y^{-3}x^2dx, \quad \omega_5=y^{-2}dx, \quad \omega_6=y^{-2}xdx$$

(ii) *A basis of* $H^0(X, \Omega^{\otimes 2})$ *is given by*

$$\Omega_1=\omega_1^2=y^{-2}(dx)^2, \quad \Omega_2=\omega_1\omega_2=y^{-4}(dx)^2, \quad \Omega_3=\omega_1\omega_3=y^{-4}x(dx)^2$$
$$\Omega_4=\omega_1\omega_4=y^{-4}x^2(dx)^2, \quad \Omega_5=\omega_2^2=y^{-6}(dx)^2, \quad \Omega_6=\omega_2\omega_3=y^{-6}x(dx)^2$$
$$\Omega_7=\omega_2\omega_4=y^{-6}x^2(dx)^2, \quad \Omega_8=\omega_3\omega_4=y^{-6}x^3(dx)^2, \quad \Omega_9=\omega_4^2=y^{-6}x^4(dx)^2$$
$$\Omega_{10}=\omega_1\omega_5=y^{-3}(dx)^2, \quad \Omega_{11}=\omega_1\omega_6=y^{-3}x(dx)^2, \quad \Omega_{12}=\omega_2\omega_5=y^{-5}(dx)^2$$
$$\Omega_{13}=\omega_2\omega_6=y^{-5}x(dx)^2, \quad \Omega_{14}=\omega_3\omega_6=y^{-5}x^2(dx)^2, \quad \Omega_{15}=\omega_4\omega_6=y^{-5}x^3(dx)^2$$

(iii) *A basis of* $H^1(X, \check{\Omega}_X)$ *is given by*

$$\theta_1=y^{-2}x^4\partial_x, \quad \theta_2=x^4\partial_x, \quad \theta_3=x^3\partial_x, \quad \theta_4=x^2\partial_x, \quad \theta_5=x^4y^2\theta_x$$

$$\theta_6=x^3y^2\partial_x,\quad \theta_7=x^2y^2\partial_x,\quad \theta_8=xy^2\partial_x,\quad \theta_9=y^2\partial_x,\quad \theta_{10}=y^{-1}x^4\partial_x$$
$$\theta_{11}=y^{-1}x^3\partial_x,\quad \theta_{12}=x^4y\partial_x,\quad \theta_{13}=x^3y\partial_x,\quad \theta_{14}=x^2y\partial_x,\quad \theta_{15}=xy\partial_x$$

PROOF. The proof is omitted as lemma 2.

We look for derivations over C

$$\mathcal{D}_1: \mathbf{C}[t_1][x, y] \longrightarrow \mathbf{C}[t_1][x, y],\qquad \tilde{\mathcal{D}}_1: \mathbf{C}[t_1][u, v] \longrightarrow \mathbf{C}[t_1][u, v]$$

such that

$$\mathcal{D}_1(t_1)=1,\qquad \tilde{\mathcal{D}}_1(t_1)=1.$$

Since

$$y^4=(x^5-1)-t_1(x^4+x^3+x^2+x+1)$$

and

$$v^5=u^5+u+t_1(uv^4+u^2v^3+u_8v^2+u^4v+u^5),$$

we get the following

(16) $4y^3\mathcal{D}_1(y)=\{5x^4-t_1(4x^3+3x^2+2x+1)\}\mathcal{D}_1(x)-(x^4+x^3+x^2+x+1)$

(16′) $\{5v^4-t_1(4uv^3+3u^2v^2+2u^3v+u^4)\}\tilde{\mathcal{D}}_1(v)-(uv^4+u^2v^3+u^3v^2+u^4v+u^5)$
$$=\{(5u^4+1)+t_1(v^4+2uv^3+3u^2v^2+4u^3v+5u^4)\}\tilde{\mathcal{D}}_1(u)$$

Put

(17)
$$\begin{cases} \mathcal{D}_1(x)=A+By+Cy^2+Dy^3+t_1(E+Fy+Gy^2+Hy^3) \\ \mathcal{D}_1(y)=I+Jy+Ky^2+Ly^3+t_1(M+Ny+Py^2+Qy^3) \end{cases}$$

