TSUKUBA J. MATH.
Vol. 3, No. 2 (1979), 17–29

ON CODIMENSION ONE ISOMETRIC IMMERSIONS BETWEEN INDEFINITE SPACE FORMS

By

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Introduction

This paper considers codimension one isometric immersions between manifolds which carry nondegenerate, though possibly indefinite, metrics and which have the same constant sectional curvature. Its major purpose is to study the completeness properties of the relative nullity foliation of such an immersion in the event that the source manifold is geodesically complete. In addition, those immersions in the case of zero curvature (and source manifold completeness) are classified.

In [G], [HN], and [N1], a similar program has been accomplished for flat Euclidean and Lorentz spaces. For Riemannian manifolds, the completeness properties of the relative nullity foliation have been studies extensively. See, e.g., [N2].

Section 1 of the present paper presents notation and necessary preliminary results. Symmetric tensors of a relevant type and associated nullity distributions, including the relative nullity foliation, are examined in Section 2. The completeness properties of the relative nullity foliations of the immersions under consideration are developed in the third section, Theorem (3.11) being the major result. This theorem is combined with techniques from [G] to classify the immersions between flat indefinite spaces in Section 4; the classification appears as Theorem (4.4).

1. Preliminaries

Consider an isometric immersion $f: M^n \rightarrow M^{n+1}(c)$ between manifolds carrying nondegenerate metrics, denoted unambiguously by $\langle\, ,\rangle$, and where the target manifold has constant sectional curvature c. Since the metrics are nondegenerate, each point of M has a neighborhood (in M) on which is defined a vector field, denoted by ξ, of unit normals (i.e., $|\langle \xi, \xi\rangle| = 1$).

Received January 10, 1979

Let the corresponding Levi-Civita connections on the source and target manifolds be denoted by ∇ and ∇' respectively. Then ∇ and ∇' satisfy the following formulas. If X and Y are vector fields on (an open set in) M, then the *Gauss formula*

$$\nabla'_X Y = f_*(\nabla_X Y) + h(X, Y)\xi \tag{1.1}$$

gives an orthogonal decomposition of $\nabla'_X Y$ into components tangential and normal to M. In (1.1), h is a symmetric bilinear form, the *second fundamental form.* A field A of tangent space endomorphisms, called the *second fundamental tensor*, is defined by the following *Weingarten formula*:

$$\nabla'_X \xi = -f_*(AX). \tag{1.2}$$

A and h satisfy

$$h(X, Y)\cdot\langle\xi, \xi\rangle = \langle AX, Y\rangle \tag{1.3}$$

and

$$\langle AX, Y\rangle = \langle X, AY\rangle. \tag{1.4}$$

If R is the curvature tensor of the connection ∇ on M, then the *equation of Gauss* relates R, A, and the curvature c of the target manifold:

$$c(X\wedge Y) = R(X, Y) - \langle\xi, \xi\rangle\ (AX\wedge AY), \tag{1.5}$$

where the operation $\wedge$ is defined by

$$(X\wedge Y)Z = \langle Z, Y\rangle X - \langle Z, X\rangle Y. \tag{1.6}$$

Finally, the second fundamental tensor satisfies the *equation of Codazzi*:

$$\nabla_X(AY) = \nabla_Y(AX) + A([X, Y]). \tag{1.7}$$

We recite some standard facts about nondegenerate metrics. If V is a finite-dimensional vector space, with nondegenerate metric (inner product) $\langle\ ,\ \rangle$, of which W is a (non-empty) subspace, then

$$W^\perp = \{v\in V : \langle v, w\rangle = 0 \quad \text{for all } w\in W\}$$

is a subspace of V whose dimension complements that of W:

$$\dim W + \dim W^\perp = \dim V.$$

Moreover, $(W^\perp)^\perp = W$. However, $V = W\oplus W^\perp$ if and only if the metric induced on W is nondegenerate.

The following lemma is from [GN]; see also, e.g., [W]. A subspace of an inner product space is *(non) degenerate* if the inner product induced on that subspace is (non) degenerate.

(1.8) *Degeneracy Lemma.* Suppose X_1 and X_2 are linearly independent vectors. Then Span $\{X_1, X_2\}$ is nondegenerate if and only if the 2×2 "degeneracy determinant"

$$\det||\langle X_i, X_j\rangle|| \qquad i, j=1, 2$$

is nonzero.

