TSUKUBA J. MATH.
Vol. 18 No. 1 (1994), 21–37

ON THE STRUCTURE OF THE MATRIX CORRESPONDING TO A FINITE TOPOLOGY

Dedicated to Professor Ryosuke Nakagawa on his 60th birthday

By

Shōji OCHIAI

1. Introduction and preliminaries

In connection with enumeration of finite topologies on an n-set, we introduced a number $\alpha(K)$ which is defined on the matrix K corresponding to a finite topology on an n-set and investigated its properties [2]. In this paper, we shall carry on our studies on the structure of the matrix corresponding to a finite topology and show the following inequality for an arbitrary $n\times n$ matrix K which corresponds to a T_0 topology:

$$n(n+5)/2+1\leqq\alpha(K)\leqq 2^{n+1}+n-1\,.$$

Let $X=\{x_1, x_2, \cdots, x_n\}$ be a finite set and T a topology on X. Let U_i be the smallest open set containing x_i. Then we define an $n\times n$, zero-one matrix $K=[k_{ij}]$ by

$$k_{ij}=\begin{cases}1 & x_j\in U_i\\ 0 & x_j\notin U_i\end{cases}$$

We denote this matrix K corresponding to the topology T by $M(T)$.

Let $B_0=\{0, 1\}$ be an ordered set $(0<1)$ with two binary operations $+$ and $*$ defined as follows: $0+0=0*1=1*0=0*0=0$, $1+0=0+1=1+1=1*1=1$. Let V_n be the set of all n-tuples $[a_1, a_2, \cdots, a_n]$ over B_0. An element of V_n is called a Boolean vector of dimension n. The system V_n together with the operation of componentwise addition is called the Boolean vector space of dimension n. A subspace of V_n is a subset which contains $[0, 0, \cdots, 0]$ and is closed under addition of vectors. The span of a set U of vectors is the intersection of all subspace containing U. The row (column) space of a matrix K is the span of the set of all rows (columns) of K. We denote the row (column) space of K by $R(K)(C(K))$ and write simply 0 for $[0, 0, \cdots, 0]$ or ${}^t[0, 0, \cdots, 0]$. In this paper except Section 6, all matrices are ones over B_0, i.e. Boolean matrices

Received December 12, 1991. Revised March 19, 1993.

where matrix multiplication involves Boolean operation.

The following is given by Sharp, H. J. R. [3].

THEOREM A. *A reflexive* $n\times n$ *matrix* K *corresponds to a topology on an* n*-set if and only if* $K^2=K$.

DEFINITION 1. Let K be an $n\times n$ matrix corresponding to a topology on an n-set and $L(K)$ the set of all $(n+1)\times(n+1)$ matrices of the form $\begin{bmatrix}1 & A\\ B & K\end{bmatrix}$ corresponding to a topology on an $(n+1)$-set, where A is a $1\times n$ matrix and B is an $n\times 1$ matrix. Then we define $\alpha(K)$ by $\alpha(K)=|L(K)|$, the number of elements of the set $L(K)$.

From this definition, we can directly derive the following result (see [2]).

THEOREM B. *Let* $T(n+1)$ *be the number of topologies on an* $(n+1)$*-set. Then we have* $T(n+1)=\sum_K \alpha(K)$ *where* K *runs over all* $n\times n$ *matrices, each of which corresponds to a topology.*

Many other properties on α are found in [2].

Throughout the rest of the paper, K denotes an $n\times n$ matrix corresponding to a topology on an n-set unless the contrary is mentioned.

2. Structure of matrices containing K

Let $T_0(n+1)$ be the number of T_0 topologies on an $(n+1)$-set.

Then we have the following result.

THEOREM 1. $T_0(n+1)=\sum_{i=0}^{n-1}(-1)^i[n]_i(\sum_{k=1}^{k(i)}\alpha(K_{n-i,k}))+(-1)^n[n]_n$, *where* $K_{n-i,k}$ *runs over all matrices of order* $n-i$ *corresponding to a* T_0 *topology and* $[n]_i=n(n-1)\cdots(n-i+1)$ *for* $0<i\leqq n$, $[n]_0=1$.

PROOF. Let K' be a matrix of order $n+1$ of the form $K'=\begin{bmatrix}1 & A\\ B & K\end{bmatrix}$ corresponding to a topology where A is a $1\times n$ matrix, B is an $n\times 1$ matrix and K is an $n\times n$ matrix. If K corresponds to a non-T_0 topology, then K' corresponds to a non-T_0 topology. Suppose that K is a matrix corresponding to a T_0 topology. Then K' corresponds to a non-T_0 topology if and only in $A=K_{i*}$ and $B=K_{*i}$ for some i, where $K_{i*}(K_{*i})$ is i-th row (column) of K. Therefore, we get $T_0(n+1)=\sum_K\alpha(K)-nT_0(n)$, where K runs over all matrices of order n, each of which corresponds to a T_0 topology.

Then we get

$$T_0(n+1)=\sum_{i=0}^{n-1}(-1)^i[n]_i(\sum_{k=1}^{k(i)}\alpha(K_{n-i,k}))+(-1)^n[n]_n,$$

where $K_{n-i,k}$ runs over all matrices of order $n-i$, each of which corresponds to a T_0 topology. This concludes the proof.

DEFINITION 2. Let $M=[m_{ij}]$ and $N=[n_{ij}]$ be $n\times m$ matrices. By $M\leqq N$, we mean that if $m_{ij}=1$, then $n_{ij}=1$ for each i and j. If M and N are incomparable, that is neither $M\leqq N$ nor $N\leqq M$, then we write $M\|N$. Define a function ζ by $\zeta(M, N)=1$ if $M\leqq N$ and otherwise $\zeta(M, N)=0$, where 0 and 1 are elements of B_0.

LEMMA 1. *Let M be an $m\times m$ reflexive matrix. Partition M into the following blocks;*

$$M=\begin{bmatrix} \overbrace{M_{11}}^{m_1} & \overbrace{M_{12}}^{m_2}, & \cdots, & \overbrace{M_{1p}}^{m_p} \\ M_{21} & M_{22}, & \cdots, & M_{2p} \\ & \cdots\cdots\cdots\cdots & & \\ & \cdots\cdots\cdots\cdots & & \\ & \cdots\cdots\cdots\cdots & & \\ M_{p1} & M_{p2}, & \cdots, & M_{pp} \end{bmatrix}\begin{matrix} \}\ m_1 \\ \}\ m_2 \\ \vdots \\ \vdots \\ \vdots \\ \}\ m_p \end{matrix}$$

where M_{ij} is an $m_i\times m_j$ matrix for each i and j.

