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MOVABILITY OF MAPS AND SHAPE FIBRATIONS II

By

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Abstract. This paper contains some results concerning completely movable maps and shape fibrations. It is shown that if $f: X \to Y$ is a completely movable map and X is a finite dimensional ANR then Y is an ANR iff Y is finite dimensional. It is also shown that if X is an ANR and $f: X \to Y$ is a shape fibration, then Y is an ANR iff Y is semi-locally contractible.

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1. Introduction.

This paper is a continuation of the preceding papers [$Ya_{2,3}$], in which we introduced the notion of global movability of maps and studied its relation to shape fibrations and completely movable maps. This paper contains further applications to these objects. The followings are our main results.

THEOREM 1.1. *Suppose X is a finite dimensional* ANR *and $f: X \to Y$ is a completely movable map. Then Y is an* ANR *iff Y is finite dimensional.*

THEOREM 1.2. *Suppose X is an* ANR *and $f: X \to Y$ is a proper onto shape fibration. If Y is semi-locally contractible then Y is an* ANR.

All spaces in this paper are assumed to be locally compact separable metric spaces. ANR's are ones for metric spaces ([Hu]). A map is proper if the inverse image of every compact subset of the range is compact. A proper onto map $f: X \to Y$ is said to be *completely movable* ([CD_2, Ya_2]) if for some (eq. any) closed embedding of X into an ANR M, the following holds:

For each $y \in Y$ and each neighborhood U of $f^{-1}(y)$ in M there exists a neighborhood V of $f^{-1}(y)$ in U such that for each fiber $f^{-1}(z) \subset V$ and

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each neighborhood W of $f^{-1}(z)$ in V there exists a deformation of V in U into W keeping $f^{-1}(z)$ fixed.

For example, if $f: X \to Y$ is a proper onto map with FANR fibers and if either Y is a finite dimensional ANR or both X and Y are ANR's, then f is a shape fibration iff f is completely movable ([CD_2, Proposition 3.6], [Ya_1, Proposition 3.6]). In § 2 we will show that every movable map is locally approximately invertible and does not raise the dimension (see § 2 for the definitions). Theorem 1.1 follows from this observation. Since every cell-like map is completely movable, Theorem 1.1 generalizes the well-known fact on cell-like maps ([Ko]) to a non cell-like case.

As for the definition of approximate fibrations and shape fibrations, we refer to [$CD_{1,2}$] and [Ma], [$MR_{1,2}$]. In [Fa], it is shown that if X is an ANR and $f: X \to Y$ is a Hurewicz fibration and if Y is semi-locally contractible, then Y is an ANR. Here Y is semi-locally contractible if each $y \in Y$ admits a neighborhood which contracts in Y. Theorem 1.2 is the shape version of this result. In § 2 we will show that any locally shape trivial maps preserve ANR's, from which follows Theorem 1.2.

Finally we list some notations and conventions. By id_X we denote the identity map on the space X. A map as $p_Y: Y \times M \to Y$ always denotes the projection onto the first factor. Let $f: X \to Y$ be a map. For a subset $B \subset Y$, f_B denotes the restriction $f|_{f^{-1}(B)}: f^{-1}(B) \to B$. A map $i: X \to Y \times M$ is said to be fiber preserving (f. p.) if $p_Y i = f$. Let $\mathcal{U}$ be an open cover of Y. Two maps $f, g: X \to Y$ are $\mathcal{U}$-near if each $x \in X$ admits a $U \in \mathcal{U}$ with $f(x), g(x) \in U$.

We refer to [MS] for the shape theory and to [$Ka_{1,2,3}$], [$Ya_{1,2}$] for the fiber shape theory.

2. Local approximate invertibility.

In this section, we will show that every movable map and every locally shape trivial map is locally approximately invertible. Some applications to completely movable maps are contained in the next section.

First we will quote some results on approximately invertible maps. A proper onto map $f: X \to Y$ is said to be *approximately invertible* ([An]) if for some (eq. any) f. p. closed embedding $i: X \to Y \times M$ with M an ANR, the following holds:

For each neighborhood U of $i(X)$ in $Y \times M$, there exists a map $s: Y \to U$ with $p_Y s = id_Y$.

A map $f: X \to Y$ is said to be *locally* approximately invertible if each $y \in Y$ admits an (necessarily) open neighborhood V for which f_V is approximately invertible.

LEMMA 2.1. *Suppose* $f: X \to Y$ *is a locally approximately invertible map.*

(i) ([An]) $\dim Y \leqq \dim X$.