3. Nullity Distributions of Symmetric (1, 1)-Type Tensors of Rank One

Let $f\colon M^n \to M^{n+1}$ be an isometric immersion between manifolds with non-degenerate metrics and the same constant sectional curvature c. Then the equation of Gauss (1.5) implies that the second fundamental tensor A has rank one when it is nontrivial. According to (1.4), it is symmetric with respect to the metric on M. In view of these facts, we turn our attention to symmetric (1, 1)-type tensors with rank at most one on M.

Let A denote such a tensor; if $x\in M^n$, then define $T_0(x)$ to be the kernel of A_x. Then $T_0(x)$ has dimension n or $n-1$. If W is the set of all $x\in M_n$ such that $T_0(x)$ has dimension $n-1$, then W is precisely the set of all $x\in M^n$ such that A_x is nonzero. Therefore, W is open, and

$$x \longmapsto T_0(x)$$

defines an $(n-1)$-dimensional distribution on W, called the *nullity distribution* of the tensor A. In the case where A is the second fundamental tensor of an immersion $f\colon M^n \to M^{n+1}$ as described above, T_0 is called the *relative nullity distribution* of f, and W may be called the "umbilic-free" set (see [G], §4).

(2.1) Lemma. If $x\in W$, then the image of A_x is precisely the orthogonal complement of $T_0(x)$ in T_xM.

Proof. Both spaces are one-dimensional, and if $X\in T_0(x)$, then $\langle AZ, X\rangle = \langle Z, AX\rangle=0$. QED.

(2.2) Lemma. For a symmetric (1, 1)-type tensor A with rank one, the following statements are equivalent.

(i) The kernel of A is degenerate.

(ii) The image of A is a light line.

(iii) $A^2\equiv 0$.

Proof. (i)$\Rightarrow$(ii). If $X\in T_0$ satisfies $\langle X, Y\rangle=0$ for all $Y\in T_0$, then $X\in T_0{}^{\perp}$, which is the image of A. Thus, the image of A is Span$\{X\}$, but $\langle X, X\rangle=0$.

(ii)$\Rightarrow$(iii). Denote the ambient (tangent) space by $\boldsymbol{R}^n$. Write $\boldsymbol{R}^n$ as an algebraic direct sum

$$\boldsymbol{R}^n = T_0 \oplus \mathrm{Span}\,\{L\}.$$

If $X \in T_0$, then $\langle A^2L, X\rangle = \langle AL, AX\rangle = 0$; but also $\langle A^2L, L\rangle = \langle AL, AL\rangle = 0$. Thus, A^2L is orthogonal to $\boldsymbol{R}^n$ and so $A^2L=0$, and $A^2 \equiv 0$.

(iii)$\Rightarrow$(i). If $A^2 \equiv 0$, then the image of A is contained in T_0. But the image of A is orthogonal to T_0, and hence is the axis of degeneracy in T_0. QED.

(2.3) COROLLARY. If G is the set of those x in W for which $T_0(x)$ is nondegenerate, then G is open.

(2.4) PROPOSITION. T_0 is a differentiable distribution on W.

PROOF. Let $y \in W$. Choose $L \notin T_0(y)$ such that $\mathrm{Span}\,\{L, AL\}$ is a nondegenerate plane. If L is extended to a vector field near y, then the degeneracy determinant for $\{L, AL\}$ remains nonzero (perhaps in some smaller neighborhood of y). So, for points near y, $\mathrm{Span}\,\{L, AL\}$ has a nondegenerate, hence algebraically complementary, orthogonal complement E, of dimension $n-2$. By Lemma (2.1), E is contained in T_0.

Now, $L \notin T_0$ so $\langle L, AL\rangle_x = \lambda(x)$ is nonzero at points x near y. Define $V \in \mathrm{Span}\,\{L, AL\}$ by

$$V = AL - (\langle AL, AL\rangle / \langle L, AL\rangle)L.$$

That $\langle V, AL\rangle = 0$ implies that V spans $T_0 \cap \mathrm{Span}\,\{L, AL\}$. It follows that

$$T_0 = E \oplus \mathrm{Span}\,\{V\}$$

near y, where the direct sum is an orthogonal sum as well. To see that T_0 is differentiable, it now suffices to see that the $(n-2)$-dimensional distribution $x \mapsto E(x)$ is differentiable.