Then the matrix M corresponds to a topology if and only if for each i and j with $1\leqq i$, $j\leqq p$, $M_{ik}M_{kj}\leqq M_{ij}$ holds for k, $1\leqq k\leqq p$.

PROOF. By Theorem A, M corresponds to a topology if and only if $M^2=M$, that is,

$$M_{i1}M_{1j}+M_{i2}M_{2j}+\cdots+M_{ip}M_{pj}=M_{ij}$$

for each i and j with $1\leqq i$, $j\leqq p$. Further, for fixed i and j, the validity of this equation is equivalent to that $M_{ik}M_{kj}\leqq M_{ij}$ holds for each k, $1\leqq k\leqq p$.

COROLLARY. *Let A be a $1\times n$ matrix and B an $n\times 1$ matrix. The matrix $\begin{bmatrix}1 & A\\ B & K\end{bmatrix}$ corresponds to a topology if and only if $A\in R(K)$, $B\in C(K)$ and $BA\leqq K$.*

We write simply (A, B) for a matrix $\begin{bmatrix}1 & A\\ B & K\end{bmatrix}$. That is,

$$L(K)=\{(A, B)\mid A\in R(K),\ B\in C(K),\ BA\leqq K\}.$$

The proof of the following lemma is easy and omitted. Let $[n]$ be the set $[n]=\{1, 2, \cdots, n\}$.

LEMMA 2. *Let* $A=[a_1, \cdots, a_i, \cdots, a_n]$, ${}^tB=[b_1, \cdots, b_j, \cdots, b_n]$ *and* $K'=(A, B)\in L(K)$. *Define subsets* Ω *and* Ψ *of* $[n]$ *by* $\Omega=\{i \mid a_i=0\}$ *and* $\Psi=\{i \mid b_i=0\}$ *respectively. If* $\Omega\neq\emptyset$ *and* $\Omega\neq[n]$, *then* $k_{ij}=0$, *for* $i\in[n]-\Omega$, $j\in\Omega$ *and if* $\Psi\neq\emptyset$ *and* $\Psi\neq[n]$, *then* $k_{ij}=0$, *for* $i\in\Psi$, $j\in[n]-\Psi$, *where* k_{ij} *is an* (i, j) *element of* K.

LEMMA 3. *For any two elements* (A, B), (C, D) *of* $L(K)$, *the following facts hold.*

(1) *If either* $A\|C$ *or* $D\|B$, *then* $CB=AD=0$,

(2) *If either* $A>C$ *or* $D>B$, *then* $CB=0$,

(3) *If either* $A<C$ *or* $D<B$, *then* $AD=0$.

PROOF. (1) Suppose that A and C are incomparable. If $a_i=1$, there exists j such that $c_j=1$ and $a_j=0$. By Lemma 2, we can see $k_{ij}=0$ in the matrix $K=[k_{ij}]$. If $d_i=1$, then $DC\not\leqq K$, a contradiction. Thus we have $d_i=0$, which implies $AD=0$. We can similarly obtain $CB=0$. (2) and (3) can be proved by the same method.

THEOREM 2. *Let* K' *be an* $(n+2)\times(n+2)$ *matrix of the form*

$$K'=\begin{bmatrix} 1 & e & C \\ f & 1 & A \\ D & B & K \end{bmatrix}$$

where $e, f\in B_0$, A *and* C *are* $1\times n$ *matrices and* B *and* D *are* $n\times 1$ *matrices. Then* K' *corresponds to a topology if and only if the following conditions are satisfied*

(1) (A, B) *and* (C, D) *are elements of* $L(K)$.

(2) $CB\leqq e\leqq\zeta(A, C)*\zeta(D, B)$.

(3) $AD\leqq f\leqq\zeta(C, A)*\zeta(B, D)$.

PROOF. Using Lemma 1, K' corresponds to a topology if and only if the following conditions are satisfied.

(a) $AK=A$ and $CK=C$.

(b) $KB=B$ and $KD=D$.

(c) $BA\leqq K$ and $DC\leqq K$.

(d) $CB\leqq e$ and $AD\leqq f$.

(e) $eA\leqq C$ and $De\leqq B$.

(f) $fC\leqq A$ and $Bf\leqq D$.

The condition (a) is equivalent to that both A and C are elements of the row space of K and (b) to that both B and D are elements of the column space of K. Hence (a), (b) and (c) are equivalent to that (A, B) and (C, D) are elements of $L(K)$.

The condition (e) is equivalent to $e \leqq \zeta(A, C)$ and $e \leqq \zeta(D, B)$ and hence to $e \leqq \zeta(A, C) * \zeta(D, B)$. By the same way, the condition (f) is equivalent to $f \leqq \zeta(C, A) * \zeta(B, D)$ and the proof is completed.

By Lemma 3, note that if (A, B) and (C, D) are elements of $L(K)$, then $\zeta(A, C) * \zeta(D, B) = 0$ implies $CB = 0$ and $\zeta(C, A) * \zeta(B, D) = 0$ implies $AD = 0$.

3. $N((A, B), (C, D), K)$

DEFINITION 3. Let A and C be $1 \times n$ matrices, and B and D, $n \times 1$ matrices. We denote the number of $(n+2) \times (n+2)$ matrices K' of the form $K' = \begin{bmatrix} 1 & * & C \\ * & 1 & A \\ D & B & K \end{bmatrix}$ corresponding to a topology by $N((A, B), (C, D), K)$.

By this definition, we get the following lemmas, which are easily proved.

LEMMA 4. *For any two elements* $(A, B), (C, D)$ *of* $L(K)$, $N((A, B), (C, D), K) = 1, 2$ *or* 4 *and* $N((A, B), (C, D), K) = N((C, D), (A, B), K)$.

LEMMA 5. *Let* $K' = (A, B)$ *be an element of* $L(K)$. *Then we get* $\alpha((A, B)) = \sum_{(C, D) \in L(K)} N((A, B), (C, D), K)$. *The number of terms on the righthand side is equal to* $\alpha(K)$.

