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A NOTE ON THE TITS SYSTEMS OF KAC-MOODY STEINBERG GROUPS

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Abstract. Let $G(H)$ be the Chevalley (Steinberg) Kac-Moody group of the Kac-Moody Lie algebra L. If σ is the canonical homomorphism of H onto G, and $\{B_G, N_G\}$ is the Tits system in G, then $\{\sigma^{-1}(B_G), \sigma^{-1}(N_G)\}$ is a Tits system in the Weyl-simple subgroup of H.

More than twenty years have passed since the study of Kac-Moody groups was begun by Moody and Teo [M−T] and Marcuson [Mar]. During this period, interest in the subject has swelled ([A−M],[Mat],[Mor$_1$],[Mor$_2$], [Mor$_3$],[M−R] and[T$_2$]) along with that for the general theory of Kac-Moody Lie algebras [K]. One focus of this research has been to describe the Tits systems, or B-N pairs, within the groups.

For both the Chevalley (adjoint) and Steinberg (nonadjoint) Kac-Moody groups, the methods for constructing Tits systems have closely resembled those used in the classical finite dimensional case. We note here that, for the Steinberg Kac-Moody groups, it is possible to construct a second Tits system within a naturally arising subgroup by elementary means.

For ϕ a field of characteristic 0, let L be a Kac-Moody ϕ-Lie algebra with Weyl group W. Let Δ, Π, and P denote the set of roots, the set of simple roots, and the set of positive roots respectively. We have $\Delta W = \Delta, \Pi W \subseteq \Delta$, and $P\,W \subseteq \Delta$. If $\alpha \in \Pi W$, we say that α is Weyl-simple [Mar]. Denote by L_α the root space corresponding to the root α.

Let G be the Kac-Moody Chevalley group of L, i.e. the group generated by all $\exp(ad\, te_\alpha)$,

$$G = \langle \exp(ad\, te_\alpha) : \alpha \in \Pi W, e_\alpha \in L_\alpha, t \in \phi \rangle .$$

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Assume that we have a representation of L on the e-extreme module $\mathfrak{M}$ with dominant highest weight. In particular, e_α acts on $\mathfrak{M}$. The Kac-Moody Steinberg group of L associated with $\mathfrak{M}$ is then

$$H = \langle \exp(te_\alpha) : \alpha \in PW, e_\alpha \in L_\alpha, t \in \phi \rangle .$$

Let σ be given by $\exp(te_\alpha) \rightarrow \exp(ad\, te_\alpha)$.

Let H_Π be the Weyl-simple subgroup of H, i.e. the subgroup generated by all $\exp(te_\alpha)$ where α is Weyl-simple. Define B'_H to be $B_H \cap H_\Pi$ and N'_H to be N_H.

THEOREM 1. σ is a canonical group map of H_Π onto G.

PROOF. See Marcuson [Mar].

The notion of a Tits system, or $B-N$ pair $[\mathrm{T}_1]$, plays an important role in the simplicity proofs for the classical Chevalley groups [C].

DEFINITION. A group G, subgroups B and N, and a subset S of $N/(B \cap N)$ is a Tits system if

i. $\langle B \cup N \rangle = G$,

ii. $B \cap N$ is normal in G,

iii. S is a set of involutions which generate $W \equiv N/(B \cap N)$,

iv. for all $s \in S$ and all $w \in W$, $wBs \subseteq BwB \cup BwsB$, and

v. for all $s \in S$ $sBs \not\subseteq B$.

In the sequel, we will say that $\{B,N\}$ forms a Tits system when the nature of G and S are clear.

In 1972, Moody and Teo showed by construction that a Tits system $\{B_G, N_G\}$ exists in G . Soon afterward, Marcuson (1975) constructed the Tits system $\{B_H, N_H\}$ generalizing that of Steinberg [S] for the representation of L on $\mathfrak{M}$. In 1983, Peterson and Kac [$P-K$] studied the theory of $B-N$ pairs $\{B_{PK}, N_{PK}\}$ in Kac-Moody groups for general integral representations. It follows from their work that $\{B_G, N_G\}$ coincides with $\{B_{PK}, N_{PK}\}$ for adjoint representations and that $\{B'_H, N'_H\}$ coincides with $\{B_{PK}, N_{PK}\}$ for highest weight representations. To see this, note first that $B_{PK} \subseteq B_G$ and $B_{PK} \subseteq B'_H$. Then $B_{PK} = B_G$, and $B_{PK} = B'_H$. Now, because $B'_H \cap N'_H$ contains the kernel of $\sigma\ [P-K]$, it follows that $\{\sigma^{-1}(B_G), \sigma^{-1}(N_G)\}$ coincides with $\{B_{PK}, N_{PK}\}$.