Give $E(y)$ an orthonormal basis $\{Y_j\}$ $(j=1,\cdots,n-2)$, and then extend Y_j near y to a vector field Z_j, for each j. Because the degeneracy determinant of $\{L, AL\}$ is nonzero near y, the solutions of the linear system

$$\langle L, Z_j\rangle = c_j\langle L, L\rangle + d_j\langle AL, L\rangle$$

$$\langle AL, Z_j\rangle = c_j\langle AL, L\rangle = d_j\langle AL, AL\rangle$$

are smooth functions of inner products among L, AL, and Z_j. Hence,

$$Z_j' = Z_j - c_jL - d_jAL \qquad j=1,\cdots,n-2$$

are smooth vector fields near y. Moreover, $(Z_j')_y = Y_j$ so the set $\{Z_j'\}$ $(j=1,\cdots,n-2)$

is linearly independent near y. Since each Z_j' is orthogonal to Span$\{L, AL\}$, the set spans E. QED.

In the case when A is a relative nullity distribution of such a codimension one isometric immersion as described at the beginning of this section, we can say more.

(2.5) PROPOSITION. The relative nullity distribution of an isometric immersion $f: M^n(c) \to M^{n+1}(c)$ is integrable.

PROOF. If $X, Y \in T_0$, then $A([X, Y]) = \nabla_X(AY) - \nabla_Y(AX) = 0$, so $[X, Y] \in T_0$. QED.

Proposition (2.5) says that T_0, the relative nullity distribution of f, is a *foliation* of the umbilic-free set W. We shall refer to T_0 as the *relative nullity foliation.*

(2.6) PROPOSITION. The image of A is parallel in any T_0 direction.

PROOF. If $X \in T_0$ and $L \notin T_0$, then

$$\begin{aligned}\nabla_X(AL) &= \nabla_L(AX) + A([X, L]) \\ &= A([X, L])\end{aligned}$$

and so lies in the image of A. QED.

(2.7) PROPOSITION. If $X, Y \in T_0$, then $\nabla_X Y \in T_0$.

PROOF. Choose $L \notin T_0$. Then Proposition (2.6) and the fact that A has rank one imply that

$$\begin{aligned}\langle \nabla_X Y, AL \rangle &= X \cdot \langle Y, AL \rangle - \langle Y, \nabla_X(AL) \rangle \\ &= k \cdot \langle Y, AL \rangle\end{aligned}$$

for some constant k. By Lemma (2.1), $\langle Y, AL \rangle$ vanishes, and then also $\nabla_X Y \in T_0$. QED.

A distribution (or foliation) D which satisfies $\nabla_X Y \in D$ whenever $X, Y \in D$ is said to be *totally geodesic*. (If D is a foliation, which is so in the presence of zero torsion, then its integral submanifolds, called its *leaves*, are totally geodesic as submanifolds. For a discussion of totally geodesic submanifolds, see [KN] or [N2].)

We now have the following result for the relative nullity foliation T_0 of an isometric immersion $f: M^n(c) \to M^{n+1}(c)$. W denotes the umbilic-free set.

(2.8) THEOREM. T_0 is a totally geodesic foliation of W.

3. Completeness Properties of the Relative Nullity Foliation

This section will study the relative nullity foliation of an isometric immersion $f: M^n(c) \to M^{n+1}(c)$, where the source manifold M^n is (geodesically) complete; that is, any geodesic in M^n may be extended to all values of its affine parameter.

A totally deodesic foliation F has the following property (see, e.g., [KN]). Given a point x_0, if x_t is a geodesic whose tangent vector $\dot{x}_0$ at x_0 lies in $F(x_0)$, then x_t lies in the leaf of F through x_0 for all t in some neighborhood of 0. A totally geodesic foliation is called *complete* if every affinely-parametrized geodesic which is tangent to the foliation can be extended to all values of the parameter and still lie in a leaf of the foliation.

It has been seen (Theorem (2.8)) that the relative nullity foliation under consideration is totally geodesic. The purpose of this section is to show that (if M^n is complete) this foliation is complete.