THEOREM 3. *Let* $K' = (A, B)$ *be an element of* $L(K)$. *Then* $AB = 1$ *if and only if* $A = K_{i*}$ *and* $B = K_{*i}$ *for some* i.

PROOF. To show the "only if" part, assume that $A = [a_1, \cdots, a_i, \cdots, a_n]$, $B = {}^t[b_1, \cdots, b_i, \cdots, b_n]$ and $a_i = b_i = 1$ for some i. Then we get $K'_{1*} \geqq K'_{i+1*}$ and $K'_{*1} \geqq K'_{*i+1}$. These imply $K'_{*1} \leqq K'_{*i+1}$, $K'_{1*} \leqq K'_{i+1*}$ and so $K'_{1*} = K'_{i+1*}$, $K'_{*1} = K'_{*i+1}$ [3]. These facts imply $A = K_{i*}$ and $B = K_{*i}$ for some i. The "if" part is obvious.

THEOREM 4. *Let* K *be a matrix corresponding to a* T_0 *topology,* A *a* $1 \times n$ *matrix and* B *an* $n \times 1$ *matrix. Then* $K' = (A, B)$ *corresponds to a* T_0 *topology if and only if* $K' \in L(K)$ *and* $AB = 0$.

PROOF. Let (A, B) be an element of $L(K)$ such that $AB = 1$. Then, by using Theorem 3, we obtain $A=K_{i*}$ and $B=K_{*i}$ for some i and hence K' does not correspond to a T_0 topology. Conversely, if $K'=(A, B)$ is an element of $L(K)$ such that $AB=0$, then K' corresponds to a topology and by using Theorem 3, we see $(A, B)\neq(K_{i*}, K_{*i})$ for each i. Therefore K' corresponds to a T_0 topology.

THEOREM 5. *For any two elements* (A, B), (C, D) *of* $L(K)$,

(1) $N((A, B), (A, B), K)=1$ *if and only if* $A=K_{i*}$ *and* $B=K_{*i}$ *for some* i.

(2) $N((A, B), (C, D), K)=4$ *if and only if* $A=C$, $B=D$ *and* $(A, B)\neq(K_{i*}, K_{*i})$ *for each* i.

PROOF. (1) Assume that $N((A, B), (A, B), K)=1$. Then a pair (e, f) is uniquely determined, so that $\begin{bmatrix}1 & e & A\\ f & 1 & A\\ B & B & K\end{bmatrix}$ corresponds to a topology. Apply Theorem 2 to the case $A=C$, $B=D$ and we get $AB=CB=AD=1$. Since $K'=(A, B)$ is the matrix corresponding to a topology, we get $A=K_{i*}$ and $B=K_{*i}$ for some i by Theorem 3. Conversely, if $A=K_{i*}$ and $B=K_{*i}$, then we get $e=f=1$ by Theorem 2.

(2) If $N((A, B), (C, D), K)=4$, then we obtain $\zeta(A, C)*\zeta(D, B)=\zeta(C, A)*\zeta(B, D)=1$ and $CB=AD=0$ by Theorem 2. These facts imply $A=C$, $D=B$ and so $AB=0$. By Theorem 3, we get $(A, B)\neq(K_{i*}, K_{*i})$ for each i. Conversely, if $(A, B)\neq(K_{i*}, K_{*i})$ for each i and $A=C$, $B=D$, then we obtain $AB=0$ by Theorem 3. By assumption, we obtain $CB=AB=AD=0$ and $\zeta(A, C)*\zeta(D, B)=\zeta(C, A)*\zeta(B, D)=1$. By Theorem 2, we get $N((A, B), (C, D), K)=4$.

For a given matrix K, let $\sim$ be an equivalence relation on $[n]$ defined by $i\sim j$ if and only if $K_{i*}=K_{j*}$. Choose a representative $k(i)$, $i=1, \cdots, l$, for each equivalence class. Then we have an $l\times l$ matrix $\tilde{K}$ such that $\tilde{K}=[k_{k(i),k(j)}]$, $1\leqq i, j\leqq l$. It is easily verified that the matrix $\tilde{K}$ corresponds to a T_0 topology on an l-set. We shall call $\tilde{K}$ a reduced matrix of K.

LEMMA 6 ([2]). *If* $\tilde{K}$ *is a reduced matrix of a matrix* K, *then we have* $\alpha(K)=\alpha(\tilde{K})$.

LEMMA 7. *Let* (A, B) *be an element of* $L(K)$. *Then* $N((A, B), (C, D), K)=1$ *holds for all elements* (C, D) *of* $L(K)$ *if and only if* $A=K_{i*}$ *and* $B=K_{*i}$ *for some* i.

PROOF. If $(A, B)\neq(K_{i*}, K_{*i})$ for all i, then by Theorem 5, we get

$N((A, B), (A, B), K)=4$. Conversely, if $A=K_{i*}$ and $B=K_{*i}$ hold for some i, then K is a reduced matrix of (K_{i*}, K_{*i}). By using Lemmas 5 and 6, we get

$$\alpha(K)=\alpha((K_{i*}, K_{*i}))=\sum_{(C,D)\in L(K)} N((K_{i*}, K_{*i}), (C, D), K).$$

Therefore, we conclude $N((K_{i*}, K_{*i}), (C, D), K)=1$ for all elements (C, D) of $L(K)$ by Lemma 4.

4. An inequality $\alpha(K)<\alpha(I_n)$

LEMMA 8. *Let* $\begin{bmatrix} 1 & C \\ D & I_n \end{bmatrix}$ *be an element of* $L(I_n)$, *where* I_n *is the identity matrix of order* n. *If* $(C, D)\neq(0, 0)$ *and* $(C, D)\neq((I_n)_{i*}, (I_n)_{*i})$ *for each* i, *then* $N((0, 0), (C, D), I_n)=2$.

PROOF. Under the assumption, we shall consider three cases.

(a) $C=0,\ D\neq 0$.

(b) $C\neq 0,\ D=0$.

(c) $C\neq 0,\ D\neq 0$.