(ii) ([CMT, Theorem 2]) *If* X *is an* ANR *and* Y *is* LC^∞ (*locally n-connected for each* $n \geqq 0$), *then* Y *is an* ANR.

In [Ya_2], we introduced the notion of global movability of maps. A map $f: X \to Y$ is said to be *movable* if for some (eq. any) f.p. closed embedding $i: X \to Y \times M$, with M an ANR, the following holds:

> For each neighborhood U of $i(X)$ in $Y \times M$ there exists a neighborhood V of $i(X)$ in U such that for each neighborhood W of $i(X)$ in V there exists a homotopy $\phi: V \times [0, 1] \to U$ such that $\phi_0 = id$, $\phi_1(V) \subset W$ and $p_Y \phi_t = p_Y$ for $0 \leqq t \leqq 1$.

Furthermore, if we can take ϕ so that $\phi_t|_{i(X)} = id_{i(X)}$ for $0 \leqq t \leqq 1$, we say f is *strongly* movable. We call such a neighborhood V as above a (strong) movability choice for U.

PROPOSITION 2.2. *If* $f: X \to Y$ *is a movable map then* f *is locally approximately invertible and* $\dim Y \leqq \dim X$.

PROOF. Let $i: X \to Y \times M$ be an f.p. closed embedding, where M is an ANR. Define $U_{-1} = Y \times M$ and let U_0 be a movability choice for U_{-1}.

Let $y \in Y$. Fix a point $(y, x) \in if^{-1}(y)$ and take an open neighborhood V of y in Y with $V \times \{x\} \subset U_0$. We will show that f_V is approximately invertible. Note that $i|_{f^{-1}(V)}: f^{-1}(V) \to V \times M$ is an f.p. closed embedding.

Let U be any neighborhood of $if^{-1}(V)$ in $V \times M$. Write $V = \cup \{K_i : i \geqq 1\}$, where K_i is closed in Y and $K_i \subset \operatorname{int} K_{i+1}$ for each $i \geqq 1$. Since f is movable, there exist neighborhoods U_i ($i \geqq 1$) of $i(X)$ in $Y \times M$ such that $U_i \cap K_i \times M \subset U$ and U_i is a movability choice for U_{i-1} for each $i \geqq 1$. Since there exists an f.p. deformation of U_i in U_{i-1} into U_{i+1} for each $i \geqq 0$, we have a map $\phi: U_0 \times [0, \infty) \to Y \times M$ such that $\phi_0 = id$, $p_Y \phi_t = p_Y$ for $t \geqq 0$ and $\phi(U_0 \times [i, \infty)) \subset U_{i-1}$ for each $i \geqq 0$. Take a map $\lambda: V \to [0, \infty)$ such that $\lambda(V - K_i) \subset [i+2, \infty)$ for each $i \geqq 0$, where $K_0 = \phi$. The required section $s: V \to U$ is defined by $s(z) = \phi(z, x, \lambda(z))$ for $z \in V$.

The latter assertion follows from Lemma 2.1 (i).

A proper onto map $f: X \to Y$ is *locally shape trivial* ([Ka_2]) if each $y \in Y$ admits a (compact) neighborhood K for which f_K is fiber shape equivalent to the projection $q_K: K \times f^{-1}(y) \to K$.

PROPOSITION 2.3. *Let $f: X \to Y$ be a locally shape trivial map. Then,*
(i) *f is locally approximately invertible and hence* $\dim Y \leqq \dim X$.
(ii) *If X is an* ANR *then so is Y.*

PROOF. Let $i: X \to Y \times M$ be an f.p. closed embedding, with M an ANR. Let $y \in Y$. By the assumption there exists an open neighborhood V of y in Y such that the closure K of V is compact and f_K is fiber shape equivalent to the projection $q_K: K \times f^{-1}(y) \to K$.

Consider the composition: $K \xrightarrow{s} K \times f^{-1}(y) \to f^{-1}(K)$, where s is a section of q_K and the unlabeled map denotes a fiber shape morphism over K. By the definition of the fiber shape morphism (see [Ya_1, § 2]), we have a decreasing neighborhood base U_k ($k \geqq 0$) of $if^{-1}(K)$ in $K \times M$ and maps $g_k: K \to U_k$ ($k \geqq 0$) such that g_k and g_{k+1} are fiber homotopic in U_k (w.r.t. $p|_{U_k}$) for each $k \geqq 0$. Using these homotopies, we have a map $G: K \times [0, \infty) \to K \times M$ such that $G(K \times [k, \infty)) \subset U_k$ for each $k \geqq 0$ and $p_K G_t = id_K$ for $0 \leqq t \leqq \infty$.

