

## ON DUAL-BIMODULES

(Dedicated to Prof. G. Azumaya for his seventieth birthday)

By

Y. KURATA and K. HASHIMOTO

A ring  $R$  with identity in which  $I = r_R l_R(I)$  for every right ideal  $I$  and  $J = l_R r_R(J)$  for every left ideal  $J$  of  $R$  is called a *dual ring*. This ring has been investigated by many authors. As is well-known, an Artinian dual ring is a QF-ring and, recently, Hajarnavis and Norton [4] have studied dual rings and pointed out that certain properties well-known for QF-rings are also seen to hold without the Artinian assumption.

In this paper, we shall introduce the notion of dual-bimodules and try to give a module-theoretic characterization of dual rings. Let  $R$  and  $S$  be rings with identity and  ${}_R Q_S$  an  $(R, S)$ -bimodule. We shall call  $Q$  a *left dual-bimodule* if

- (1)  $l_R r_Q(A) = A$  for every left ideal  $A$  of  $R$ , and
- (2)  $r_Q l_R(Q') = Q'$  for every  $S$ -submodule  $Q'$  of  $Q$ .

A right dual-bimodule is similarly defined and we shall call  $Q$  a *dual-bimodule* if it is a left dual-bimodule and is a right dual-bimodule as well. A left dual-bimodule need not be a right dual-bimodule in general (see Example 4.2).

Trivially a dual ring is a dual-bimodule. A bimodule which defines a Morita duality is a dual-bimodule [1, Exercise 24.7]. Furthermore, a dual-bimodule is a quasi-Frobenius bimodule in the sense of Azumaya [2] (cf. also [5, Theorem 4]).

In Section 1, we shall study basic properties of left dual-bimodules and show that, among other things, an  $(R, S)$ -bimodule  $Q$  such that the mapping

$$\lambda: R \longrightarrow \text{End}(Q_S)$$

given by  $a \rightarrow a_L$ , the left multiplication by  $a$ , is surjective is a left dual-bimodule if and only if every factor module of  ${}_R R$  and  $Q_S$  is  $Q$ -torsionless (Theorem 1.4), for a left dual-bimodule  ${}_R Q_S$  the ring  $R$  is semilocal (Theorem 1.10) and that for every  $R$ -module  ${}_R Q \neq 0$ ,  ${}_R Q_S$  is a left dual-bimodule with  $S = \text{End}({}_R Q)$  if and

only if  $R$  is simple Artinian (Theorem 1.16). Finally, in closing this section, we shall show that the notion of left dual-bimodules is closed under Morita equivalence (Theorem 1.20).

We shall treat in Section 2 dual-bimodules. It is shown that for an  $(R, S)$ -bimodule  $Q$  with both  ${}_R Q$  and  $Q_S$  finitely generated,  $Q$  is a dual-bimodule if and only if  ${}_R R$  and  $S_S$  are  $Q$ -reflexive and every factor module of  ${}_R R$ ,  $S_S$ ,  ${}_R Q$  and  $Q_S$  is  $Q$ -torsionless (Theorem 2.8). Furthermore, if  $Q_S$  is finitely generated and  $\text{rad}({}_R Q) \leq \text{rad}(Q_S)$ , then the ring  $R$  is semiperfect (Theorem 2.10).

In Section 3, we shall consider a duality defined by a left dual-bimodule  ${}_R Q_S$ . It is shown that, in case  ${}_R Q$  is finitely generated, a duality defined by  $Q$  exists if and only if  $Q_S$  is quasi-injective and  $\lambda$  is surjective (Theorem 3.3) and that the duality is one between the full subcategory of  $R$ -mod of finitely generated  $Q$ -reflexive  $R$ -modules and the full subcategory of  $\text{mod-}S$  of finitely cogenerated  $Q$ -reflexive  $S$ -modules (Proposition 3.4).

Finally we shall provide, in Section 4, some examples of left dual-bimodules to illustrate the results given in this paper.

Throughout this paper,  $R$  and  $S$  will denote rings with identity. If  ${}_R M$  is a left  $R$ -module and  $M'$  is a submodule of  $M$ , then we shall write  $M' \leq_R M$ , in particular,  $A \leq_R R$  will mean  $A$  is a left ideal of  $R$ . For  $M' \leq_R M$ ,  $M' \leq_{e(S)} M$  will mean  $M'$  is essential (small) in  $M$ . We shall use similar notations for right  $S$ -modules. For an  $(R, S)$ -bimodule  $Q$ , we write  $( )^* = \text{Hom}(-, Q)$  to denote the  $Q$ -dual functor.

For notations, definitions and familiar results concerning the ring theory we shall mainly follow [1].

### 1. Left Dual-Bimodules.

We shall begin with the following

LEMMA 1.1 [1, Exercise 24.3]. *Let  $Q$  be a left dual-bimodule. Then for each indexed set  $(A_\lambda)_\lambda$  of left ideals of  $R$  and each indexed set  $(Q'_\lambda)_\lambda$  of submodules of  $Q_S$*

$$r_Q(\bigcap_\lambda A_\lambda) = \sum_\lambda r_Q(A_\lambda) \quad \text{and} \quad l_R(\bigcap_\lambda Q'_\lambda) = \sum_\lambda l_R(Q'_\lambda).$$

The preceding lemma implies that if  $Q$  is a left dual-bimodule, then the mapping  $A \rightarrow r_Q(A)$  is a lattice anti-isomorphism with inverse  $Q' \rightarrow l_R(Q')$  between the submodule lattices of  ${}_R R$  and  $Q_S$ . In particular, we have  $l_R(Q) = 0$ , i. e.  ${}_R Q$  is faithful.

LEMMA 1.2. *Let  $Q$  be an  $(R, S)$ -bimodule. Then for  $A \leq_R R$  the following conditions are equivalent:*

- (1)  $l_R r_Q(A) = A$ .
- (2)  $R/A$  is a  $Q$ -torsionless  $R$ -module.

PROOF. This follows from the fact that  $l_R r_Q(A)/A = \text{Rej}_{R/A}(Q)$  for every  $A \leq_R R$  [1, Lemma 24.4], where  $\text{Rej}_{R/A}(Q) = \bigcap \{ \text{Ker } h \mid h \in \text{Hom}_R(R/A, Q) \}$  [1, p. 109].

Let  $Q$  be a left dual-bimodule. Then by (1.2)  ${}_R R$  is  $Q$ -torsionless. Hence, not only cyclic  $R$ -modules, but also left ideals of  $R$  are  $Q$ -torsionless.

Note that if a bimodule  $Q$  defines a Morita duality, then every left ideal of  $R$  is  $Q$ -reflexive [1, p. 278]. However, there is a dual-bimodule  $Q$  which has no  $Q$ -reflexive left ideal of  $R$  (see Example 4.1). Hence a dual-bimodule need not define a Morita duality, in general.

Recall that  $\lambda: R \rightarrow \text{End}(Q_S)$  is the mapping given by  $a \rightarrow a_L$ , the left multiplication by  $a$ . If  $Q$  is a left dual-bimodule, then  ${}_R Q$  is faithful and hence  $\lambda$  is injective.

LEMMA 1.3. *Let  $Q$  be an  $(R, S)$ -bimodule. Then for  $Q' \leq Q_S$  the following conditions are equivalent:*

- (1)  $r_Q l_R(Q') = Q'$ .
- (2)  $Q' \cong \phi(R/l_R(Q'))^*$ , where  $\phi: Q' \rightarrow (R/l_R(Q'))^*$  denotes the monomorphism given by  $\phi(u)(a + l_R(Q')) = au$  for  $u \in Q'$ ,  $a \in R$ .

Furthermore, (1) implies

- (3)  $Q/Q'$  is  $Q$ -torsionless,

and if  $\lambda$  is surjective, then (3) implies (1).