But the case (c) dose not occur, because $DC\leqq I_n$. Applying Theorem 2 to the case (a), we have

$$0\leqq e\leqq \zeta(0, 0)*\zeta(D, 0)=0.$$

$$0\leqq f\leqq \zeta(0, 0)*\zeta(0, D)=1.$$

In case of (b), we obtain similarly,

$$0\leqq e\leqq \zeta(0, C)*\zeta(0, 0)=1.$$

$$0\leqq f\leqq \zeta(C, 0)*\zeta(0, 0)=0.$$

Therefore, we conclude $N((0, 0), (C, D), I_n)=2$.

LEMMA 9. *Let* (A, B) *be an* $(n+1)\times(n+1)$ *non-identity matrix belonging to* $L(K)$ *where* A *and* B *are* $1\times n$ *and* $n\times 1$ *matrices respectively. Then there exists an element* (C, D) *of* $L(K)$ *such that* $(C, D)\neq(K_{i*}, K_{*i})$ *for all* i *and* $N((A, B), (C, D), K)=1$.

PROOF. The proof is divided into the following four cases.

(a) $A=0,\ B=0$.

By assumtion, K is a non-identity matrix. Hence there exists a comparable pair K_{i*}, K_{j*} such that $i\neq j$. Without loss of generality, we may assume $K_{i*}\leqq K_{j*}$. If we put $C=K_{i*}$ and $D=K_{*j}$, then these Boolean vectors C, D satisfy $DC\leqq K$. This implies $(C, D)\in L(K)$. We see easily $\zeta(D, B)=\zeta(K_{*j}, 0)=0$, $\zeta(C, A)=\zeta(K_{i*}, 0)=0$. By Theorem 2, we can obtain $e=f=0$.

(b) $A=0$, $B\neq 0$.

Let $D=0$ and $B={}^t[b_1, \cdots, b_j, \cdots, b_n]$ be an element of $C(K)$ such that $b_j=1$. If we put $C=K_{j*}$, then $DC=0\leqq K$ and $CB=1$. By Theorem 2, we obtain $CB=1\leqq e\leqq\zeta(0, K_{j*})*\zeta(0, B)=1$. $AD=0\leqq f\leqq\zeta(K_{j*}, 0)*\zeta(B, 0)=0$. Then we get $e=1$ and $f=0$.

(c) $A\neq 0$, $B=0$.

Let $C=0$ and D be an element of $C(K)$ such that $AD=1$. By the same method as above, we get $e=0$ and $f=1$.

(d) $A\neq 0$, $B\neq 0$.

Let $C=0$ and $D=0$. Then we similarly obtain $e=f=0$. These facts imply the lemma.

THEOREM 6. *Let K be an $n\times n$ non-identity matrix corresponding to a topology. Then we obtain $\alpha(K)<\alpha(I_n)$.*

PROOF. By Lemma 6, if K corresponds to a non-T_0 topology, there exists a reduced $m\times m$ $(m<n)$ matrix $\tilde{K}$ corresponding to T_0-topology and satisfying $\alpha(\tilde{K})=\alpha(K)$. Since $\alpha(I_n)=2^{n+1}+n-1$ [Lemma 6, [2]], we have $\alpha(I_m)<\alpha(I_n)$. Therefore, we prove this theorem for a matrix K corresponding to a T_0-topology. The proof of our theorem will be done by induction on the order n of K. If $n=2$, then $\alpha(K)=8$ and $\alpha(I_2)=9$ and so this inequality is true. Assume that $\alpha(K)<\alpha(I_k)$ holds for a non-identity matrix K $(=K_k)$ of order k corresponding to a T_0 topology $(2\leqq k)$. Note that I_{k+1} is expressed by an element $(0, 0)$ of $L(I_k)$. By Lemma 5, $\alpha(I_{k+1})=\alpha((0, 0))=\sum_{(C,D)\in L(I_k)} N((0, 0), (C, D), I_k)$. By Lemma 8 and Theorem 5, the number of (C, D) with $N((0, 0), (C, D), K)=1$ is k. Assume that K' is a $(k+1)\times(k+1)$ non-identity matrix K_{k+1} of the form $K'=(A, B)$. Then

$$\alpha(K')=\alpha((A, B))=\sum_{(C,D)\in L(K)} N((A, B), (C, D), K)\,.$$

By Lemma 9, the number m of (C, D) with $N((A, B), (C, D), K)=1$ satisfies $k+1\leqq m$. From these facts, we obtain $\alpha(I_{k+1})=2\alpha(I_k)-k+2$ and $\alpha(K_{k+1})=2\alpha(K_k)-m+2$. Since $\alpha(K_k)<\alpha(I_k)$ and $k+1\leqq m$, we obtain $\alpha(K_{k+1})<\alpha(I_{k+1})$. This completes the inductive step and the proof of the theorem.

5. A lattice $L(K)$

Let $A=[a_1, a_2, \cdots, a_n]$, $C=[c_1, c_2, \cdots, c_n]$ and define $A\cdot C$ by $A\cdot C=[a_1*c_1, a_2*c_2, \cdots, a_n*c_n]$. Similarly, define ${}^tA\cdot{}^tC$ by ${}^tA\cdot{}^tC={}^t(A\cdot C)$ and $A+C$ as usual. Define an order in $L(K)$ by $(A, B)\leqq_L(A', B')$ if and only if $A\leqq A'$

and $B \geqq B'$, and a join operation $\vee$ and a meet operation $\wedge$ in $L(K)$ by

$$(A, B) \vee (A', B') = (A+A', B \cdot B'), \qquad (A, B) \wedge (A', B') = (A \cdot A', B+B').$$

Then, it is easily checked that $L(K)$ is a distributive lattice. For terminology of lattice theory refer to [4].

LEMMA 10. *Let $K=M(T)$ and $J=M(T')$ be the matrices corresponding to topologies T and T' respectively. If two matrices K and J are isomorphic (i. e. $K={}^tPJP$, with a permutation matrix P), then $L(K)$ and $L(J)$ are isomorphic as lattices.*

The proof is easy and so omitted. Note that T and T' are homeomorphic if and only if K and J are isomorphic [2], [3].

As a matter of convenience, we call that (C, D) satisfies a condition $C_{A,B}$ for a fixed element (A, B) of $L(K)$, if (C, D) satisfies the following conditions:

(1) (C, D) is an element of $L(K)$ and is different from (A, B).

(2) $AD=CB=0$.