Write $V = \cup \{K_i : i \geqq 1\}$, where K_i is compact and $K_i \subset \operatorname{int} K_{i+1}$ for each $i \geqq 1$. Let $\lambda: V \to [0, 1)$ be a map such that $\lambda(K_k - K_{k-1}) \subset [k, \infty)$ for each $k \geqq 1$.

Now we proceed to the verification of (i) and (ii).

(i) We will show that f_V is approximately invertible. Let U be any neighborhood of $if^{-1}(V)$ in $V \times M$. Passing to a subsequence, we may assume $U_k \cap K_k \times M \subset U$ for each $k \geqq 1$. The desired section $s: V \to U$ is defined by $s(x) = G(x, \lambda(x))$ for $x \in V$.

(ii) Suppose X is an ANR. Then $f^{-1}(V)$ is also an ANR. We will show that for each open cover $\mathcal{W}$ of V there exists a map $g: V \to f^{-1}(V)$ and a homotopy $F: V \times [0, 1] \to V$ such that $F_0 = id_V$, $F_1 = fg$ and $F(x \times [0, 1]) \subset st(x, \mathcal{W})$ $(= \cup \{W \in \mathcal{W} : x \in W\})$ for each $x \in V$. If this is done then by [Hu, p. 139, Theorem 6.3] V is an ANR and by [Hu, p. 98, Theorem 8.1] Y itself is an ANR.

The map g is obtained as follows. Since $f^{-1}(V)$ is an ANR, there exists an open neighborhood U of $if^{-1}(V)$ in $V \times M$ and a retraction $r: U \to f^{-1}(V)$ such that fr and $p_V|_U$ are $\mathcal{W}$-near. Again we may assume $U_k \cap K_k \times M \subset U$ for each $k \geqq 1$. Define $g: V \to f^{-1}(V)$ and $F: V \times [0, 1] \to V$ by

$$g(x)=rG(x, \lambda(x)+1) \qquad (x\in V),$$

$$F(x, t)=\begin{cases} x & x\in V,\ t=0 \\ frG(x, \lambda(x)+1/t) & x\in V,\ 0<t\leqq 1. \end{cases}$$

This completes the proof.

Theorem 1.2 follows from Proposition 2.3 (ii), since every shape fibration with a semi-locally contractible base is locally shape trivial (cf. [Ka_2, Proposition 1.3]).

3. Some applications to completely movable maps.

This section contains some consequences of Proposition 2.2 and 2.3, combined with Lemma 2.1.

COROLLARY 3.1. *If a map* $f: X\to Y$ *is completely movable and* $\dim Y<\infty$ *then* $\dim Y\leqq\dim X$.

PROOF. By [Ya_2, Theorem 1.3] f is movable. Then the conclusion follows from Proposition 2.2.

The next corollary improves a result in [Ya_3].

COROLLARY 3.2. *Suppose* X *is an* ANR *and* $f: X\to Y$ *is a proper onto map. Then the following conditions are equivalent.*

(i) Y *is an* ANR *and* f *is completely movable (eq. an approximate fibration).*

(ii) f *is completely movable and locally approximately invertible.*

(iii) f *is an approximate fibration with* FANR *fibers and locally approximately invertible.*

(iv) f *is locally shape trivial.*

(v) f *is (strongly) movable.*

PROOF. (i)$\leftrightarrow$(iv). In [Ka_2, Theorem 1.4] it is shown that a proper onto map between ANR's is an approximate fibration iff it is locally shape trivial. The conclusion follows from this and Proposition 2.3 (ii).

(ii) or (iii)$\to$(i). By [CD_3, Theorem 3.4] or [DS, Corollary 4.12] Y is LC^∞. Therefore by Lemma 2.1 (ii) Y is an ANR.

(i)$\to$(ii) and (iii). This follows from (i)$\leftrightarrow$(iv) and Proposition 2.3 (i).

(i)$\leftrightarrow$(v). This is contained in [Ya_3]. However, for the sake of completeness, we will verify (i)$\to$(v).

First note that the ANR Y admits a local equiconnecting function $\lambda: V\times[0, 1]\to Y$ ([Fo]), that is,

(a) V is a neighborhood of the diagonal $\Delta(Y)=\{(y, y): y\in Y\}$ in $Y\times Y$.