PROOF. Since  $Q' \leq r_Q l_R(Q')$  and the composite map of the canonical isomorphism  $(R/l_R(Q'))^* \cong r_Q l_R(Q')$  with  $\phi$  is the identity map of  $Q'$ , the equality holds if and only if  $\phi$  is onto. This means that (1) and (2) are equivalent.

(1)  $\Rightarrow$  (3) follows from the fact that  $\text{Rej}_{Q/Q'}(Q) \leq r_Q l_R(Q')/Q'$ . If  $\lambda$  is surjective, these are the same and (3) implies (1).

Clearly  $Q_S$  is  $Q$ -torsionless. Hence, for a left dual-bimodule  $Q$  by (1.3) not only submodules of  $Q_S$ , but also factor modules of  $Q_S$  are  $Q$ -torsionless.

Combining these two lemmas, we have

THEOREM 1.4. *Let  $Q$  be an  $(R, S)$ -bimodule. If  $\lambda$  is surjective, then the*

following conditions are equivalent:

- (1)  $Q$  is a left dual-bimodule.
- (2) Every factor module of  ${}_R R$  and  $Q_S$  is  $Q$ -torsionless.

As we shall show in (2.7), if  $Q$  is a dual-bimodule and  $Q_S$  is finitely generated, then  $\lambda$  is surjective. However, in case  $\lambda$  is not surjective, though every factor module of  ${}_R R$  and  $Q_S$  is  $Q$ -torsionless, we can not conclude that  $Q$  is a left dual-bimodule, in general (see Example 4.4).

The following lemma is often useful.

LEMMA 1.5. *Let  $Q$  be a left dual-bimodule,  $A \leq_s {}_R R$  and  $Q' \leq Q_S$ . Then we have*

- (1)  $A \leq_{e(s)} {}_R R$  if and only if  $\imath_Q(A) \leq_{s(e)} Q_S$ .
- (2)  $Q' \leq_{e(s)} Q_S$  if and only if  $l_R(Q') \leq_{s(e)} {}_R R$ .

PROOF. (1) Suppose that  $A \leq_s {}_R R$  and  $\imath_Q(A) \cap Q' = 0$  for some  $Q' \leq Q_S$ . Then by (1.1)  $A + l_R(Q') = R$  and hence  $l_R(Q') = R$ . Thus we have  $Q' = 0$ , from which we see that  $\imath_Q(A) \leq_e Q_S$ .

Conversely, suppose that  $\imath_Q(A) \leq_e Q_S$  and  $A + A' = R$  for some  $A' \leq {}_R R$ . Then  $\imath_Q(A) \cap \imath_Q(A') = 0$  and hence  $\imath_Q(A') = 0$ . Thus we have  $A' = R$ , which shows that  $A \leq_s {}_R R$ .

(2) follows from (1) at once.

From this lemma, we can see that the socle corresponds to the radical to each other under the lattice anti-isomorphism between the submodule lattices of  ${}_R R$  and  $Q_S$ . Indeed, we have

PROPOSITION 1.6. *Let  $Q$  be a left dual-bimodule. Then*

- (1)  $Z({}_R Q) = \text{rad}(Q_S) = \imath_Q(\text{soc}({}_R R))$  where  $Z({}_R Q)$  denotes the singular submodule of  ${}_R Q$ .
- (2)  $\text{rad}({}_R R) = l_R(\text{soc}(Q_S))$ .

PROOF. (1) If  $u \in Z({}_R Q)$ , then by (1.5)  $uS \leq_s Q_S$  and hence  $u \in \text{rad}(Q_S)$ . Conversely, if  $u \in \text{rad}(Q_S)$ , then  $u$  is contained in some small submodule  $Q'$  of  $Q_S$ . Hence,  $uS$  is also small in  $Q_S$ . Again by (1.5)  $l_R(u) \leq_e {}_R R$  and  $u$  is in  $Z({}_R Q)$ .

Furthermore,  $\text{rad}(Q_S) = \bigcap \{Q' \leq Q_S \mid Q' \text{ is maximal in } Q_S\} = \bigcap \{\imath_Q(A) \mid A \text{ is minimal in } {}_R R\} = \imath_Q(\text{soc}({}_R R))$ .

Likewise (2) follows from (1.1).

PROPOSITION 1.7. *Let  $Q$  be a left dual-bimodule. Then  $Q_S$  has finite Goldie dimension.*

PROOF. Let  $0 \neq u \in Q$ . If there is no nonzero submodule of  $Q_S$  not containing  $u$ , then  $Q_S$  is indeed uniform. Otherwise there exists a submodule  $Q_u$  of  $Q_S$  maximal with respect to not containing  $u$  by Zorn's lemma. Then  $Q/Q_u$  is uniform.

Now clearly  $\bigcap_{0 \neq u \in Q} Q_u = 0$ . Therefore  $R = \sum_{0 \neq u \in Q} l_R(Q_u)$  and hence there exist  $u_1, \dots, u_n$  in  $Q$  such that  $l_R(Q_{u_1}) + \dots + l_R(Q_{u_n}) = R$ . We therefore have  $\bigcap_{i=1}^n Q_{u_i} = 0$ . Thus  $Q$  is embedded into  $Q/Q_{u_1} \oplus \dots \oplus Q/Q_{u_n}$ , from which we see that  $Q_S$  has finite Goldie dimension.

From this proof we see at once

PROPOSITION 1.8. *Let  $Q$  be a left dual-bimodule. Then*

- (1)  $Q_S$  is finitely cogenerated.
- (2)  $\text{soc}(Q_S)$  is finitely generated and is the smallest essential submodule of  $Q_S$  [1, Proposition 10.7].
- (3) There are only finitely many non-isomorphic simple submodules of  $Q_S$ .

The preceding proposition is based on the fact that  ${}_R R$  is finitely generated. Hence, we have

PROPOSITION 1.9. *Let  $Q$  be a left dual-bimodule. Then  $Q_S$  is finitely generated if and only if  ${}_R R$  is finitely cogenerated.*

*If this is the case,  $\text{soc}({}_R R)$  is finitely generated and is the smallest essential left ideal of  $R$ .*

PROOF. The proof of the "only if" part is similar to that of (1.8). To prove the "if" part, suppose that  ${}_R R$  is finitely cogenerated. Since  $Q_S = \sum_{u \in Q} uS$ , it follows that  $0 = \bigcap_{u \in Q} l_R(uS)$ . By assumption there exist  $u_1, \dots, u_n$  in  $Q$  such that  $0 = \bigcap_{i=1}^n l_R(u_i S)$  and hence we have  $Q = \sum_{i=1}^n u_i S$ . This shows that  $Q$  is finitely generated.

THEOREM 1.10. *Let  $Q$  be a left dual-bimodule. Then  $R$  is semilocal, i.e.  $R/\text{rad}(R)$  is semisimple.*

PROOF. Let  $\text{soc}(Q_S) = \bigoplus_{i=1}^n Q_i$ , where each  $Q_i$  is a simple submodule of  $Q_S$ . Then  $\text{rad}(R) = \bigcap_{i=1}^n l_R(Q_i)$ . Since each  $l_R(Q_i)$  is a maximal left ideal of  $R$  and  $0 \rightarrow R/\text{rad}(R) \rightarrow \bigoplus_{i=1}^n R/l_R(Q_i)$  is exact,  $R/\text{rad}(R)$  is semisimple.

In particular, we have by [1, Proposition 15.17]

PROPOSITION 1.11. *For a left dual-bimodule  $Q$ , we have*

$$\text{soc}({}_R Q) = \imath_Q(\text{rad}(R)) = \text{soc}(Q_S).$$

Henceforth we shall denote  $\text{soc}({}_R Q) = \text{soc}(Q_S)$  simply by  $\text{soc}(Q)$ .