(17′)
$$\begin{cases} \tilde{\mathcal{D}}_1(u)=\tilde{A}+\tilde{B}v+\tilde{C}v^2+\tilde{D}v^3+\tilde{R}v^4+t_1(\tilde{E}+\tilde{F}v+\tilde{G}v^2+\tilde{H}v^3+\tilde{S}v^4) \\ \tilde{\mathcal{D}}_1(v)=\tilde{I}+\tilde{J}v+\tilde{K}v^2+\tilde{L}v^3+\tilde{W}v^4+t_1(\tilde{M}+\tilde{N}v+\tilde{P}v^2+\tilde{Q}v^3+\tilde{Z}v^4). \end{cases}$$

From (16), (17), (16′) and (17′), we get

(18)
$$\begin{cases} 4J(x^5-1)=5Ax^4-(x^4+x^3+x^2+x+1) \\ 4K(x^5-1)=5Bx^4 \\ 4L(x^5-1)=5Cx^4 \\ 4I=5Dx^4 \\ -4J(x^4+x^3+x^2+x+1)+4N(x^5-1)=-A(4x^3+3x^2+2x+1)+5Ex^4 \\ -4K(x^4+x^3+x^2+x+1)+4P(x^5-1)=-B(4x^3+3x^2+2x+1)+5Fx^4 \\ -4L(x^4+x^3+x^2+x+1)+4Q(x^5-1)=-C(4x^3+3x^2+2x+1)+5Gx^4 \\ 4M=-D(4x^3+3x^2+2x+1)+5Hx^4 \end{cases}$$

$$
(18')\quad \begin{cases}
\tilde{A}(5u^4+1)+u^5=5\tilde{J}(u^5+u) \\
\tilde{B}(5u^4+1)+u^4=5\tilde{K}(u^5+u) \\
\tilde{C}(5u^4+1)+u^3=5\tilde{L}(u^5+u) \\
\tilde{D}(5u^4+1)+u^2=5\tilde{W}(u^5+u) \\
\tilde{R}(5u^4+1)+u=5\tilde{I} \\
5\tilde{A}u^4+\tilde{B}(u^5+u)+2\tilde{C}u(u^5+u)+3\tilde{D}u^2(u^5+u)+4\tilde{R}u^3(u^5+u)+\tilde{E}(5u^4+1) \\
\qquad =-\tilde{I}u^4+5\tilde{J}u^5+\tilde{K}u(u^5+u)+2\tilde{L}u^2(u^5+u)+3\tilde{W}u^3(u^5+u)+5\tilde{N}(u^5+u) \\
4\tilde{A}u^3+5\tilde{B}u^4+\tilde{C}(u^5+u)+2\tilde{D}u(u^5+u)+3\tilde{R}u^2(u^5+u)+\tilde{F}(5u^4+1) \\
\qquad =-2\tilde{I}u^3+4\tilde{J}u^4+5\tilde{K}u^5+\tilde{L}u(u^5+u)+2\tilde{W}u^2(u^5+u)+5\tilde{P}(u^5+u) \\
3\tilde{A}u^2+4\tilde{B}u^3+5\tilde{C}u^5+\tilde{D}(u^5+u)+2\tilde{R}u(u^5+u)+\tilde{G}(5u^4+1) \\
\qquad =-3\tilde{I}u^2+3\tilde{J}u^3+4\tilde{K}u^4+5\tilde{L}u^5+\tilde{W}u(u^5+u)+5\tilde{Q}(u^5+u) \\
2\tilde{A}u+3\tilde{B}u^2+4\tilde{C}u^3+5\tilde{D}u^4+\tilde{R}(u^5+u)+\tilde{H}(5u^4+1) \\
\qquad =-4\tilde{I}u+2\tilde{J}u^2+3\tilde{K}u^3+4\tilde{L}u^4+5\tilde{W}u^5+5\tilde{Z}(u^5+u) \\
\tilde{A}+2\tilde{B}u+3\tilde{C}u^2+4\tilde{D}u^3+5\tilde{R}u^4+\tilde{S}(5u^4+1) \\
\qquad =\tilde{J}u+2\tilde{K}u^2+3\tilde{L}u^3+4\tilde{W}u^4+5\tilde{M}
\end{cases}
$$