(3) (C, D) is comparable to (A, B) in $L(K)$.

LEMMA 11. *Let (A, B) and (C, D) be two elements of $L(K)$. Then $N((A, B), (C, D), K)=2$ holds if and only if (C, D) satisfies $C_{A,B}$.*

PROOF. Assume that (C, D) satisfies $C_{A,B}$. Then, one of two equations $\zeta(A, C)*\zeta(D, B)=1$ and $\zeta(C, A)*\zeta(B, D)=1$ is true. Hence, under the assumption $CB=AD=0$, we obtain $N((A, B), (C, D), K)=2$ by Theorem 2. Conversely, if $N((A, B), (C, D), K)=2$ holds, then the following four cases can be considered.

(a) $1=CB=\zeta(A, C)*\zeta(D, B)=1$,
$0=AD \leqq \zeta(C, A)*\zeta(B, D)=1$.

(b) $0=CB \leqq \zeta(A, C)*\zeta(D, B)=1$,
$1=AD=\zeta(C, A)*\zeta(B, D)=1$.

(c) $0=CB=\zeta(A, C)*\zeta(D, B)=0$,
$0=AD \leqq \zeta(C, A)*\zeta(B, D)=1$.

(d) $0=CB \leqq \zeta(A, C)*\zeta(D, B)=1$,
$0=AD=\zeta(C, A)*\zeta(B, D)=0$.

In cases of (a) and (b), we can conclude $(A, B)=(C, D)$. But, these cases occur only if $N((A, B), (C, D), K)=1$ or 4. In case (c), we get $(A, B) \neq (C, D)$, $C \leqq A$, $B \leqq D$ and so $(C, D) \leqq_L (A, B)$, that is, (C, D) satisfies $C_{A,B}$. In case (d), we get the same conclusion.

LEMMA 12. *Let K be a matrix corresponding to T_0 topology. If $(A, B) \neq (K_{i*}, K_{*i})$ for each i, then the subset of $L(K)$ defined by $\{(C, D) \mid AD=CB=0$ and (C, D) is comparable to (A, B) in $L(K)\}$ contains (A, B) and is a sublattice of $L(K)$.*

PROOF. Since $(A, B) \neq (K_{i*}, K_{*i})$ for each i, the above subset contains (A, B) by Theorem 4. The remaining part of the lemma can be proved by easy computations.

DEFINITION 4. Let v be a Boolean vector of dimension n. We define a complement v^c of v by a vector such that $v_i^c=1$ if and only if $v_i=0$, where v_i v_i^c denote the i-th component of v and v^c respectively.

THEOREM 7. *Let K be an $n \times n$ matrix corresponding to a T_0 topology, $A=[a_1, \cdots, a_i, \cdots, a_n]$, $B={}^t[b_1, \cdots, b_i, \cdots, b_n]$ and $S=\{i \mid a_i=b_i=0\}$, $s=|S|$. Suppose that $(A, B) \in L(K)$ and $(A, B) \neq (K_{i*}, K_{*i})$ for each i. Then there exists a chain in $L(K)$ satisfying the following conditions.*

(a) *The chain contains (A, B) and its length is $n+s$.*

(b) *Each element (C, D) of the chain, which is different from (A, B), satisfies $C_{A,B}$.*

(c) *The maximal element of the chain is $(({}^tB)^c, 0)$ and the minimal element of the chain is $(0, ({}^tA)^c)$.*

PROOF. Since K corresponds to a T_0 topology, K is isomorphic to a triangular matrix [3]. Therefore, by Lemma 10, we may assume that K is an upper triangular matrix.

If $(A, B)=([1, 1, \cdots, 1], 0)$, $(0, {}^t[1, 1, \cdots, 1])$, or $(0, 0)$, there exists clearly a chain satisfying the conditions of the theorem. Therefore, we may assume $(A, B) \neq ([1, 1, \cdots, 1], 0)$, $(0, {}^t[1, 1, \cdots, 1])$, $(0, 0)$. Let $R=\{i \mid a_i=0,\ b_i=1\}$, $r=|R|$, $T=\{i \mid a_i=1,\ b_i=0\}$, $t=|T|$, $u:[r] \to R$, $v:[s] \to S$, $w:[t] \to T$ be strictly order reversing maps, where each order of these sets is natural one, and let

$$R(m)=\{u(m+1), u(m+2), \cdots, u(r)\},$$

$$S(m)=\{v(m+1), v(m+2), \cdots, v(s)\},$$

$$T(m)=\{w(m+1), w(m+2), \cdots, w(t)\}.$$

Now we shall construct a chain $(A_0, B_0), (A_1, B_1), \cdots, (A_{n+s}, B_{n+s})$ as follows.

First define the following Boolean vectors. $A_0=A_1=\cdots=A_s=0$, $A_{s+t+1}=\cdots$

$=A_n=A$, $B_0={}^t[b_1^0, \cdots, b_n^0]$ with $b_i^0=a_i^c$, $i=1, \cdots, n$, $B_{s+1}=\cdots=B_{s+t}=B$, $B_{n+1}=\cdots=B_{n+s}=0$.

Note that (A_0, B_0) satisfies $C_{A,B}$ by the corollary of Lemma 1 and Lemma 2.

(1). If $S=\emptyset$, then go to the next step (2). If $S\neq\emptyset$, then we shall define the Boolean vector B_1 by replacing the $v(1)$-th component $b_{v(1)}^0=1$ of B_0 by 0. Then (A_1, B_1) covers (A_0, B_0) (denoted by $(A_0, B_0)<\cdot(A_1, B_1)$) and (A_1, B_1) satisfies $C_{A,B}$. Assume that a saturated chain $(A_0, B_0)<\cdot(A_1, B_1)<\cdot\cdots<\cdot(A_m, B_m)$ is constructed and each (A_i, B_i) satisfies $C_{A,B}$, $1\leqq i\leqq m$. If $S(m)=\emptyset$, go to the next step (2). If $S(m)\neq\emptyset$, then we shall define the Boolean vector B_{m+1} by replacing the $v(m+1)$-th component $b_{v(m+1)}^{(m)}=1$ of $B_m=[b_1^{(m)}, b_2^{(m)}, \cdots, b_n^{(m)}]$ by 0. Then $(A_m, B_m)<\cdot(A_{m+1}, B_{m+1})$ and (A_{m+1}, B_{m+1}) satisfies $C_{A,B}$. This completes the inductive step and so we obtain $(A_0, B_0)<\cdot(A_1, B_1)<\cdot\cdots<\cdot(A_s, B_s)=(0, B)$ and each element (A_i, B_i) satisfies $C_{A,B}$, $0\leqq i\leqq s$.