Using [1, Corollary 15.18], for any  $R$ -module  ${}_R M$ ,

$$\text{rad}({}_R M) = \text{rad}(R) \cdot M$$

and  $M/\text{rad}({}_R M)$  is semisimple, i. e.  ${}_R M$  is semisimple if and only if  $\text{rad}({}_R M) = 0$ .

As an application of (1.6) and (1.11), we have

PROPOSITION 1.12. *Let  $Q$  be a left dual-bimodule. Then the following conditions are equivalent:*

- (1)  $R$  is semisimple.
- (2)  $Q_S$  is semisimple.
- (3)  ${}_R Q$  is semisimple.
- (4)  $Z({}_R Q) = 0$ .
- (5)  $\text{rad}(Q_S) = 0$ .

LEMMA 1.13. *Let  $Q$  be a left dual-bimodule. Then every  $R$ -homomorphism from a left ideal of  $R$  to  $Q$  with finitely generated image is given by a right multiplication of an element of  $Q$ .*

PROOF. Cf. [4, Proposition 5.2].

The preceding lemma implies that, for every finitely generated left ideal  $A$  of  $R$ , every diagram of the form

$$\begin{array}{c} A \leq R \\ \downarrow \\ Q \end{array}$$

is completed by an  $R$ -homomorphism  $R \rightarrow Q$ .

Hence, by [6, Proposition 2.8] we have

COROLLARY 1.14. *Let  $Q$  be a left dual-bimodule. If either  ${}_R Q$  or  ${}_R R$  is Noetherian, then  ${}_R Q$  must be an injective cogenerator.*

In general, for a finitely generated left ideal  $A$  of  $R$ , the mapping  $Q/\imath_Q(A) \rightarrow A^*$  given by  $u + \imath_Q(A) \rightarrow u_R|_A$  is an  $S$ -monomorphism. The lemma also shows that this mapping is surjective and hence

$$Q/\imath_Q(A) \cong A^* .$$

Let  $Q$  be a left dual-bimodule and  $Q'$  an  $(R, S)$ -submodule. If  ${}_R Q'$ 's is also a left dual-bimodule, then  ${}_R Q'$  must be faithful. Hence,  $Q' = \imath_Q l_R(Q') = \imath_Q(0) = Q$ . Thus, there is no proper  $(R, S)$ -submodule which is also a left dual-bimodule. However, we have

PROPOSITION 1.15. *Let  $Q$  be a left dual-bimodule,  $Q'$  an  $(R, S)$ -submodule and  $\bar{R} = R/l_R(Q')$ . Then  ${}_{\bar{R}} Q'$ 's is a left dual-bimodule.*

PROOF. Since  $Q'$  can be regarded as an  $\bar{R}$ -module by defining  $a + l_R(Q') \cdot u' = au'$  for  $a \in R$  and  $u' \in Q'$ , we have  $\imath_{Q'}(A/l_R(Q')) = \imath_{Q'}(A)$  for  $A/l_R(Q') \leq_{\bar{R}} \bar{R}$  and  $l_{\bar{R}}(Q'') = l_R(Q'')/l_R(Q')$  for  $Q'' \leq Q'$ 's. Therefore, we have  $l_{\bar{R}} \imath_{Q'}(A/l_R(Q')) = l_{\bar{R}} \imath_{Q'}(A) = l_R \imath_{Q'}(A)/l_R(Q') = (l_R \imath_Q(A) + l_R(Q'))/l_R(Q') = l_R \imath_Q(A)/l_R(Q') = A/l_R(Q')$  and  $\imath_{Q'} l_{\bar{R}}(Q'') = \imath_{Q'}(l_R(Q'')/l_R(Q')) = \imath_{Q'} l_R(Q'') = \imath_Q l_R(Q'') \cap Q' = Q'' \cap Q' = Q''$ .

In particular, for a left dual-bimodule  $Q$ ,  $\text{soc}(Q)$  is an  $(R, S)$ -submodule and hence  ${}_{\bar{R}} \text{soc}(Q)_S$  is a left dual-bimodule satisfying the equivalent condition of (1.12), where  $\bar{R} = R/\text{rad}(R)$ .

The following theorem characterizes simple Artinian rings by means of the notion of left dual-bimodules.

THEOREM 1.16. *For a ring  $R$  the following conditions are equivalent:*

- (1)  *$R$  is simple Artinian.*
- (2) *For every  $R$ -module  ${}_R Q \neq 0$ ,  ${}_R Q_S$  is a left dual-bimodule with  $S = \text{End}({}_R Q)$ .*
- (3) *For every finitely generated  $R$ -module  ${}_R Q \neq 0$ ,  ${}_R Q_S$  is a left dual-bimodule with  $S = \text{End}({}_R Q)$ .*
- (4) *For every simple  $R$ -module  ${}_R Q$ ,  ${}_R Q_S$  is a left dual-bimodule with  $S = \text{End}({}_R Q)$ .*
- (5) *There exists a simple  $R$ -module  ${}_R Q$  such that  ${}_R Q_S$  is a left dual-bimodule with  $S = \text{End}({}_R Q)$ .*

*If this is the case,  $R \cong {}^\lambda \text{End}(Q_S)$  for every  $R$ -module  ${}_R Q \neq 0$  with  $S = \text{End}({}_R Q)$ . Furthermore, in case  ${}_R Q$  is finitely generated,  $S$  is also simple Artinian.*

PROOF. (1) $\Rightarrow$ (2). Let  $R$  be a simple Artinian ring,  ${}_R Q \neq 0$  and  $S = \text{End}({}_R Q)$ . Then by [1, Exercise 13.10]  ${}_R Q$  is a cogenerator. Hence every (cyclic)  $R$ -module is  $Q$ -torsionless.

Furthermore,  ${}_R Q$  is balanced by [1, Exercise 18.32] which means that  $\lambda$  is surjective. However,  $\text{Ker } \lambda$  must be zero, since  $R$  is a simple ring and  $Q \neq 0$ . Thus, we have  $R \cong {}^\lambda \text{End}(Q_S)$ .

Since  ${}_R R$  is semisimple, we can write  $R$  as  $R = m_1 \oplus \cdots \oplus m_n$  with each  $m_i$  a minimal left ideal of  $R$ . Using this decomposition,  $Q_S \cong \text{Hom}_R(R, Q) \cong \text{Hom}_R(m_1, Q) \oplus \cdots \oplus \text{Hom}_R(m_n, Q)$ , where each  $\text{Hom}_R(m_i, Q)$  is either simple or zero by [1, Exercise 16.18]. It follows that  $Q_S$  is semisimple.

Now let  $Q' \leq Q_S$ . Then  $Q/Q'$  is isomorphic to a submodule of  $Q_S$  and hence is  $Q$ -torsionless. Thus, by (1.4)  ${}_R Q_S$  is a left dual-bimodule.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are clear.

(5) $\Rightarrow$ (1). Let  ${}_R Q$  be a simple  $R$ -module such that  ${}_R Q_S$  is a left dual-bimodule with  $S = \text{End}({}_R Q)$ . Then for any ideal  $A$  of  $R$   ${}_R Q(A)$  is an  $(R, S)$ -submodule of  $Q$  and is either  $Q$  or 0. Hence  $A$  must be either 0 or  $R$  and  $R$  is a simple ring. Since  $\text{rad}(R) = 0$ ,  ${}_R R$  is semisimple by (1.10). Thus  $R$  is simple Artinian by [1, Proposition 13.5].

Note that in the preceding theorem each condition of (1) to (5) is also equivalent to each one of the following

(3') For every finitely generated  $R$ -module  ${}_R Q \neq 0$ ,  ${}_R Q_S$  is a dual-bimodule with  $S = \text{End}({}_R Q)$ .

(4') For every simple  $R$ -module  ${}_R Q$ ,  ${}_R Q_S$  is a dual-bimodule with  $S = \text{End}({}_R Q)$ .