T_0 will denote the foliation, and W will denote the "umbilic-free" set, on which the $(n-1)$-dimensional T_0 lives. Let $x_0 \in W$ and let $X_0 \in T_0(x_0)$. Let x_t be an affinely-parametrized geodesic such that $\dot{x}_0 = X_0$, and let x_t be extended to all values of t in the complete manifold M^n. Suppose $b > 0$ satisfies that x_t lies in the leaf of T_0 through x_0 for $0 \leq t < b$; such a b exists by Theorem (2.8). The proof of the following lemma is essentially that of Lemma (5.9) of [G], which argument also appears in [N1].

(3.1) LEMMA. If $x_b \in W$, then there exists a positive ε such that for $0 \leq t < b+\varepsilon$, x_t lies in the leaf of T_0 through x_0.

Hence, to show that T_0 is complete, it is imperative to show that $x_b \in W$. In what follows, $y = x_{t_1}$ will always denote a fixed point of the geodesic x_t at which some pertinent differentiation or function evaluation will occur. The second fundamental tensor at a point x will be denoted $A(x)$.

Let Ω generate the image of $A(x_0)$. Extend Ω as a parallel vector field along all of x_t to a vector field Ω_t. By Proposition (2.6), Ω_t generates the image of $A(x_t)$ if $t < b$. Choose $L \notin T_0(x_0)$ such that $\langle L, \Omega \rangle = -1$. Extend L as a parallel vector field along x_t, for all t. For $t < b$, $L_t \notin T_0(x_t)$. Also define a smooth function p: $\boldsymbol{R} \to \boldsymbol{R}$ by

$$p(t) = \langle AL, L \rangle |_{x_t}. \tag{3.2}$$

p is defined for all t, but if $t < b$, then

$$AL_{x_t} = -p(t)\Omega_t. \tag{3.3}$$

Near any point y on the geodesic, we may extend $\dot{x}(t_1)$ to a T_0-field X such that $X(x_t)=\dot{x}_t$ for each point x_t of the geodesic near y. X will be called a T_0-*extension* (of $\dot{x}(t_1)$) *near* y, and is constructed as follows. Choose a normal coordinate system $(x^1,\cdots,x^n)$ at y such that the geodesic x_t is described by $x^2=\cdots=x^n=0$. Choose a T_0-field on $x^1=0$ extending $\dot{x}_0$, and consider the geodesics in those directions. Their tangent vectors form a T_0-extension near y.

We will also need extensions of L and Ω in certain directions transverse to the geodesic x_t. Define a map $h: \boldsymbol{R}^2 \to M^n$ by

$$h(t, u)=\exp_{x_t}(uL_t).$$

Since

$$h_{*(t,0)}(\partial/\partial t)=\dot{x}_t,\ h_{*(t,0)}(\partial/\partial u)=L_t,$$

for each t, there is a neighborhood U of $(t, 0)$ such that $h: U \to h(U)$ is an embedding. These U-neighborhoods form a neighborhood V of $\{(t, 0): t \in \boldsymbol{R}\}$. The vector field $h_*(\partial/\partial u)$ is an extension of L to $h(V)$. By shrinking the neighborhood V, if need be, and restricting t to $[0, b)$, we may assume that $AL \neq 0$; since $L \notin T_0$, $\langle AL, L\rangle \neq 0$. Now let

$$\Omega=(-1/\langle AL, L\rangle)AL.$$

This extends Ω to $h(V)$. Finally, note that if Z is a vector field near $y=x(t_1)$, then $(\nabla_L Z)_y$ depends only on the behavior of Z along the curve $u \to h(t_1, u)$. In particular, with the above extensions, $(\nabla_L \Omega)_{x_t}$ is a well-defined vector field along the geodesic x_t.

Now we examine Codazzi's equation (1.7) near y, using L and a T_0-extension X near y. Using zero torsion, the Codazzi equation reduces to

$$\nabla_X(AL)=-A(\nabla_L X),$$

since L is parallel along x_t, and $X \in T_0$.