(b) $\lambda(y, y', 0)=y$, $\lambda(y, y', 1)=y'$ for $(y, y')\in V$ and $\lambda(y, y, t)=y$ for $y\in Y$ and $0\leqq t\leqq 1$.

To see that f is strongly movable, consider the f. p. embedding $i: X\to Y\times X$ defined by $i(x)=(f(x), x)$ $(x\in X)$ ($i(X)$ is the graph of f). Let U' be any neighborhood of $i(X)$ in $Y\times X$. Since $id_Y\times f$ is proper and $i(X)=(id_Y\times f)^{-1}(\Delta(Y))$, we may assume $U'=(id_Y\times f)^{-1}(U)$ for some neighborhood U of $\Delta(Y)$ in $Y\times Y$. Take an open cover $\mathcal{A}$ of Y such that $U_1=\cup\{\mathrm{st}(A, \mathcal{A})^2: A\in\mathcal{A}\}\subset U$. By (b) we may assume that for each $(y, y', t)\in V\times[0, 1]$ there exists an $A\in\mathcal{A}$ with y, $\lambda(y, y', t)\in A$.

We will show that $V'=(id_Y\times f)^{-1}(V)$ is a strong movability choice for U'. Let W' be any neighborhood of $i(X)$ in V'. We may assume $W'=(id_Y\times f)^{-1}(W)$, where W is a neighborhood of $i(X)$ in V. Take an open cover $\mathcal{B}$ of Y such that $\cup\{B^2: B\in\mathcal{B}\}\subset W$ and $\mathcal{B}$ refines $\mathcal{A}$. By the regular approximate homotopy lifting property of f ([CD_1, Proposition 1.5]) we have a map $H: V'\times[0, 1]\to X$ such that $H(y, x, 1)=x$ $((y, x)\in V')$, $H(y, x, t)=x$ $((y, x)\in i(X),\ 0\leqq t\leqq 1)$, fH and $\lambda\circ(id_Y\times f\times id_{[0,1]})$ are $\mathcal{B}$-near. Then the required f. p. deformation $\phi: V'\times[0, 1]\to U'$ of V' into W' rel $i(X)$ is defined by $\phi(y, x, t)=(y, H(y, x, 1-t))$ $((y, x)\in V',\ 0\leqq t\leqq 1)$. To see that $\phi_t(V')\subset U'$ for $0\leqq t\leqq 1$ and $\phi_1(V')\subset W'$, let $(y, x)\in V'$. Note that $fH(y, x, 1-t)$, $\lambda(y, f(x), 1-t)\in A$, y, $\lambda(y, f(x), 1-t)\in A'$ for some $A, A'\in\mathcal{A}$, and $y=\lambda(y, f(x), 0)$, $fH(y, x, 0)\in B$ for some $B\in\mathcal{B}$. Therefore $(id_Y\times f)\phi_t(y, x)=(y, fH(y, x, 1-t))\in\mathrm{st}(A, \mathcal{A})^2\subset U$ and $(id_Y\times f)\phi_1(y, x)=(y, fH(y, x, 0))\in B^2\subset W$, from which follows the conclusion. This completes the proof.

At this point, one can easily verify Theorem 1.1. However, to give a more precise statement (Corollary 3.5), we have a preliminary.

Let $n\geqq 0$. A proper onto map $f: X\to Y$ is said to be *n-movable* ([CD_1]) if for some (eq. any) closed embedding $X\subset M$ into an ANR, the following holds:

(a) For each $y\in Y$ and each neighborhood U of $f^{-1}(y)$ in M there exists a neighborhood V of $f^{-1}(y)$ in U such that for each fiber $f^{-1}(z)\subset V$ (and for each base point in $f^{-1}(z)$)

(i) $\mathrm{Im}(\check{\pi}_i f^{-1}(z)\to\pi_i(U))=\mathrm{Im}(\pi_i(V)\to\pi_i(U))$ for $i=0, \cdots, n$

(ii) $\check{\pi}_i f^{-1}(z)\to\pi_i(V)$ is a monomorphism for $i=0, \cdots, n-1$,

where the unlabeled homomorphisms are induced from the inclusions.

As for the shape group $\check{\pi}_i$, see [MS, Ch. II, §3]. We will show that the complete movability is reduced to the n-movability if the domain is an n-dimensional ANR.