(5') There exists a simple  $R$ -module  ${}_R Q$  such that  ${}_R Q_S$  is a dual-bimodule with  $S = \text{End}({}_R Q)$ .

To see this, assume that  $R$  is simple Artinian and  ${}_R Q \neq 0$  is a finitely generated  $R$ -module with  $S = \text{End}({}_R Q)$ . As was shown in the proof of (1) $\Rightarrow$ (2) of the preceding theorem,  ${}_R Q_S$  is a left dual-bimodule and  $R \cong {}^2 \text{End}(Q_S)$ . Hence, to prove (3') it is sufficient to show that  $S$  is simple Artinian. Since  ${}_R Q$  is semisimple by (1.12) and is finitely generated, we can write  ${}_R Q$  as  $Q = Q_1 \oplus \cdots \oplus Q_n$  with each  ${}_R Q_i$  simple and  $Q_i \cong Q_j$  for all  $i$  and  $j$  [1, Exercise 13.1]. Therefore,  $S$  is isomorphic to the ring of all  $n \times n$  matrices over the division ring  $\text{End}({}_R Q_i)$  and thus it is simple Artinian. This shows that (1) $\Rightarrow$ (3') and (3') $\Rightarrow$ (4') $\Rightarrow$ (5') $\Rightarrow$ (5) are evident.

As we shall show in Example 4.5, the condition (2') corresponding to the condition (2) of (1.16) does not hold in general.

The following proposition follows from (1.14) and the proof of (1) $\Rightarrow$ (2) of (1.16).

**PROPOSITION 1.17.** *Let  $R$  be a semisimple ring and  ${}_R Q$  an  $R$ -module with  $S = \text{End}({}_R Q)$ . Then  ${}_R Q_S$  is a left dual-bimodule if and only if  ${}_R Q$  is a cogenerator.*



Let  $Q$  be a left dual-bimodule. Then  $Q_S$  is finitely cogenerated and hence by [1, Exercise 10.15]  $Q_S$  has a finite indecomposable decomposition  $Q_S = Q_1 \oplus \cdots \oplus Q_n$  with each  $Q_i$  indecomposable. Each  $Q_i$  can be written as  $Q_i = r_Q(A_i)$  for some  $A_i \leq_R R$  and  $R/A_i$  is indecomposable. For, if  $R/A_i$  is decomposable and  $R/A_i = A'/A_i \oplus A''/A_i$  for  $A_i \leq A'$ ,  $A'' \leq_R R$ , then we have  $Q_i = r_Q(A') \oplus r_Q(A'')$ , a contradiction. Since  $r_Q(A_i + \bigcap_{j \neq i} A_j) = Q_i \cap \sum_{j \neq i} Q_j = 0$ ,  $(A_i)_{1 \leq i \leq n}$  is coindependent and hence the  $R$ -homomorphism  $f: R \rightarrow \bigoplus_{i=1}^n R/A_i$  defined by  $f(a) = (a + A_i)$  for  $a \in R$  is surjective by [1, Exercise 6.18]. Furthermore,  $\text{Ker } f = \bigcap_{i=1}^n A_i = l_R(Q) = 0$ . Thus, we have  $R \cong^f \bigoplus_{i=1}^n R/A_i$  and  $R$  has a finite indecomposable decomposition.

PROPOSITION 1.18. *Let  $Q$  be a left dual-bimodule. Then both  $Q_S$  and  ${}_R R$  have finite indecomposable decompositions. In particular,  $Q_S$  is indecomposable if and only if  ${}_R R$  is indecomposable.*

Finally, in closing this section, we shall show that the notion of left dual-bimodules is closed under Morita equivalence.

To see this, let  ${}_R Q_S$  be a left dual-bimodule and  $T$  a ring equivalent to  $S$  via an equivalence  $H: \text{mod-}S \rightarrow \text{mod-}T$ . There exists a  $(T, S)$ -bimodule  $P$  such that  ${}_T P$  and  $P_S$  are progenerators and  $H$  is given by  $H \cong \text{Hom}_S(P, -)$  [1, Theorem 22.1]. We assume that for simplicity  $H = \text{Hom}_S(P, -)$ . Using [1, Proposition 21.7], each submodule of  $H(Q)_T$  is of the form  $\text{Im } H(\nu)$  for some  $Q' \leq Q_S$  and the inclusion map  $\nu: Q' \rightarrow Q$ .

LEMMA 1.19. *With the same notation as above, we have*

- (1)  $l_R(\text{Im } H(\nu)) = l_R(Q')$ .
- (2)  $r_{H(Q)} l_R(\text{Im } H(\nu)) = \text{Im } H(\nu)$ .

*For a left ideal  $A$  of  $R$  and the inclusion map  $\mu: r_Q(A) \rightarrow Q$ ,*

- (3)  $r_{H(Q)}(A) = \text{Im } H(\mu)$ .
- (4)  $l_R r_{H(Q)}(A) = A$ .

PROOF. (1) Suppose that  $a \in l_R(Q')$ . Then for any  $f \in H(Q')$  and any  $p \in P$   $(a \cdot \nu f)(p) = a \cdot f(p) \in aQ' = 0$  and hence  $l_R(Q') \leq l_R(\text{Im } H(\nu))$ . Conversely, suppose that  $a \in l_R(\text{Im } H(\nu))$ . Since  $P_S$  is a generator, there exists a set  $\Lambda$  such that  $P^{(\Lambda)} \rightarrow^{\alpha} Q' \rightarrow 0$  is exact. For the injection map  $\nu_\lambda: P \rightarrow P^{(\Lambda)}$ ,  $\lambda \in \Lambda$ ,  $\alpha \nu_\lambda$  is in  $H(Q')$  and hence by assumption  $(a \cdot \alpha \nu_\lambda)(p) = 0$  for each  $p \in P$  and each  $\lambda \in \Lambda$ . Let  $u' \in Q'$  and let  $x \in P^{(\Lambda)}$  such that  $\alpha(x) = u'$ . Then  $x$  can be written as  $x = \nu_{\lambda_1}(p_1) + \cdots + \nu_{\lambda_k}(p_k)$  for some  $\lambda_1, \dots, \lambda_k \in \Lambda$  and  $p_1, \dots, p_k \in P$ . Then  $au' = a \cdot \alpha(x) = (a \cdot \alpha \nu_{\lambda_1})(p_1) + \cdots + (a \cdot \alpha \nu_{\lambda_k})(p_k) = 0$ . Hence,  $a \in l_R(Q')$  and thus  $l_R(\text{Im } H(\nu))$

$\leq l_R(Q')$ .

(2) Let  $f \in \imath_{H(Q)} l_R(\text{Im } H(\nu)) = \imath_{H(Q)} l_R(Q')$ . Then for each  $a \in l_R(Q')$  and each  $p \in P$   $a \cdot f(p) = (a \cdot f)(p) = 0$ . Hence  $f(p) \in \imath_Q l_R(Q') = Q'$ , since  ${}_R Q_S$  is a left dual-bimodule. It follows that  $f(p) = \nu(f(p)) = (\nu f)(p)$  and thus  $f = \nu f \in \text{Im } H(\nu)$ . Hence we have  $\imath_{H(Q)} l_R(\text{Im } H(\nu)) \leq \text{Im } H(\nu)$  and thus (2) follows.

(3) Let  $f \in \imath_{H(Q)}(A)$ . Then for each  $a \in A$  and each  $p \in P$   $a \cdot f(p) = (af)(p) = 0$ . It follows that  $f(p) \in \imath_Q(A)$  and hence  $f = \mu f \in \text{Im } H(\mu)$ . Conversely, let  $\mu f \in \text{Im } H(\mu)$ , where  $f \in H(\imath_Q(A))$ . Then for each  $a \in A$  and each  $p \in P$   $(a \cdot \mu f)(p) = a \cdot f(p) = 0$ . Hence,  $a \cdot \mu f = 0$  and  $\mu f \in \imath_{H(Q)}(A)$ . Thus, we have  $\imath_{H(Q)}(A) = \text{Im } H(\mu)$ .