In these equalities, we may put

$$
(19)\quad \begin{cases}
A=(x^4+x^3+x^2+x+1)/5 \qquad J=(x^3+x^2+x+1)/4 \\
N=-(10x^3+9x^2+7x+4)/20 \qquad E=-2(x^4+x^3+x^2+x+1)/5 \\
B=C=D=F=G=H=I=K=L=M=P=Q=0
\end{cases}
$$

$$
(19')\quad \begin{cases}
\tilde{A}=u^5/4 \quad \tilde{B}=u^4/4 \quad \tilde{C}=u^3/4 \quad \tilde{D}=u^2/4 \quad \tilde{R}=-u \\
\tilde{J}=u^4/4 \quad \tilde{K}=u^3/4 \quad \tilde{L}=u^2/4 \quad \tilde{W}=u/4 \quad \tilde{I}=-u^5 \\
\tilde{E}=u^5/4 \quad \tilde{F}=u^4/2 \quad \tilde{G}=3u^3/4 \quad \tilde{H}=u^2 \quad \tilde{Z}=0 \\
\tilde{S}=0 \quad \tilde{M}=-u^5.
\end{cases}
$$

Therefore, by (17), (17′), (19) and (19′), we get

$$
(20)\quad \begin{cases}
\mathcal{D}_1(x)=(x^4+x^3+x^2+x+1)/5-2t_1(x^4+x^3+x_2+x+1)/5 \\
\mathcal{D}_1(y)=(x^3+x^2+x+1)y/4-t_1(10x^3+9x^2+7x+4)y/20
\end{cases}
$$

$$
(20')\quad \begin{cases}
\mathcal{D}_1(u)=(u^5+u^4v+u^3v^2+u^2v^3-4uv^4)/4+t_1(u^5+2u^4v+3u^3v^2+4u^2v^3)/4 \\
\tilde{\mathcal{D}}_1(v)=(-4u^5+u^4v+u^3v^2+u^2v^3+uv^4)/4+t_1(-4u^5-3u^4v-2u^3v^2-u^2v^3)/4
\end{cases}
$$

Now we put

$$\mathscr{D}_1 \equiv \mathscr{D}_1 \qquad \bar{\mathscr{D}}_1 \equiv \tilde{\mathscr{D}}_1 \pmod{t_1}.$$

Then we get

$$\bar{\mathscr{D}}_1(x)=\bar{\mathscr{D}}(uv^{-1})=u^{-2}\{u\bar{\mathscr{D}}_1(v)-v\bar{\mathscr{D}}_1(u)\}$$
$$=u^{-2}\{(-4u^6+u^5v+u^4v^2+u^2v^3+u^2v^4)-(u^5v+u^4v^2+u^3v^3+u^2v^4-4uv^5)\}/4$$
$$=-u^4+u^{-1}v^5=-u^4+u^{-1}(u^5+u)=1$$
$$\bar{\mathscr{D}}_1(y)=\bar{\mathscr{D}}_1(u^{-1})=-u^{-2}\bar{\mathscr{D}}_1(u)=-(1+x+x^2+x^3-4x^4)y^{-3}/4.$$

Hence on $\mathcal{U}_0 \cap \mathcal{V}_0$, we get

$$(21)\quad \begin{cases} (\mathscr{D}_1-\bar{\mathscr{D}}_1)(x)=(x^4+x^3+x^2)/5+(x-4)/5 \\ (\mathscr{D}_1-\bar{\mathscr{D}}_1)(y)=(x^3+x^2+x+1)y/4+(1+x+x^2+x^3-4x^4)y^{-3}/4 \\ \qquad =(x^4+x^3+x^2)(5x^4)(4y^3)^{-1}/5+(x-4)(5x^4)(4y^3)^{-1}/5 \end{cases}$$

Hence $\kappa((\partial/\partial t_1)_0)=\mathscr{D}_1-\bar{\mathscr{D}}_1$ is determined by the part $(x^4+x^3+x^2)\partial_x/5$ and $(x-4)\partial_x/5$ is a boundary component from lemma 3 (iii), that is

$$(22)\quad \kappa((\partial/\partial t_1)_0)=(x^4+x^3+x^2)\partial_x/5=(\theta_2+\theta_3+\theta_4)/5.$$

Next, we reset

$$X=\tilde{R}_{0,t_2,0}=\tilde{R}_{t_2}:\ y^4=(x-1)(x-\rho-\rho t_2)(x-\rho^2)(x-\rho^3)(x-\rho^4).$$