(2). If $T=\emptyset$, go to the next step (3). If $T\neq\emptyset$, then we define the Boolean vector $A_{s+1}=[a_1^{(s+1)}, \cdots, a_i^{(s+1)}, \cdots, a_n^{(s+1)}]$ by $w(1)$-th component $a_{w(1)}^{(s+1)}=1$, $a_i^{(s+1)}=0$, $i\neq w(1)$. Then we obtain $(A_s, B_s)<\cdot(A_{s+1}, B_{s+1})$ and (A_{s+1}, B_{s+1}) satisfies $C_{A,B}$. Assume that a saturated chain $(A_s, B_s)<\cdot(A_{s+1}, B_{s+1})<\cdot\cdots<\cdot(A_{s+m}, B_{s+m})$ is constructed and each (A_i, B_i) satisfies $C_{A,B}$, $s+1\leqq i\leqq s+m$. If $T(m)=\emptyset$, go to the next step (3). If $T(m)\neq\emptyset$, then we define the Boolean vector A_{s+m+1} by replacing the $w(m+1)$-th component $a_{w(m+1)}^{(s+m)}=0$ of $A_{s+m}=[a_1^{(s+m)}, \cdots, a_i^{(s+m)}, \cdots, a_n^{(s+m)}]$ by 1. Then $(A_{s+m}, B_{s+m})<\cdot(A_{s+m+1}, B_{s+m+1})$ holds and (A_{s+m+1}, B_{s+m+1}) satisfies $C_{A,B}$. This completes the inductive step and we obtain the chain

$$(A_s, B_s)<\cdot(A_{s+1}, B_{s+1})<\cdot\cdots<\cdot(A_{s+t}, B_{s+t})=(A, B)$$

and each (A_{s+i}, B_{s+i}) satisfies $C_{A,B}$, $1\leqq i\leqq t-1$.

(3). If $R=\emptyset$, go to the next step (4). If $R\neq\emptyset$, then we define the Boolean vector B_{s+t+1} by replacing the $u(1)$-th component $b_{u(1)}=1$ of $B=[b_1, \cdots, b_i, \cdots, b_n]$ by 0. Then $(A_{s+t}, B_{s+t})<\cdot(A_{s+t+1}, B_{s+t+1})$ holds and (A_{s+t+1}, B_{s+t+1}) satisfies $C_{A,B}$. Now, suppose that the saturated chain $(A_{s+t}, B_{s+t})<\cdot(A_{s+t+1}, B_{s+t+1})<\cdot\cdots<\cdot(A_{s+t+m}, B_{s+t+m})$ is constructed and each element (A_{s+t+i}, B_{s+t+i}) satisfies $C_{A,B}$, $1\leqq i\leqq m$. If $R(m)=\emptyset$, go to the next step (4). If $R(m)\neq\emptyset$, then we define the Boolean vector $B_{s+t+m+1}$ by replacing the $u(m+1)$-th component $b_{u(m+1)}^{(s+t+m)}=1$ of $B_{s+t+m}=[b_1^{(s+t+m)}, \cdots, b_i^{(s+t+m)}, \cdots, b_n^{(s+t+m)}]$ by 0.

Then we see $(A_{s+t+m}, B_{s+t+m})<\cdot(A_{s+t+m+1}, B_{s+t+m+1})$ and $(A_{s+t+m+1}, B_{s+t+m+1})$ satisfies $C_{A,B}$. Then we obtain the saturated chain

$$(A, B)=(A_{s+t}, B_{s+t})<\cdot(A_{s+t+1}, B_{s+t+1})<\cdot\cdots<\cdot(A_n, B_n)=(A, 0)$$

and each (A_{s+t+i}, B_{s+t+i}) satisfies $C_{A,B}$, $1\leqq i\leqq r$. Note that $n=s+t+r$.

(4). If $S=\emptyset$, then the construction of the chain concludes. If $S\neq\emptyset$, then we define the Boolean vector A_{n+1} by replacing the $v(1)$-th component $a_{v(1)}=0$ of $A=[a_1, \cdots, a_i, \cdots, a_n]$ by (1) and then we obtain $(A_n, B_n)<\cdot(A_{n+1}, B_{n+1})$ and (A_{n+1}, B_{n+1}) satisfies $C_{A,B}$. Now suppose that the saturated chain $(A_n, B_n)<\cdot(A_{n+1}, B_{n+1})<\cdot\cdots<\cdot(A_{n+m}, B_{n+m})$ is constructed and each (A_{n+i}, B_{n+i}) satisfies $C_{A,B}$, $1\leqq i\leqq m$. If $S(m)=\emptyset$, then concludes the construction of the chain. If $S(m)\neq\emptyset$, then we define the Boolean vector A_{n+m+1} by replacing the $v(m+1)$-th component $a_{v(m+1)}^{(n+m)}=0$ of $A_{n+m}=[a_1^{(n+m)}, a_2^{(n+m)}, \cdots, a_n^{(n+m)}]$ by replacing 1. Then we see $(A_{n+m}, B_{n+m})<\cdot(A_{n+m+1}, B_{n+m+1})$ and (A_{n+m+1}, B_{n+m+1}) satisfies $C_{A,B}$. This completes the inductive step and we have a chain $(A_n, B_n)<\cdot(A_{n+1}, B_{n+1})<\cdot\cdots<\cdot(A_{n+s}, B_{n+s})=({}^t(B^c), 0)$.

From above facts, we constructed the chain satisfying the desired condition of the theorem;

$$(0, {}^t(A^c))=(A_0, B_0)<\cdot(A_1, B_1)<\cdot\cdots<\cdot(A_{n+s}, B_{n+s})=({}^t(B^c), 0)\,.$$

COROLLARY. *Let $(A, B)\neq(K_{i*}, K_{*i})$ be a fixed element of $L(K)$, where K is an $n\times n$ matrix corresponding to T_0 topology. In the multiset $\{N((A, B), (C, D), K)\,|\,DC\leqq K, C\in R(K), D\in C(K)\}$, the number of 2's is equal or greater than $n+s$, where the number s is mentioned above.*

PROOF. By Theorem 7, at least $n+s$ elements of $L(K)$ satisfy the conditions of Lemma 11. This implies the corollary.