(4) Using (1),  $l_R(\text{Im } H(\mu)) = l_R \imath_Q(A)$  and hence by (3)  $l_R \imath_{H(Q)}(A) = l_R \imath_Q(A) = A$ , since  ${}_R Q_S$  is a left dual-bimodule.

**THEOREM 1.20.** *Let  $Q$  be a left dual-bimodule and let  $T$  be a ring equivalent to  $S$  via an equivalence  $H: \text{mod-}S \rightarrow \text{mod-}T$ . Then  ${}_R H(Q)_T$  is also a left dual-bimodule.*

As is well-known,  $S$  and the ring  $(S)_n$  of all  $n \times n$  matrices over  $S$  are equivalent via  $H = - \otimes_S S^n: \text{mod-}S \rightarrow \text{mod-}(S)_n$ . Hence, we have

**COROLLARY 1.21.** *Let  $Q$  be a left dual-bimodule. Then for each  $n > 0$ ,  ${}_R Q_{(S)_n}^n$  is also a left dual-bimodule.*

In particular, if  $R$  is a dual ring, then for each  $n > 0$ ,  ${}_R R_{(R)_n}^n$  is a left dual-bimodule.

## 2. Dual-Bimodules.

If  $Q$  is a left dual-bimodule, then there are only finitely many non-isomorphic simple submodules of  $Q_S$ . However, in case  $Q$  is a dual-bimodule, by (1.10) there are only finitely many non-isomorphic simple right  $S$ -modules and each of which is isomorphic to a submodule of  $Q_S$  [6, Proposition 2.8]. Furthermore, we have

**THEOREM 2.1.** *Let  $Q$  be a dual-bimodule. Then*

- (1) *The  $Q$ -dual of every simple left  $R$ -module as well as that of every simple right  $S$ -module is simple.*
- (2) *Every simple left  $R$ -module as well as every simple right  $S$ -module is  $Q$ -reflexive.*
- (3) *There is a bijection between the irredundant sets of representatives of the simple left  $R$ -modules and the simple right  $S$ -modules.*

PROOF. Suppose first that  $Q$  is a left dual-bimodule and  ${}_R M$  is a simple  $R$ -module. Then by [6, Proposition 2.8]  $M$  is isomorphic to a simple submodule  $Ru$  of  ${}_R Q$  for some  $u \in Q$ . Therefore,  $M^* \cong (R/l_R(u))^* \cong r_Q l_R(u) = uS$  and hence  $M_S^*$  is simple, since  $l_R(u)$  is a maximal left ideal. Thus, for each  $u \in Q$ ,  ${}_R Ru$  is simple if and only if  $uS_S$  is simple and further we have  $uS \cong (Ru)^*$  via  $us \rightarrow s_R|_{Ru}$ .

However, if in addition  $Q$  is a right dual-bimodule, then every simple right  $S$ -module is of the form  $uS$  for some  $u \in Q$ . Since  $(uS)^* \cong (S/r_S(u))^* \cong l_Q r_S(u) = Ru$ ,  $Ru \cong (uS)^*$  via  $au \rightarrow a_L|_{uS}$  and thus for each  $u \in Q$  the mapping  $Ru \rightarrow uS$  can be seen as a bijection between irredundant sets of representatives of the simple left  $R$ -modules and the simple right  $S$ -modules.

Finally it is easy to see that isomorphisms mentioned above yield the condition (2).

More precisely, we have

PROPOSITION 2.2. *For a dual-bimodule  $Q$ , let  $\bar{e}_1, \dots, \bar{e}_m$  and  $\bar{f}_1, \dots, \bar{f}_m$  be basic sets of idempotents of the semisimple ring  $\bar{R} = R/\text{rad}(R)$  and  $\bar{S} = S/\text{rad}(S)$ , respectively. Then*

$$e_1 \cdot \text{soc}(Q), e_2 \cdot \text{soc}(Q), \dots, e_m \cdot \text{soc}(Q)$$

and

$$\text{soc}(Q) \cdot f_1, \text{soc}(Q) \cdot f_2, \dots, \text{soc}(Q) \cdot f_m$$

*exhaust non-isomorphic simple right  $S$ -modules and that of simple left  $R$ -modules, respectively.*

PROOF. For each  $i$ ,  $l_R(e_i \cdot \text{soc}(Q)) = \{a \in R \mid ae_i \in \text{rad}(R)\}$  and hence the mapping  $R \rightarrow \bar{R}\bar{e}_i$ , given by  $a \rightarrow a\bar{e}_i$ , is an  $R$ -epimorphism with kernel  $l_R(e_i \cdot \text{soc}(Q))$ . Therefore,  $e_i \cdot \text{soc}(Q)$  is a simple submodule of  $Q_S$ . Furthermore,  $e_i \cdot \text{soc}(Q) \cong (R/l_R(e_i \cdot \text{soc}(Q)))^* \cong (\bar{R}\bar{e}_i)^*$ . Thus, the proposition follows from (2.1).

THEOREM 2.3. *Let  $Q$  be a dual-bimodule. Then every finitely generated submodule of  $Q_S$  as well as that of  ${}_R Q$  is  $Q$ -reflexive.*

To see this, we need a lemma which is shown by a similar way as in [4, Proposition 5.2].

LEMMA 2.4. *Let  $Q$  be a dual-bimodule and  $Q' \leq Q_S$ . Then every  $S$ -homomorphism from  $Q'$  to  $Q$  with finitely generated image is given by a left multiplication of an element of  $R$ .*

It follows from this lemma that if  $Q_S$  is Noetherian, then  $Q_S$  is quasi-injective.

PROOF OF (2.3). For every finitely generated submodule  $Q'$  of  $Q_S$ , the  $R$ -homomorphism  $R/l_R(Q') \rightarrow Q'^*$  given by  $a+l_R(Q') \rightarrow a_L|_{Q'}$  yields by (2.4)

$$R/l_R(Q') \cong Q'^*$$

Therefore, using the natural isomorphism  $Q' = {}_Q l_R(Q') \cong (R/l_R(Q'))^*$ , we see that  $Q'$  is  $Q$ -reflexive.

The preceding theorem is not true without the assumption that  $Q'$  is finitely generated (see Example 4.1).

Since  $\text{soc}(Q_S)$  is finitely generated, the above isomorphism  $R/l_R(Q') \cong Q'^*$  yields, in particular,

$$R/\text{rad}(R) \cong \text{End}(\text{soc}(Q)_S)$$

as rings.

From (2.3) and [1, Proposition 20.14] we have

COROLLARY 2.5. *Let  $Q$  be a dual-bimodule. Then for every finitely generated submodule  $Q'$  of  $Q_S$ ,  $R/l_R(Q')$  is  $Q$ -reflexive.*

The proof of [4, Theorem 5.3] carries over almost word for word to the case of dual-bimodules.

PROPOSITION 2.6. *Let  $Q$  be a dual-bimodule. Then for each  $n > 0$  every factor module of  $Q_S^n$  has finite Goldie dimension.*

*In particular, in case where  $Q_S$  is a generator, every finitely generated right  $S$ -module has finite Goldie dimension.*

PROOF. First we shall prove by induction on  $n$  that every semisimple submodule of any factor module of  $Q_S^n$  is finitely generated.