Next we consider the derivative of the function p of (3.2) and (3.3):

$$\begin{aligned}\frac{dp}{dt}&=X\cdot\langle AL, L\rangle\\ &=\langle \nabla_X(AL), L\rangle+\langle AL, \nabla_X L\rangle\\ &=-\langle A(\nabla_L X), L\rangle\\ &=-\langle \nabla_L X, AL\rangle,\end{aligned}$$

so

$$\frac{dp}{dt}=p(t)\langle \nabla_L X, \Omega\rangle_t. \tag{3.4}$$

Since p is well-defined and smooth for all t, (3.4) implies that the function $Q(t)=$

$\langle \nabla_L X, \Omega\rangle_t$ is well-defined and smooth on the interval $0 \leq t < b$. (In particular, $Q(t)$ is independent of the T_0-extension near x_t, for each t.) Let us pause to study the behavior of $Q(t)$ along the geodesic x_t.

By the Gauss equation (1.5) of the immersion,

$$R(X, L)X = c\langle X, L\rangle X - c\langle X, X\rangle L. \tag{3.5}$$

Since there is no torsion,

$$R(X, L)X = \nabla_X \nabla_L X - \nabla_L \nabla_X X - (\nabla_{\nabla_X L - \nabla_L X} X). \tag{3.6}$$

Equations (3.5) and (3.6) can be combined to give

$$\nabla_X \nabla_L X = c\langle X, L\rangle X - c\langle X, X\rangle L + \nabla_L \nabla_X X + \nabla_{\nabla_X L} X - \nabla_{\nabla_L X} X. \tag{3.7}$$

Now,

$$\begin{aligned}\frac{dQ}{dt} &= X\cdot\langle \nabla_L X, \Omega\rangle \\ &= \langle \nabla_X \nabla_L X, \Omega\rangle + \langle \nabla_L X, \nabla_X \Omega\rangle \\ &= \langle \nabla_X \nabla_L X, \Omega\rangle,\end{aligned}$$

since Ω is parallel along x_t. From (3.7) the following equation obtains:

$$\begin{aligned}\frac{dQ}{dt} &= c\langle X, L\rangle\langle X, \Omega\rangle - c\langle X, X\rangle\langle L, \Omega\rangle + \langle \nabla_L \nabla_X X, \Omega\rangle \\ &\quad + \langle \nabla_{\nabla_X L} X, \Omega\rangle - \langle \nabla_{\nabla_L X} X, \Omega\rangle.\end{aligned} \tag{3.8}$$

(3.9) LEMMA. The function Q satisfies the following differential equation on the interval $0 \leq t < b$:

$$\frac{dQ}{dt} = Q(t)^2 + c\langle \bar{x}_t, \bar{x}_t\rangle.$$

REMARK. Note that c and $\langle \bar{x}_t, \bar{x}_t\rangle$ are constants.

PROOF. Each term of the right-hand side of (3.8) is to be evaluated at $y = x(t_1)$ where $0 \leq t_1 < b$.

Since Ω_y generates the image of A_y, the first term vanishes. By design, $\langle L, \Omega\rangle_t = -1$; since $X_t = \bar{x}_t$, the second term equals $c\langle \bar{x}_t, \bar{x}_t\rangle$. At y, $\nabla_X L$ is zero (L is parallel along x_t); therefore, $(\nabla_{\nabla_X L} X)_y$, and the fourth term, vanish.

That the third term vanishes can be seen as follows. At y, near which a fixed T_0-extension has been established, consider the equation

$$\begin{aligned}\langle \nabla_L \nabla_X X, \Omega\rangle_y &= L_y\langle \nabla_X X, \Omega\rangle - \langle (\nabla_X X)_y, (\nabla_L \Omega)_y\rangle \\ &= L_y\langle \nabla_X X, \Omega\rangle.\end{aligned}$$

Near y, $\nabla_X X \in T_0$. Since Ω generates the image of A for points on the integral curves of L constructed previously, $\langle \nabla_X X, \Omega \rangle$ vanishes on those curves. Therefore, $L_y \langle \nabla_X X, \Omega \rangle$ is zero.