The following lemma is easily proved by using Lemma 11 and the proof is omitted.

LEMMA 13. *Let L_n be the upper triangular matrix of order n defined as follows;*

$$L_n=\begin{bmatrix} 1 & 1\cdots\cdots & 1 \\ 0 & 1\cdots\cdots & 1 \\ \vdots & \ddots & \vdots \\ 0\cdots\cdots & 0 & 1 \end{bmatrix}$$

and A an n-dimensional Boolean vector where each component of A is 1. Then $N((A, 0), (C, D), L_n)=2$ holds if and only if $(C, D)=(C_i, 0)$ holds where C_i is the n-dimensional Boolean vector such that $C_i=[c_1^i, \cdots, c_n^i]$ such that $c_j^i=0$ for $j\leqq i$ and $c_j^i=1$ for $j>i$ for some i, $1\leqq i\leqq n$.

This lemma implies that in the multiset $\{N((A, 0), (C, D), L_n\,|\,(C, D)\in L(L_n)\}$, the number of 2's is n, the number of 1's is $\alpha(L_n)-(n+1)$ and remaining element is 4

THEOREM 8. *Let K be an $n\times n$ matrix corresponding to a T_0 topology. Then*

$$\alpha(L_n)\leqq\alpha(K).$$

PROOF. The proof is carried out by induction on the order of K. If $n=2$, then $\alpha(L_2)=8$, $\alpha(K)=8$ or 9 and so theorem is true. Assume that $\alpha(L_k)\leqq\alpha(K)$, $(2\leqq k)$ holds for a $k\times k$ matrix K corresponding to a T_0 topology. Let K' be a $(k+1)\times(k+1)$ matrix of the form (A, B) corresponding to a T_0 topology. By using Lemma 5, we obtain $\alpha(K')=\alpha((A, B))=\sum_{(C,D)\in L(K)} N((A, B), (C, D), K)$. By using the corollary of Theorem 7, on the righthand side of this equation, the number of 2's is equal or greater than $k+s$. If we denote this number of 2's by m, the number of 1's is $\alpha(K)-(m+1)$ and the remaining term is 4. Therefore, $\alpha(K')=\alpha(K)+m+3$. Since $\alpha(L_{k+1})=\alpha(L_k)+k+3$ and $\alpha(L_k)\leqq\alpha(K)$ hold, we obtain $\alpha(L_{k+1})\leqq\alpha(K)$. This completes the inductive step and concludes the proof.

THEOREM 9. *For an arbitrary matrix K of order n which corresponds to a T_0 topology, we have*

$$n(n+5)/2+1\leqq\alpha(K)\leqq2^{n+1}+n-1.$$

PROOF. It is known that $\alpha(L_n)=n(n+5)/2+1$ and $\alpha(I_n)=2^{n+1}+n-1$ [2]. By Theorems 6 and 8, we get the conclusion.

6. Table of $N((A, B), (C, D), K)$

In this section, we state the results of computations of $N((A, B), (C, D), K)$ for several matrices K by using previous results.

They are expressed by matrices, whose elements are positive integers, and rows and columns are indexed by $L(K)$. For each K, we give a matrix $[m_{\lambda\mu}]$ defined by

$$m_{\lambda\mu}=N((A, B), (C, D), K) \qquad \text{for } \lambda=(A, B),\ \mu=(C, D).$$

This is symmetric by Lemma 4. In the following tables, each index $\lambda=(A, B)$ is expressed by ${}^{t}{}^{A}_{B}$ and the numbers on the left side or the upper side of $\lambda={}^{t}{}^{A}_{B}$ are $\alpha\left(\begin{bmatrix}1 & A\\ B & K\end{bmatrix}\right)$.

$$K=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

		18	15						9	
		$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&0\\0&1\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$	$\begin{smallmatrix}1&0\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$	$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$
18	$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	4	2	2	2	2	2	2	1	1
15	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	2	4	1	2	2	1	1	1	1
	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	2	1	4	1	1	2	2	1	1
	$\begin{smallmatrix}0&0\\0&1\end{smallmatrix}$	2	2	1	4	1	2	1	1	1
	$\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$	2	2	1	1	4	1	2	1	1
	$\begin{smallmatrix}1&0\\0&0\end{smallmatrix}$	2	1	2	2	1	4	1	1	1
	$\begin{smallmatrix}0&1\\0&0\end{smallmatrix}$	2	1	2	1	2	1	4	1	1
9	$\begin{smallmatrix}0&1\\0&1\end{smallmatrix}$	1	1	1	1	1	1	1	1	1
	$\begin{smallmatrix}1&0\\1&0\end{smallmatrix}$	1	1	1	1	1	1	1	1	1

Table 1.

$$K=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

		15			13			8	
		$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	$\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	$\begin{smallmatrix}0&1\\1&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\1&0\end{smallmatrix}$
15	$\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$	4	2	2	2	1	2	1	1
	$\begin{smallmatrix}0&1\\0&0\end{smallmatrix}$	2	4	2	2	2	1	1	1
	$\begin{smallmatrix}0&0\\1&0\end{smallmatrix}$	2	2	4	1	2	2	1	1
13	$\begin{smallmatrix}1&1\\0&0\end{smallmatrix}$	2	2	1	4	1	1	1	1
	$\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$	1	2	2	1	4	1	1	1
	$\begin{smallmatrix}0&0\\1&1\end{smallmatrix}$	2	1	2	1	1	4	1	1
8	$\begin{smallmatrix}0&1\\1&1\end{smallmatrix}$	1	1	1	1	1	1	1	1
	$\begin{smallmatrix}1&1\\1&0\end{smallmatrix}$	1	1	1	1	1	1	1	1

Table 2.