Let  $n=1$  and  $K \leq Q' \leq Q_S$ . Suppose that  $Q'/K$  is semisimple, which is not finitely generated. Then by (2.2)  $Q'/K$  contains a countably infinite direct sum  $\bigoplus_{i \geq 1} (u_i S + K)/K$ , where each  $(u_i S + K)/K$  is simple and is isomorphic to the same simple  $S$ -module  $e \cdot \text{soc}(Q)$ . Let  $f_i: u_i S + K \rightarrow e \cdot \text{soc}(Q)$  be the composite of the canonical map  $\pi_i: u_i S + K \rightarrow (u_i S + K)/K$  with the isomorphism. Then by (2.4)  $f_i = a_{iL}$  for some  $a_i \in l_R(K)$  and hence we have  $a_i u_i S = e \cdot \text{soc}(Q)$ .

We now define for any subset  $A$  of  $N$ , where  $N$  denotes the set of positive integers, an  $S$ -homomorphism

$$h_A: \sum_{i \geq 1} (u_i S + K) \longrightarrow e \cdot \text{soc}(Q)$$

to be  $h_A(u_i) = a_i u_i$  whenever  $i \in A$ ,  $h_A(u_i) = 0$  whenever  $i \notin A$ ,  $h_A(K) = 0$  and extending this definition by linearity. By (2.4)  $h_A = b_{AL}$  for some  $b_A \in l_R(K)$  and hence for each  $A$  and  $i \in N$  we have  $eb_A u_i = b_A u_i$ , since the image of  $h_A$  is  $e \cdot \text{soc}(Q)$ .

Using [4, Lemma 5.1] there is an uncountable independent collection  $C$  of subsets of  $N$ . We shall show that

$$\sum_{A \in C} (Reb_A + l_R(Q')) / l_R(Q')$$

is a direct sum. To this end, let  $A_1, \dots, A_n$  be distinct elements of  $C$  and let  $c_1 eb_{A_1} + \dots + c_n eb_{A_n} \in l_R(Q')$  where  $c_1, \dots, c_n \in R$ . For each  $j$ ,  $1 \leq j \leq n$ , take an  $t_j \in A_j \cap (A_1^{-1} \cap \dots \cap A_{j-1}^{-1} \cap A_{j+1}^{-1} \cap \dots \cap A_n^{-1})$ , where  $A_i^{-1}$  means the set  $N \setminus A_i$ . Since  $u_{t_j} \in Q'$ ,  $c_1 eb_{A_1} u_{t_j} + \dots + c_n eb_{A_n} u_{t_j} = 0$ . But if  $k \neq j$ , then  $t_j \notin A_k$ . Hence  $b_{A_k} u_{t_j} = 0$  and therefore we have  $c_j eb_{A_j} u_{t_j} = 0$  for  $1 \leq j \leq n$ . We now show that  $c_j e \in \text{rad}(R)$  for  $1 \leq j \leq n$ . Suppose that  $c_j e \notin \text{rad}(R)$  for some  $j$ . Then  $Rc_j e + \text{rad}(R) \neq \text{rad}(R)$  and hence  $\bar{R}\bar{e} = (Re + \text{rad}(R)) / \text{rad}(R) \cong (Rc_j e + \text{rad}(R)) / \text{rad}(R) \neq 0$ . Since  $\bar{R}\bar{e}$  is simple,  $Re + \text{rad}(R) = Rc_j e + \text{rad}(R)$  and therefore we have  $eb_{A_j} u_{t_j} \in (Re + \text{rad}(R))b_{A_j} u_{t_j} = (Rc_j e + \text{rad}(R))b_{A_j} u_{t_j} = Rc_j eb_{A_j} u_{t_j} + \text{rad}(R)b_{A_j} u_{t_j} = 0$ , since  $c_j eb_{A_j} u_{t_j} = 0$  and  $b_{A_j} u_{t_j} \in e \cdot \text{soc}(Q) \subseteq \text{soc}(Q) = {}_{\nu_Q}(\text{rad}(R))$ . However,  $eb_{A_j} u_{t_j} = b_{A_j} u_{t_j} = a_{t_j} u_{t_j} \neq 0$ , a contradiction.

Since  $Q'/K$  is semisimple,  $Q'/K \cdot \text{rad}(S) \subseteq \text{rad}(Q'/K) = 0$  and so we have  $Q' \cdot \text{rad}(S) \subseteq K$ , which means that  $l_R(K) \cdot Q' \cdot \text{rad}(S) = 0$  and  $l_R(K) \cdot Q' \subseteq l_Q(\text{rad}(S)) = {}_{\nu_Q}(\text{rad}(R))$  by (1.11). Therefore,  $\text{rad}(R) \cdot l_R(K) \cdot Q' = 0$  and we have  $\text{rad}(R) \cdot l_R(K) \subseteq l_R(Q')$ . Since  $c_j e \in \text{rad}(R)$  and  $b_{A_j} \in l_R(K)$ ,  $c_j eb_{A_j} \in l_R(Q')$ . This shows that  $\sum_{A \in C} (Reb_A + l_R(Q')) / l_R(Q')$  is a direct sum.

As we have shown above,  $\text{rad}(R) \cdot l_R(K) \subseteq l_R(Q')$ , from which we see that  $l_R(K) / l_R(Q')$  is an  $\bar{R}$ -module and is a semisimple  $R$ -module. On the other hand,  $b_A \in l_R(K)$  implies that  $(Reb_A + l_R(Q')) / l_R(Q')$  is a submodule of  $l_R(K) / l_R(Q')$ . Hence it is semisimple and so  $\bigoplus_{A \in C} (Reb_A + l_R(Q')) / l_R(Q')$  is also semisimple. Thus, we see that  $\dim(l_R(K) / l_R(Q')) \geq |C| > |N|$  (see [4, p. 259] for the definition). A symmetrical argument now gives  $\dim(Q'/K) > |N|$ . But this holds whenever  $Q'/K$  is a non finitely generated semisimple  $S$ -module and in particular when  $Q'/K = \bigoplus_{i \geq 1} (u_i S + K) / K$ . However, clearly in this case  $\dim(Q'/K) = |N|$ , a contradiction. Thus, we have established that every semisimple submodule of any factor module of  $Q_S$  is finitely generated.

Now suppose that, for  $k \leq n-1$ , every semisimple submodule of any factor module of  $Q_S^k$  is finitely generated. Let  $k = n$  and  $K \subseteq Q_S^n$ . Then  $(Q^{n-1} + K) / K \subseteq$

$Q^n/K$  and we have  $\text{soc}(Q^n/K) \cong \text{soc}((Q^{n-1}+K)/K) \oplus \text{soc}(Q^n/K)/\text{soc}((Q^{n-1}+K)/K)$ . Since  $(Q^{n-1}+K)/K \cong Q^{n-1}/(Q^{n-1} \cap K)$ , it follows that by induction hypothesis  $\text{soc}((Q^{n-1}+K)/K)$  is finitely generated. On the other hand,  $\text{soc}(Q^n/K)/\text{soc}((Q^{n-1}+K)/K) \cong \text{soc}(Q^n/K)/(\text{soc}(Q^n/K) \cap (Q^{n-1}+K)/K) \cong (\text{soc}(Q^n/K) + (Q^{n-1}+K)/K)/((Q^{n-1}+K)/K) \leq Q^n/(Q^{n-1}+K)$ . Let  $K_n$  denote the submodule of all the  $n$ -th coordinates of elements of  $K$ . Then  $Q^n/(Q^{n-1}+K)$  is isomorphic to  $Q/K_n$  via  $(v_1, \dots, v_n) \rightarrow \bar{v}_n$  and hence  $\text{soc}(Q^n/K)/\text{soc}((Q^{n-1}+K)/K)$  can be seen as a semi-simple submodule of  $Q/K_n$ . Hence, it is finitely generated. Therefore, we see that  $\text{soc}(Q^n/K)$  is also finitely generated.