Finally, the fifth term depends on $(\nabla_L X)_y$, a vector well-defined at y since the T_0-extension X and the integral curve of L through y are fixed. Now, if $Z = \nabla_L X + \langle \nabla_L X, \Omega \rangle L$, then $\langle Z, \Omega \rangle = 0$, and hence $Z \in T_0(y)$. As well, $(\nabla_Z X)_y$ lies in $T_0(y)$; so $\langle \nabla_Z X, \Omega \rangle_y = 0$. This implies that

$$\begin{aligned}\langle \nabla_{\nabla_L X} X, \Omega \rangle_y &= -\langle \nabla_{\langle \nabla_L X, \Omega \rangle L} X, \Omega \rangle_y \\ &= -\langle \nabla_L X, \Omega \rangle_y^2 \\ &= -Q(t_1)^2.\end{aligned}$$

In summary, then, equation (3.8) reduces to

$$\left.\frac{dQ}{dt}\right|_{t=t_1} = c\langle \ddot{x}_{t_1}, \dot{x}_{t_1} \rangle - (-Q(t_1))^2$$

independently of the T_0-extension near y. QED.

Now, to see whether the relative nullity is complete, which (by Lemma (3.1)) is equivalent to whether $x_b \in W$, we consider the differential equations in (3.4) and (3.9), according to different cases for the constant $c\ \langle \dot{x}_t, \dot{x}_t \rangle$.

First, suppose that $c\ \langle \dot{x}_t, \dot{x}_t \rangle = 0$. Then, from (3.9), Q satisfies the differential equation

$$\frac{dQ}{dt} = Q^2.$$

It follows that either $Q \equiv 0$ on $[0, b)$ or

$$Q(t) = \frac{Q(0)}{1 - t \cdot Q(0)}.$$

Then (3.4) implies that either $p(t) \equiv p(0)$ on $[0, b)$ or

$$p(t) = \frac{p(0)}{|1 - t \cdot Q(0)|}$$

(see [G]). In either case $\lim_{t \to b} p(t) = p(b)$ is nonzero, and $x_b \in W$.

Next, suppose there is some r such that $c\ \langle \dot{x}_t, \dot{x}_t \rangle = -r^2$. The differential equation for Q given by (3.9) is

$$\frac{dQ}{dt} = Q^2 - r^2,$$

for which $Q(t) \equiv r$ is a solution; in this event, $p(t) = p(0)e^{rt}$, so $p(b)$ is nonzero.

If $Q \neq r$, then it is easy to see that

$$Q(t)=r \coth (k-rt)$$

where k is the inverse hyperbolic cotangent of $Q(0)/r$. $Q(t)$ is well-defined for $0 \leq t < b$, so $k/r \notin [0, b)$. Now, from (3.4) it follows that for $0 \leq t < b$,

$$p(t)=p(0) \exp\left(\int_0^t Q(s)ds\right)$$

$$=p(0) \exp\left[\log\left(\frac{\sinh k}{\sinh(h-rt)}\right)\right].$$

(NOTE: since $k/r \notin [0, b)$, the quantities $\sinh k$ and $\sinh(k-rt)$ have the same sign.) So

$$p(t)=(p(0) \sinh k)/\sinh(k-rt)).$$

Since $k \neq 0$, $p(t) \neq 0$ $(0 \leq t < b)$. Were k equal to rb, $\lim_{t \to b} p(t)$ would be infinite. However, p is well-defined and smooth for all t. Therefore, the limit exists and is nonzero, and $x_b \in W$.

In the cases where $c\langle \dot{x}_t, \dot{x}_t \rangle$ is nonpositive, it has been shown that $x_b \in W$. Lemma (3.1) now implies that

$$\sup\{t : x_u \in W \quad \text{for. } 0 \leq u < t\}$$

cannot be finite. Therefore, the (complete) geodesic x_t lies in W.

Finally, consider the case where $c\langle \dot{x}_t, \dot{x}_t \rangle = r^2$ for some r. Then (3.9) implies that

$$Q(t)=r \tan(rt+k)$$

where $k=\arctan(Q(0)/r)$. Note that $k \neq \pi/2$, since $Q(t)$ is well-defined on $[0, b)$. For the same reason, the quantity

$$b'=\frac{\pi-2k}{2r}$$

does not lie in $[0, b)$. This fact and (3.4) give

$$p(t)=p(0) \exp\int_0^t r \tan(rs+k)ds$$

so

$$p(t)=(p(0) \cos k)/\cos(rt+k). \tag{3.10}$$

Now, $\cos k \neq 0$ since $Q(0)$ is not infinite. However, since M^n is complete, b' is an admissible parameter value, and so $p(b')$ must be well-defined (and smooth there).