LEMMA 3.3. *A proper onto map $f: X \to Y$ is n-movable iff for some (eq. any) closed embedding $X \subset M$ into an* ANR *the following holds:*

(b) *For each $y \in Y$ and each neighborhood U of $f^{-1}(y)$ in M there exists a neighborhood V of $f^{-1}(y)$ in U such that for each fiber $f^{-1}(z) \subset V$ and each neighborhood W of $f^{-1}(z)$ in V, there exists a neighborhood Z of $f^{-1}(z)$ in W such that if K is an n-dimensional polyhedron, L is a subpolyhedron of K and $\alpha: K \to V$ is a map with $\alpha(L) \subset Z$, then α is homotopic to a map $\beta: K \to W$ in U rel L.*

PROOF. By [CD_2, Lemma 3.1] and [DS_2, Lemma 2.4] f is n-movable iff f is 0-movable and has *the property* DUV^n, that is,

(c) For each $y \in Y$ and each neighborhood U of $f^{-1}(y)$ in M there exists a neighborhood V of $f^{-1}(y)$ in U such that for each fiber $f^{-1}(z) \subset V$ and each neighborhood W of $f^{-1}(z)$ in V,

(i) every point of V can be joined to a point of W by a path in U.

(ii) there exists a neighborhood Z of $f^{-1}(z)$ in W such that the inclusion induced homomorphism $\pi_i(V, Z) \to \pi_i(U, W)$ is the zero homomorphism for $i=1, \cdots, n$ (and for any base point in $f^{-1}(z)$).

(c)$\to$(b) follows from the induction on n and the simplexwise move using (c) (cf. [CD_2, Lemma 3.2]).

(b)$\to$(c) is obvious.

PROPOSITION 3.4. *Suppose X is an n-dimensional* ANR. *Then a proper map $f: X \to Y$ is completely movable iff f is n-movable.*

PROOF. The necessity follows from the proof of [CD_2, Proposition 3.6].

Suppose f is n-movable. Since X is an ANR, we can take $M=X$. Let $y \in Y$ and let U be any open neighborhood of $f^{-1}(y)$ in X. Let V be an open neighborhood of $f^{-1}(y)$ in U as in 3.3 (b). We will show that V satisfies the required complete movability condition for U (see §1). Let $f^{-1}(z) \subset V$ and let W be any neighborhood of $f^{-1}(z)$ in V. Let Z be an open neighborhood of $f^{-1}(z)$ in W given by 3.3 (b). Since V is an ANR and $\dim V \leqq n$, by [Hu, p. 164, Theorem 6.1], there exist an n-dimensional polyhedron K and maps

$V \xrightarrow{\alpha} K \xrightarrow{\beta} V$ such that $\beta\alpha$ is so close to id_V that $\beta\alpha(f^{-1}(z)) \subset Z$ and there exists a small homotopy $\phi: V \times [0, 1] \to V$ from id to $\beta\alpha$ with $\phi(f^{-1}(z) \times [0, 1]) \subset W$. Take a subpolyhedron L of K such that $\alpha f^{-1}(z) \subset L \subset \beta^{-1}(Z)$. By 3.3 (b) there exists a homotopy $\psi: K \times [0, 1] \to U$ such that $\psi_0 = \beta$, $\psi_1(K) \subset W$ and $\psi_t|_L = \beta|_L$ for $0 \leqq t \leqq 1$. Define $\chi: V \times [0, 1] \to U$ by

$$\chi(x, t) = \begin{cases} \phi(x, 2t) & (0 \leqq t \leqq 1/2) \\ \psi(\alpha(x), 2t-1) & (1/2 \leqq t \leqq 1). \end{cases}$$

Then χ is a deformation of V in U into W and $\chi(f^{-1}(z) \times [0, 1]) \subset W$, which is easily adjusted so that $\chi_t|_{f^{-1}(z)} = id$ for $0 \leqq t \leqq 1$ because U and W are ANR's.

The next corollary contains Theorem 1.1.

COROLLARY 3.5. *Suppose X is an n-dimensional* ANR *and $f: X \to Y$ is an n-movable map. Then the followings are equivalent.*

(i) *Y is an* ANR.

(ii) $\dim Y \leqq n$.

(iii) $\dim Y < \infty$.

PROOF. By 3.4 f is completely movable. By [DS, Corollary 4.12] Y is LC^{∞}, from which follows (iii)→(i).

(i)→(ii) follows from Corollary 3.2 and Lemma 2.1 (i).

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