In the finite dimensional case, $H_\Pi = H$, and the Tits system in H is the inverse image under σ of the Tits system in G. It is therefore natural to ask whether B_{PK} and N_{PK} form a Tits system simply as the inverse images of B_G and N_G under σ.

THEOREM 2. $B''_H \equiv \sigma^{-1}(B_G)$ and $N''_H \equiv \sigma^{-1}(N_G)$ form a Tits system in H_Π

PROOF. Let K_H = kernel σ.

i. $H_\Pi = \langle B''_H \cup N''_H \rangle$.

Let $h \in H_\Pi$, and write $\sigma(h) = g_1 \cdots g_n$ where $g_i \in B_G \cup N_G$ for $i = 1, \ldots, n$. Let $h_i \in H_\Pi$ be such that $\sigma(h_i) = g_i$. Then $h_0 = h_1 \cdots h_n \in \langle B''_H \cup N''_H \rangle$, and $h_0^{-1} h = k \in K_H \subseteq B''_H \cup N''_H$. Hence $h = h_0 k \in \langle B''_H \cup N''_H \rangle$ so that $H_\Pi \subseteq \langle B''_H \cup N''_H \rangle$. Thus $H_\Pi = \langle B''_H \cup N''_H \rangle$.

ii. $B''_H \cap N''_H$ is normal in N''_H.

This follows from $B''_H \cap N''_H = \sigma^{-1}(B_G \cap N_G)$.

iii. $N''_H / (B''_H \cap N''_H) = W$ is generated by a set of involutions.

This is a consequence of the fact that σ induces an isomorphism of $N''_H (B''_H \cap N''_H)$ onto $N_G / (B_G \cap N_G)$.

iv. For all $s \in S$, and all $w \in W$,

$$wB''_H s \subseteq B''_H s B''_H \cup B''_H ws B''_H .$$

Let w''_i and n'' be representatives of s and w respectively in $N''_H / (B''_H \cap N''_H)$, and let $b'' \in B''_H$. Assume that σ maps w''_i, n, and b'' to w_i, n and b respectively. Thus

$$\sigma : n'' b'' w''_i \to nbw_i \in wB_G s \subseteq B_G s B_G \cup B_G ws B_G .$$

Hence $n'' b'' w''_i \in \sigma^{-1}(B_G s B_G \cup B_G ws B_G)$

$$=\sigma^{-1}(B_G s B_G)\cup \sigma^{-1}(B_G ws B_G).$$

To finish, we need only show that for any $w\in W$, $\sigma^{-1}(B_G w B_G)=B''_H w B''_H$. Now, $B''_H w B''_H \subseteq \sigma^{-1}(B_G w B_G)$ is clear. Let $x''\in\sigma^{-1}(B_G w B_G)$. Then $\sigma(x'')$ = bnc where n is a representative of w in $N_G/(B_G\cap N_G)$. Choose b'' and c'' in B''_H and n'' a representative of w in $N''_H/(B''_H\cap N''_H)$ such that σ maps $x''_0\equiv b''n''c''$ to bnc. Then $x''_0{}^{-1}=k\in K_H\subseteq B''_H$, and we see that $x''=k\,x''_0\in K_H B''_H w B''_H = B''_H w B''_H$. Therefore $\sigma^{-1}(B_G w B_G)\subseteq B''_H w B''_H$.

v. For all $s\in S$, $sB''_H s\not\subseteq B''_H$.

We know $sB_G s\not\subseteq B_G$ so we have $w_{i0}b_0w_{i0}\notin B_G$ for some representative w_{i0} of s and some $b_0\in B_G$. Choose $b''\in B''_H$ and $w''_i\in N''_H$ such that σ takes b'' to b_0 and w''_i to w_{i0}. Then

$$\sigma : w''_i b'' w''_i \to w_{i0}b_0w_{i0}\notin B_G,$$

and so $w''_i b'' w''_i\notin B''_H$.

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