This equation is obtained by setting

$$x=\rho X, \qquad y=Y \quad \text{and} \quad t_2=t_1$$

in the equation of $\tilde{R}_{t_1,0,0}$: $Y^4=(X-1-t_1)(X-\rho)(X-\rho^2)(X-\rho^3)(X-\rho^4)$. Hence, $\mathscr{D}_2$ and $\tilde{\mathscr{D}}_2$ which are corresponding derivations to $\tilde{R}_{t_2}$ must satisfy the following

$$(\mathscr{D}_2-\bar{\mathscr{D}}_2)(x)=(\mathscr{D}_1-\bar{\mathscr{D}}_1)(\rho X)=\rho(\mathscr{D}_1-\bar{\mathscr{D}}_1)(X)=\rho(X^4+X^3+X^2)/5$$
$$=(\rho^2x^4+\rho^3x^3+\rho^4x^2)/5$$
$$(\mathscr{D}_2-\bar{\mathscr{D}}_2)(y)=(\mathscr{D}_1-\bar{\mathscr{D}}_1)(Y)=(X^4+X^3+X^2)(5X^4)(4Y^3)^{-1}/5$$
$$=(\rho^2x^4+\rho^3x^3+\rho^4x^2)(5x^4)(4y^3)^{-1}/5$$

Therefore

$$(23)\quad \kappa((\partial/\partial t_2)_0)=(\mathscr{D}_2-\bar{\mathscr{D}}_2)=(\rho^2\theta_2+\rho^3\theta_3+\rho^4\theta_4)/5.$$

Next, we reset again

$$X=\tilde{R}_{0,0,t_3}=\tilde{R}_{t_3}:\ y^4=(x-1)(x-\rho)(x-\rho^2-\rho^2t_3)(x-\rho^3)(x-\rho^4)$$

This equation is obtained by setting

$$x=\rho^2X, \qquad y=Y, \qquad t_3=t_1$$

in the equation of $\tilde{R}_{t_1,0,0}$.

Hence, $\mathcal{D}_3$, $\bar{\mathcal{D}}_3$ which are correspoding derivations to $\tilde{R}_{t_3}$ must satisfy the following

$$(\mathcal{D}_3-\bar{\mathcal{D}}_3)(x)=(\mathcal{D}_1-\bar{\mathcal{D}}_1)(\rho^2 X)=\rho^2(\mathcal{D}_1-\bar{\mathcal{D}}_1)(X)=\rho^2(X^4+X^3+X^2)/5$$
$$=(\rho^4 x^4+\rho x^3+\rho^3 x^2)/5$$
$$(\mathcal{D}_3-\bar{\mathcal{D}}_3)(y)=(\mathcal{D}_1-\bar{\mathcal{D}}_1)(Y)=(X^4+X^3+X^2)/5$$
$$=(\rho^4 x^4+\rho x^3+\rho^3 x^2)(5x^4)(4y^3)^{-1}/5\,.$$

Therefore

$$\kappa((\partial/\partial t_3)_0)=(\mathcal{D}_3-\mathcal{D}_3)=(\rho^4\theta_2+\rho\theta_3+\rho^3\theta_4)/5\,. \tag{24}$$

From (22), (23) and (24), we could prove the following theorem for the family $X=\tilde{R}_{t_1,t_2,t_3}$

THEOREM 6. *The Kodaira-Spencer map* κ: $T_{S,0}\to H^1(X, \mathcal{T}_X)$ *is given by*

$$\kappa\begin{pmatrix}(\partial/\partial t_1)_0\\(\partial/\partial t_2)_0\\(\partial/\partial t_3)_0\end{pmatrix}=\frac{1}{5}\begin{pmatrix}1&1&1\\\rho^2&\rho^3&\rho^4\\\rho^4&\rho&\rho^3\end{pmatrix}\begin{pmatrix}\theta_2\\\theta_3\\\theta_4\end{pmatrix}. \tag{25}$$