Finally, for any  $K \leq Q_S^n$ , we shall show that  $Q^n/K$  has finite Goldie dimension. Let  $0 \neq Q_\alpha/K \leq Q^n/K$  for  $\alpha \in A$  and suppose that  $(Q_\alpha/K)_{\alpha \in A}$  is independent. For each  $\alpha \in A$ , take  $0 \neq \bar{x}_\alpha = x_\alpha + K \in Q_\alpha/K$ . Then  $\bar{x}_\alpha \cdot \text{rad}(S) \neq \bar{x}_\alpha S$  by Nakayama's lemma and hence  $\bar{x}_\alpha S/\bar{x}_\alpha \cdot \text{rad}(S)$  is a nonzero semisimple  $S$ -module. Using [1, Exercise 6.3] we have  $\bigoplus_A \bar{x}_\alpha S/\bigoplus_A \bar{x}_\alpha \cdot \text{rad}(S) \cong \bigoplus_A (\bar{x}_\alpha S/\bar{x}_\alpha \cdot \text{rad}(S))$  and both  $\bigoplus_A \bar{x}_\alpha S$  and  $\bigoplus_A \bar{x}_\alpha \cdot \text{rad}(S)$  are submodules of  $Q^n/K$ . Hence we can see that  $\bigoplus_A (\bar{x}_\alpha S/\bar{x}_\alpha \cdot \text{rad}(S))$  is a semisimple submodule of  $Q^n/K'$  for some  $K' \leq Q_S^n$  and is finitely generated. It follows that  $A$  is a finite set, which completes the proof of the proposition.

**THEOREM 2.7.** *Let  $Q$  be a dual-bimodule. Then  $R$  is a dense subring of  $\text{End}(Q_S)$ .*

*In particular, if  $Q_S$  is finitely generated, then we have*

$$R \cong {}^\lambda \text{End}(Q_S).$$

**PROOF.** Let  $f \in \text{End}(Q_S)$ ,  $u_1, \dots, u_n$  finitely many elements of  $Q$  and  $Q' = u_1 S + \dots + u_n S$ . Then the mapping  $f|_{Q'}$  belongs to  $Q'^*$  and hence by (2.4) there exists an  $a \in R$  such that  $f|_{Q'} = a_L|_{Q'}$ . Thus,  $f(u_i) = a u_i$ ,  $1 \leq i \leq n$ , and  $R$  is dense in  $\text{End}(Q_S)$ .

If  $Q_S$  is not finitely generated, the theorem is not always true in general (see Example 4.1). We note that the last part of the theorem also follows from (2.5).

By [1, Theorem 24.1], an  $(R, S)$ -bimodule  $Q$  defines a Morita duality if and only if every factor module of  ${}_R R$ ,  $S_S$ ,  ${}_R Q$  and  $Q_S$  is  $Q$ -reflexive. However, for a dual-bimodule by (1.4) and (2.7) we have

**THEOREM 2.8.** *Let  $Q$  be an  $(R, S)$ -bimodule such that both  ${}_R Q$  and  $Q_S$  are finitely generated. Then the following conditions are equivalent:*

- (1)  $Q$  is a dual-bimodule.

(2)  ${}_R R$  and  $S_S$  are  $Q$ -reflexive and every factor module of  ${}_R R$ ,  $S_S$ ,  ${}_R Q$  and  $Q_S$  is  $Q$ -torsionless.

LEMMA 2.9. *Let  $Q$  be a dual-bimodule with  $\lambda$  surjective. Assume that  $\text{rad}({}_R Q) \leq_s Q_S$ . Then every idempotent of  $R$  can be lifted modulo  $\text{rad}(R)$ .*

PROOF. Cf. [4, Theorem 3.8].

Thus, we have

THEOREM 2.10. *Let  $Q$  be a dual-bimodule with  $Q_S$  finitely generated and  $\text{rad}({}_R Q) \leq \text{rad}(Q_S)$ . Then  $R$  is semiperfect.*

As we shall show in Example 4.1, there is a dual-bimodule  $Q$  for which  $R$  is semiperfect, but  $Q_S$  is not finitely generated.

### 3. Dualities.

For a left dual-bimodule  $Q$ , it is shown in (1.2) every cyclic  $R$ -module is  $Q$ -torsionless. The following theorem gives a criterion for every cyclic  $R$ -module being  $Q$ -reflexive. First, we need a lemma.

LEMMA 3.1 (cf. [3, Proposition 1.1]). *Let  ${}_R Q_S$  be an  $(R, S)$ -bimodule and  $N_S$  an  $S$ -module such that  $Q_S$  is  $N$ -injective and  $N_S$  is  $Q$ -reflexive. Then for  $K \leq N_S$ ,  $N/K$  is  $Q$ -torsionless if and only if  $K$  is  $Q$ -reflexive.*

PROOF. Let  $Q_S$  be  $N$ -injective and  $K \leq N_S$ . Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & N/K \longrightarrow 0 \\ & & \sigma_K \downarrow & & \downarrow \sigma_N & & \downarrow \sigma_{N/K} \\ 0 & \longrightarrow & K^{**} & \longrightarrow & N^{**} & \longrightarrow & (N/K)^{**} \end{array}$$

where  $\sigma_*$  means the evaluation map. Assume that  $N_S$  is  $Q$ -reflexive. Then by [1, Lemma 3.14] we see that  $\sigma_{N/K}$  is monic if and only if  $\sigma_K$  is epic and this is so if and only if  $\sigma_K$  is an isomorphism, since  $K$  is  $Q$ -torsionless as a submodule of  $N_S$ .

THEOREM 3.2. *Let  $Q$  be a left dual-bimodule. Then the following conditions are equivalent:*

- (1)  $Q_S$  is quasi-injective and  $\lambda$  is surjective.

- (2) Every cyclic  $R$ -module is  $Q$ -reflexive.
- (3) Every finitely generated  $Q$ -torsionless  $R$ -module is  $Q$ -reflexive.

Moreover, if each one of these conditions holds, then  $R$  is semiperfect and every submodule of  $Q_S$  is finitely cogenerated  $Q$ -reflexive.

PROOF. (1) $\Rightarrow$ (3). Let  ${}_R M$  be a finitely generated  $Q$ -torsionless  $R$ -module. Then  $R^n \rightarrow M \rightarrow 0$  is exact for some  $n > 0$ . Since  $(R^n)^* \cong Q^n$  and  $Q$  is  $Q^n$ -injective, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
 R^n & \longrightarrow & M & \longrightarrow & 0 & & \\
 \sigma_{R^n} \downarrow & & \downarrow \sigma_M & & & & \\
 (R^n)^{**} & \longrightarrow & M^{**} & \longrightarrow & 0 & & 
 \end{array}$$

Since  $\lambda$  is surjective,  $\sigma_{R^n}$  is an epimorphism and hence  $\sigma_M$  is also an epimorphism. Thus,  $M$  is  $Q$ -reflexive.

(3) $\Rightarrow$ (2). This is evident by (1.2).

(2) $\Rightarrow$ (1). For any  $A \leqslant_R R$  the mapping  $\lambda_A : R \rightarrow \imath_Q(A)^*$  given by  $a \rightarrow a_L | \imath_{Q(A)}$  is an  $R$ -homomorphism. With the canonical  $S$ -isomorphism  $h : (R/A)^* \rightarrow \imath_Q(A)$ ,  $\lambda_A$  yields a commutative diagram

$$\begin{array}{ccc}
 R & \longrightarrow & {}^\pi R/A \\
 \lambda_A \downarrow & & \downarrow \sigma_{R/A} \\
 \imath_Q(A)^* & \longrightarrow & {}_h (R/A)^{**}
 \end{array}$$

Since  $\sigma_{R/A}$  is an epimorphism, so is  $\lambda_A$ . Therefore, for every  $S$ -homomorphism  $f : \imath_Q(A) \rightarrow Q$ , there exists an  $a \in R$  such that  $f = a_L | \imath_{Q(A)}$ . Thus,  $Q_S$  is quasi-injective. In particular, if we take  $A=0$ , then we see that  $\lambda$  is surjective.