But, by (3.10), $\lim_{t\to b'} p(t)$ does not exist, engendering a contradiction. The only viable conclusion is that, if M^n is complete, the case where $c\langle \dot{x}_t, \dot{x}_t\rangle$ is positive cannot occur.

The basic completeness properties of the relative nullity foliation, derived in this section, are summarized by the following theorem.

(3.11) THEOREM. Let $f: M^n \to M^{n+1}$ be an isometric immersion, where
(i) M^n and M^{n+1} have indefinite nondegenerate metrics;
(ii) M^n and M^{n+1} have the same constant curvature c; and
(iii) M^n is (geodesically) complete.

Let W be the umbilic-free set of the immersion, and let x_t be an (affinely parametrized) geodesic passing through a point of W. Then:
(1) $c\langle \dot{x}_t, \dot{x}_t\rangle$ is a nonpositive constant;
and (2) x_t lies in W for all t.

If M^n is not complete, then it can be asserted that $x_t \in W$ for those values to which the geodesic can be extended from a particular value t_0 for which $x_{t_0} \in W$. If $c\langle \dot{x}_t, \dot{x}_t\rangle$ is positive, then the contradiction involving b' engendered during the development of Theorem (3.11) can at best imply that b' is not a value to which the geodesic can be extended.

4. Codimension One Isometric Immersions Between Indefinite Euclidean Spaces

Let $\boldsymbol{R}_s^n$ be the n-dimensional real vector space together with an indefinite metric (inner product) of signature $(s, n-s)$ given by

$$\langle x, y\rangle = -\sum_{j=1}^{s} x^j y^j + \sum_{k=s+1}^{n} x^k y^k$$

for $x=(x^1,\cdots,x^n)$ and $y=(y^1,\cdots,y^n)$. $\boldsymbol{R}_s^n$ will be called the *indefinite* (n-*dimensional*) *Euclidean space* (*with signature* s). If $s=0$, then $\boldsymbol{R}_s^n$ is just the ordinary Euclidean space $\boldsymbol{E}^n$. If $s=1$, then $\boldsymbol{R}_s^n$ is what is usually called the n-dimensional Lorentz space, $\boldsymbol{L}^n$. Note that there is a natural isomorphism between $\boldsymbol{R}_s^n$ and $\boldsymbol{R}_{n-s}^n$. Under this isomorphism, geometric properties of one correspond to geometric properties of the other. This "independence of sign convention" has long been exploited in the case of $\boldsymbol{L}^n$ versus $\boldsymbol{R}_{n-1}^n$.

The purpose of this section is to outline the classification of isometric immersions of $\boldsymbol{R}_s^n$ into $\boldsymbol{R}_s^{n+1}$. The case $s=0$ was done by Hartman and Nirenberg [HN]; a proof also appears in [N1]. The classification in the case $s=1$ appears in [G]. All classifications are based upon the completeness properties of the relative nullity foliation. The natural isomorphism between $\boldsymbol{R}_s^n$ and $\boldsymbol{R}_{n-s}^n$ may be applied to the classification of immersions $\boldsymbol{R}_s^n \to \boldsymbol{R}_s^{n+1}$ to classify the immersions $\boldsymbol{R}_s^n \to \boldsymbol{R}_{s+1}^{n+1}$ as well.

Let $M_0(x)$ denote the leaf through $x \in \boldsymbol{R}_s^n$ of the relative nullity foliation of an immersion $\boldsymbol{R}_s^n \to \boldsymbol{R}_s^{n+1}$.

(4.1) THEOREM. The umbilic-free set W is a union of parallel hyperplanes.

PROOF. By (2) of Theorem (3.11), $M_0(x)$ is contained in the $(n-1)$-dimensional subspace tangent to it at x, for $x \in W$. Now, the hyperplanes $M_0(x)$ are the maximal connected integral submanifolds of the relative nullity foliation. Distinct hyperplanes therefore cannot intersect; but nonintersecting hyperplanes must be parallel. QED.

Since parallel hyperplanes inherit the same metric from $\boldsymbol{R}_s^n$, the following theorem is immediate.

(4.2) THEOREM. Either $G=\emptyset$ or $G=W$.