$$K=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

		15			13			8	
		1 0 0 0	0 0 0 1	0 0 0 0	1 1 0 0	1 0 0 1	0 0 1 1	1 1 0 1	1 0 1 1
15	1 0 0 0	4	2	2	2	2	1	1	1
	0 0 0 1	2	4	2	1	2	2	1	1
	0 0 0 0	2	2	4	2	1	2	1	1
13	1 1 0 0	2	1	2	4	1	1	1	1
	1 0 0 1	2	2	1	1	4	1	1	1
	0 0 1 1	1	2	2	1	1	4	1	1
8	1 1 0 1	1	1	1	1	1	1	1	1
	1 0 1 1	1	1	1	1	1	1	1	1

Table 3.

$$K=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & 1 & 1\end{bmatrix}$$

		28	26		24		23				22			15		
		000 001	000 000	110 000	010 000	100 000	000 011	000 101	010 001	100 001	000 111	110 001	111 000	111 001	010 011	100 101
28	000 001	4	2	2	2	2	2	2	2	2	2	2	1	1	1	1
26	000 000	2	4	2	2	2	2	2	1	1	2	1	2	1	1	1
	110 000	2	2	4	2	2	1	1	2	2	1	2	2	1	1	1
24	010 000	2	2	2	4	1	1	2	2	1	1	1	2	1	1	1
	100 000	2	2	2	1	4	2	1	1	2	1	1	2	1	1	1
23	000 011	2	2	1	1	2	4	1	1	2	2	1	1	1	1	1
	000 101	2	2	1	2	1	1	4	2	1	2	1	1	1	1	1
	010 001	2	1	2	2	1	1	2	4	1	1	2	1	1	1	1
	100 001	2	1	2	1	2	2	1	1	4	1	2	1	1	1	1
22	000 111	2	2	1	1	1	2	2	1	1	4	1	1	1	1	1
	110 001	2	1	2	1	1	1	1	2	2	1	4	1	1	1	1
	111 000	1	2	2	2	2	1	1	1	1	1	1	4	1	1	1
15	111 001	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	010 011	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	100 101	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 4.

$$K=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}$$

		28	26		24		23				22			15		
		000 000	000 001	010 000	000 101	110 000	000 100	000 111	100 000	111 000	000 011	011 000	010 001	011 001	010 011	100 100
28	000 000	4	2	2	2	2	2	2	2	2	2	2	1	1	1	1
26	000 001	2	4	2	2	2	1	2	2	1	2	1	2	1	1	1
	010 000	2	2	4	2	2	2	1	1	2	1	2	2	1	1	1
24	000 101	2	2	2	4	1	2	2	1	1	1	1	2	1	1	1
	110 000	2	2	2	1	4	1	1	2	2	1	1	2	1	1	1
23	000 100	2	1	2	2	1	4	2	1	1	1	2	1	1	1	1
	000 111	2	2	1	2	1	2	4	1	1	2	1	1	1	1	1
	100 000	2	2	1	1	2	1	1	4	2	2	1	1	1	1	1
	111 000	2	1	2	1	2	1	1	2	4	1	2	1	1	1	1
22	000 011	2	2	1	1	1	1	2	2	1	4	1	1	1	1	1
	011 000	2	1	2	1	1	2	1	1	2	1	4	1	1	1	1
	010 001	1	2	2	2	2	1	1	1	1	1	1	4	1	1	1
15	011 001	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	010 011	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	100 100	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 5.

$$K=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

		35	28								26						18		
		000 000	000 001	000 010	000 100	000 111	001 000	010 000	100 000	111 000	000 011	000 101	000 110	011 000	101 000	110 000	001 001	010 010	100 100
35	000 000	4	2	2	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1
28	000 001	2	4	1	1	2	1	2	2	1	2	2	1	1	1	2	1	1	1
	000 010	2	1	4	1	2	2	1	2	1	2	1	2	1	2	1	1	1	1
	000 100	2	1	1	4	2	2	2	1	1	1	2	2	2	1	1	1	1	1
	000 111	2	2	2	2	4	1	1	1	1	2	2	2	1	1	1	1	1	1
	001 000	2	1	2	2	1	4	1	1	2	1	1	2	2	2	1	1	1	1
	010 000	2	2	1	2	1	1	4	1	2	1	2	1	2	1	2	1	1	1
	100 000	2	2	2	1	1	1	1	4	2	2	1	1	1	2	2	1	1	1
	111 000	2	1	1	1	1	2	2	2	4	1	1	1	2	2	2	1	1	1
26	000 011	2	2	2	1	2	1	1	2	1	4	1	1	1	1	1	1	1	1
	000 101	2	2	1	2	2	1	2	1	1	1	4	1	1	1	1	1	1	1
	000 110	2	1	2	2	2	2	1	1	1	1	1	4	1	1	1	1	1	1
	011 000	2	1	1	2	1	2	2	1	2	1	1	1	4	1	1	1	1	1
	101 000	2	1	2	1	1	2	1	2	2	1	1	1	1	4	1	1	1	1
	110 000	2	2	1	1	1	1	2	2	2	1	1	1	1	1	4	1	1	1
18	001 001	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	010 010	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	100 100	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 6.

REMARK. From these numerical results, the following conjectures seem to be true. For each matrix K corresponding to a topology and positive numbers i and j, define a multiset $\{m_{\lambda\mu}\}$, where λ runs over all λ satisfying $\alpha(\lambda)=i$ and μ runs over all μ satisfying $\alpha(\mu)=j$.

CONJECTURE 1: This multiset $\{m_{\lambda\mu}\}$ is uniquely determined by $\alpha(K)$, i and j.

CONJECTURE 2: Let M and M' be matrices constructed from matrices K and K' respectively as above. If $\alpha(K)=\alpha(K')$, then M and M' are isomorphic (that is $M'={}^tPMP$, with a permutation matrix P).

References

[1] Kim, K.H., Boolean matrix theory and its applications, Marcel Dekker, N.Y. and Basel 1982.
[2] Ochiai, S., On a topological invariant of finite topological spaces and enumerations, Tsukuba Jour. of Math. Vol. 16, No. 1 (1992), 63-74.
[3] Sharp, H.J.R., Quasi-orderings and Topologies on Finite Sets, Proc. Amer. Math. Soc., 17 (1966), 1344-1349.
[4] Stanley, R.P., Enumerative Combinatrics Vol. 1, Wadsworth, Monterey, Calif., 1986.

Department of Mathematics
Utsunomiya University
Mine-machi Utsunomiya 321
Japan