COROLLARY. κ *is injective.*

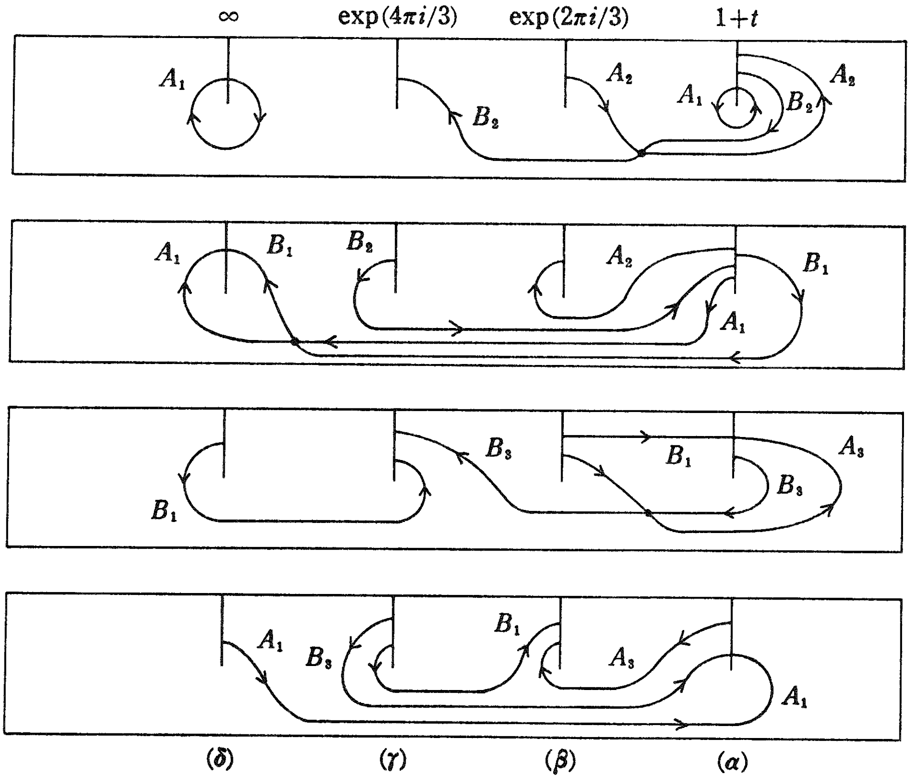
PROOF. It is clear since the matrix in (25) is nonsingular.

REMARK 5. Theorem 6 and Corollary show that t_1, t_2 and t_3 are independent parameters of Prym variety. Indeed θ_2, θ_3 and θ_4 correspond to $\Omega_2=\omega_1\omega_2$, $\Omega_3=\omega_1\omega_3$ and $\Omega_4=\omega_1\omega_4$ where ω_1, ω_2, ω_3 and ω_4 are Prym differentials and the injectivity asserts their independency.

Appendix.

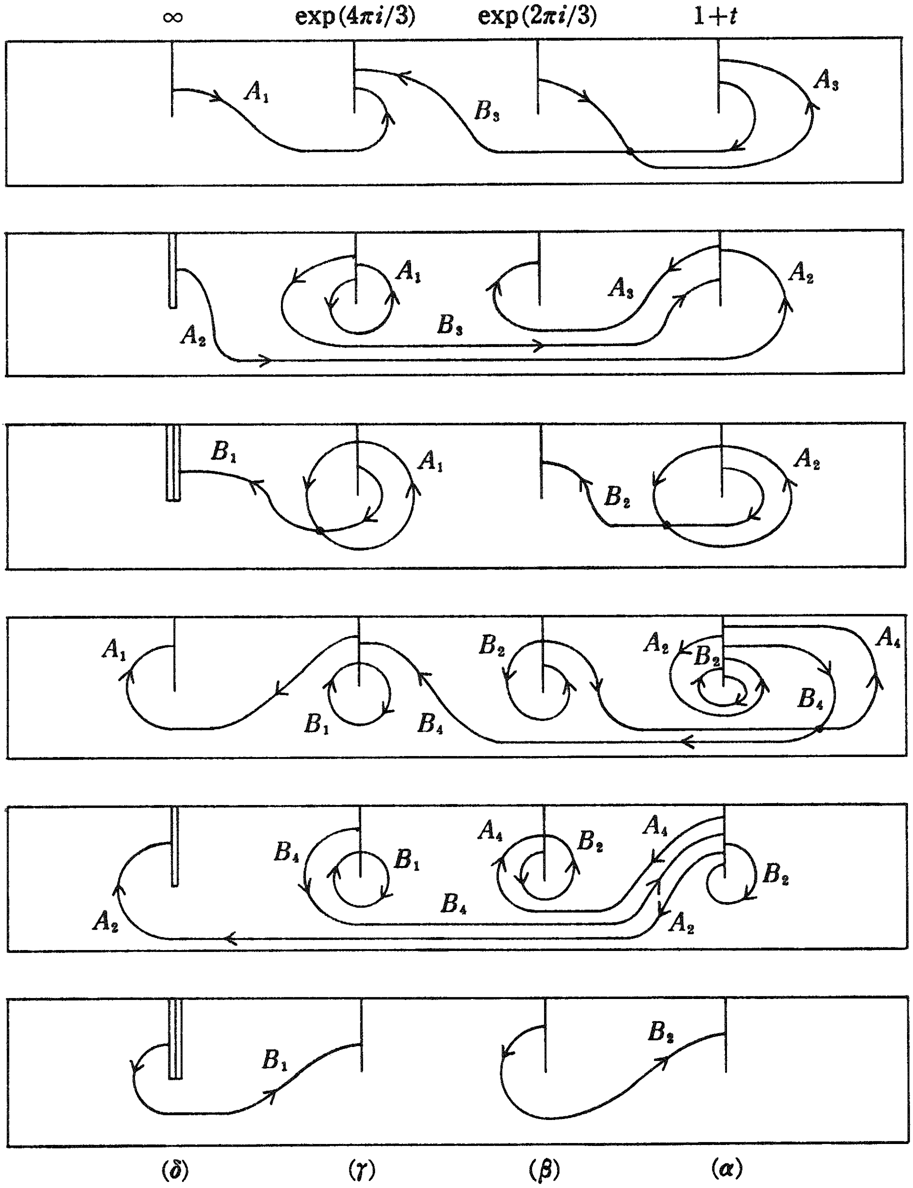
1. A canonical homology basis of the Riemann surface defined by