Now, if x_0 is a fixed origin in $W \subseteq \boldsymbol{R}_s^n$, write $M_0=M_0(x_0)$. Choose a vector L at x_0 such that $L \notin T_0(x_0)$; if $T_0(x_0)$ is nondegenerate, choose L to be orthogonal to T_0. Then

$$M_0 \oplus \operatorname{Span}\{L\} = \boldsymbol{R}_s^n \tag{4.3}$$

describes $\boldsymbol{R}_s^n$ as a direct sum of vector spaces which is orthogonal if M_0 is nondegenerate. If f is the immersion, then define $f_0: M_0 \to \boldsymbol{R}_s^{n+1}$ and $f_1: \operatorname{Span}\{L\} \to \boldsymbol{R}_s^{n+1}$ by

$$f_0(x)=f(x, 0), \quad \text{if } x \in M_0;$$

$$f_1(s)=f_1(sL)=f(x_0, s).$$

Invoking the "Moore Lemma" in [G] gives

$$f(x, s)=f_0(x)+f_1(s).$$

Using the Gauss formula (1.1), it is easy to see that f_0 is an isometry of M_0 onto an $(n-1)$-plane in $\boldsymbol{R}_s^{n+1}$. If M_0 is nondegenerate, and L is chosen orthogonal to M_0, then f_1 maps $\operatorname{Span}\{L\}$ into the orthogonal complement of $f(M_0)$ in $\boldsymbol{R}_s^{n+1}$. If M_0 is degenerate, then for some $(n-2)$-plane E and nonzero vector Ω,

$$M_0 = E \oplus \operatorname{Span}\{\Omega\}$$

describes M_0 as a direct and orthogonal sum of a nondegenerate $(n-2)$-plane and an axis of degeneracy. Let the vector L of (4.3) be chosen orthogonal to E. Since $\boldsymbol{R}_s^n$ has a nondegenerate metric, $\langle L, \Omega\rangle \neq 0$. From the additional fact that $\langle \Omega, \Omega\rangle = 0$, it follows that Span$\{L, \Omega\}$ is an indefinite Euclidean plane with signature 1, i.e., an $\boldsymbol{L}^2$, which is orthogonal to E. The isometry $f_0: M_0 \to f(M_0)$ induces an isometry of E onto an $(n-2)$-plane $f(E)$, and $f(\text{Span}\{L, \Omega\})$ lies in that $\boldsymbol{L}^3$ which is the orthogonal complement of $f(E)$ in $\boldsymbol{R}_s^{n+1}$. The methods of [G] now apply virtually verbatim to establish the following theorem.

(4.4) THEOREM. Up to a proper motion of $\boldsymbol{R}_s^{n+1}$, an isometric immersion $\boldsymbol{R}_s^n \to \boldsymbol{R}_s^{n+1}$ has one of the following forms.

(i) $\mathrm{id} \times \mathrm{c} : \boldsymbol{R}_{s-1}^{n-1} \times \boldsymbol{L}^1 \to \boldsymbol{R}_{s-1}^{n-1} \times \boldsymbol{L}^2$

(ii) $\mathrm{id} \times \mathrm{c} : \boldsymbol{R}_s^{n-1} \times \boldsymbol{E}^1 \to \boldsymbol{R}_s^{n-1} \times \boldsymbol{E}^2$

(iii) $\mathrm{id} \times \mathrm{g} : \boldsymbol{R}_{s-1}^{n-2} \times \boldsymbol{L}^2 \to \boldsymbol{R}_{s-1}^{n-2} \times \boldsymbol{L}^3$

where: in (i), $c: \boldsymbol{L}^1 \to \boldsymbol{L}^2$ is a "unit-speed" time-like curve ($\langle dc/dt, dc/dt \rangle = -1$) in $\boldsymbol{L}^2$; in (ii), $c: \boldsymbol{E}^1 \to \boldsymbol{E}^2$ is a unit-speed Euclidean plane curve; in each case "id" is the appropriate identity map; and in (iii), g is an immersion of $\boldsymbol{L}^2$ into $\boldsymbol{L}^3$ with degenerate relative nullity (as classified in [G]). Moreover, class (iii) consists of precisely those immersions $\boldsymbol{R}_s^n \to \boldsymbol{R}_s^{n+1}$ with degenerate relative nullities.

It should be noted that those immersions with nondegenerate relative nullities are cylinders over curves, in analogy with the Hartman-Nirenberg result, whereas the description (iii) is the best possible for those immersions with degenerate relative nullities.

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