$$y^4=(x-1-t)(x^2+x+1).$$



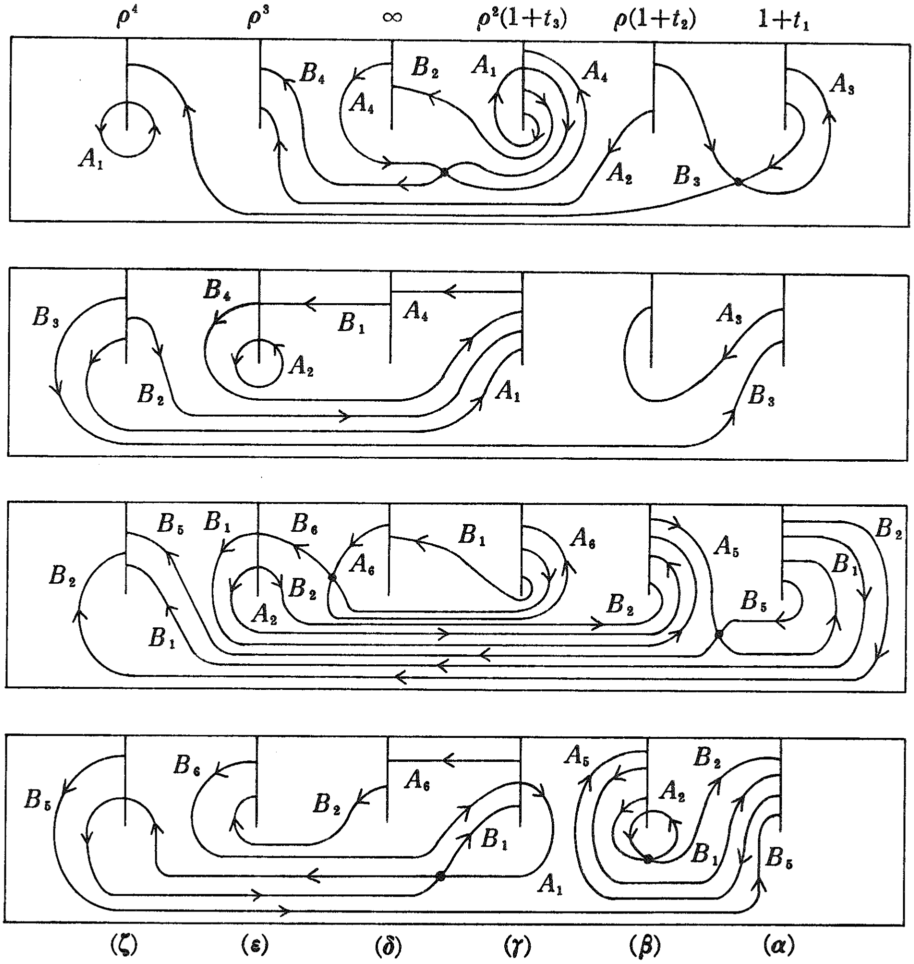
2. A canonical homology basis of the Riemann surface defined by

$$y^6=(x-1-t)(x^2+x+1).$$



3. A canonical homology basis of the Riemann surface defined by

$$y^4=(x-1-t_1)(x-\rho-\rho t_2)(x-\rho^2-\rho^2 t_3)(x-\rho^3)(x-\rho^4).$$



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Mathematical Division of General Education
College of Science and Technology
Nihon University
Narashinodai, Funabashi-shi, Chiba, 274
Japan