As was pointed out in [3, p. 120], if  $Q_S$  is quasi-injective, then  $\text{End}(Q_S)$  is semiperfect if and only if  $Q_S$  has finite Goldie dimension. Hence, the last part of the theorem follows from (1.3), (1.7) and (3.1).

As is seen from (2.4) and (2.7), if  $Q$  is a dual-bimodule and  $Q_S$  is Noetherian, then  $Q$  satisfies the equivalent condition of the preceding theorem.

It is also to be noted that the equivalence in the preceding theorem is closely related to the assumption that  $Q$  is a left dual-bimodule and without this assumption we can not prove (3) $\Rightarrow$ (1). See Example 4.6.

We shall give another criterion for every cyclic  $R$ -module being  $Q$ -reflexive. To do this, for an  $(R, S)$ -bimodule  ${}_R Q_S$ , consider the full subcategory  $\underline{M}$  of  $R$ -mod of finitely generated  $Q$ -torsionless  $R$ -modules and the full subcategory  $\underline{N}$



of  $\text{mod-}S$  whose objects are all the  $S$ -modules  $N$  such that there exists an exact sequence of the form  $0 \rightarrow N \rightarrow Q^n \rightarrow Q^I$  for some  $n > 0$  and a set  $I$ .

**THEOREM 3.3.** *For an  $(R, S)$ -bimodule  $Q$ , consider the following conditions:*

- (1)  $Q_S$  is quasi-injective and  $\lambda$  is surjective.
- (2) The pair  $(H', H'')$  of functors

$$H' = \text{Hom}_R(-, Q): \underline{M} \longrightarrow \underline{N} \quad \text{and} \quad H'' = \text{Hom}_S(-, Q): \underline{N} \longrightarrow \underline{M}$$

defines a duality between  $\underline{M}$  and  $\underline{N}$ .

Then (1) implies (2). If  $Q$  is a left dual-bimodule with  ${}_R Q$  finitely generated, then (2) implies (1) and in this case (1) and (2) are equivalent.

**PROOF.** (1) $\Rightarrow$ (2) (cf. [3, Proposition 1.3]). First, we note that from the proof of (1) $\Rightarrow$ (3) of (3.2) each  ${}_R M \in \underline{M}$  is  $Q$ -reflexive.

Next we show that  $M^* \in \underline{N}$  for every  ${}_R M \in \underline{M}$ . Since  $M$  is finitely generated,  $R^n \rightarrow M \rightarrow 0$  is exact for some  $n > 0$ . Hence  $0 \rightarrow M^* \rightarrow {}^a Q^n$  is exact. We may show that  $Q^n / \alpha(M^*)$  is  $Q$ -torsionless. Since  $\lambda$  is surjective,  $\sigma_R$  is an epimorphism and hence  $R^*$  is  $Q$ -reflexive. Therefore,  $Q$  is  $Q$ -reflexive and so is  $Q^n$ . Applying (3.1) to  $Q_S$  and  $Q_S^n$ , we see that  $Q^n / \alpha(M^*)$  is  $Q$ -torsionless, since  $M^*$  is  $Q$ -reflexive.

Now we show that  $N_S \in \underline{N}$  implies  $N^* \in \underline{M}$ . Let  $0 \rightarrow N \rightarrow Q^n \rightarrow Q^I$  be exact for some  $n > 0$  and  $I$ . Since  $Q_S$  is  $Q^n$ -injective,  $(Q^n)^* \rightarrow N^* \rightarrow 0$  is exact. Furthermore,  $\lambda$  is surjective and hence  $R^n \rightarrow (Q^*)^n \rightarrow 0$  is exact. Thus,  $R^n \rightarrow N^* \rightarrow 0$  must be exact, from which we see that  $N^*$  is finitely generated. By [1, Proposition 20.14]  $N^*$  is  $Q$ -torsionless.

Finally we see that  $N$  is  $Q$ -reflexive for  $N_S \in \underline{N}$ , applying (3.1) again.

(2) $\Rightarrow$ (1). This follows from a similar way as in the proof of [1, Theorem 23.5]. Note that, by the assumption that  ${}_R Q$  is finitely generated, we may use [1, Exercise 20.5].

As is shown above, the quasi-injectivity of  $Q_S$  implies a duality between  $\underline{M}$  and  $\underline{N}$ . The converse, however, is not the case without the assumption that  $Q$  is a left dual-bimodule. See Example 4.6.

Now let  ${}_R Q_S$  be an  $(R, S)$ -bimodule and let  $\underline{M}$  and  $\underline{N}$  be as above. Assume that  $Q_S$  is quasi-injective and  $\lambda$  is surjective. Then as is remarked in the proof of (3.3),  $\underline{M}$  is the full subcategory of finitely generated  $Q$ -reflexive  $R$ -modules. On the other hand, if we assume further that  $Q_S$  is finitely cogenerated, then  $\underline{N}$  becomes the full subcategory of  $\text{mod-}S$  of finitely cogenerated  $Q$ -reflexive  $S$ -

modules.

PROPOSITION 3.4. *Let  $Q$  be an  $(R, S)$ -bimodule such that  $Q_S$  is quasi-injective and  $\lambda$  is surjective. Assume further that  $Q_S$  is finitely cogenerated. Then*

$$\underline{M} = \{ {}_R M \mid M \text{ is finitely generated and } Q\text{-reflexive} \},$$

and

$$\underline{N} = \{ N_S \mid N \text{ is finitely cogenerated and } Q\text{-reflexive} \}.$$

PROOF. It is clear that each  $N_S \in \underline{N}$  is finitely cogenerated and  $Q$ -reflexive.

Conversely, suppose that  $N_S$  is finitely cogenerated  $Q$ -reflexive. Then there exists an  $n > 0$  for which  $0 \rightarrow N_S \rightarrow {}^\alpha Q^n$  is exact. By (3.1)  $Q^n/\alpha(N)$  is  $Q$ -torsionless and thus  $0 \rightarrow N_S \rightarrow {}^\alpha Q^n \rightarrow Q^I$  is exact for some set  $I$ .

For a dual-bimodule  $Q$ , by [6, Proposition 2.8] and [7, Lemma 4], we have

LEMMA 3.5. *For a dual-bimodule  $Q$  with  $\lambda$  surjective, the following conditions are equivalent:*

- (1)  $Q_S$  is injective.
- (2)  $Q_S$  is a cogenerator,
- (3)  $E(Q_S)$  is  $Q$ -torsionless.

Let  ${}_R FG$  and  ${}_R FC$  be the full subcategory of finitely generated and finitely cogenerated left  $R$ -modules, respectively. We shall use similar notations for right  $S$ -modules.

THEOREM 3.6. *For a dual-bimodule  $Q$  with  ${}_R Q$  finitely generated, the following conditions are equivalent:*

- (1)  $Q_S$  is injective and  $\lambda$  is surjective.
- (2)  $(H', H'')$  defines a duality between  $\underline{M}$  and  $FC_S$ .

PROOF. (1) $\Rightarrow$ (2) follows from [1, Exercise 10.3], (3.3), (3.4) and (3.5).

(2) $\Rightarrow$ (1). As is seen from the proof of (3.3) each  ${}_R M \in \underline{M}$  is  $Q$ -reflexive and hence by (3.2)  $\lambda$  is surjective. On the other hand, since  $Q_S \in FC_S$ ,  $E(Q_S) \in FC_S$ . Therefore,  $E(Q_S) \cong M^*$  for some  $M \in \underline{M}$  and thus  $E(Q_S)$  is  $Q$ -torsionless. This shows that  $Q_S$  is injective by (3.5).

Let  ${}_R Q_S$  be an  $(R, S)$ -bimodule. Then by [1, Theorem 24.1]  $Q$  defines a Morita duality if and only if  $Q$  is a balanced bimodule such that  ${}_R Q$  and  $Q_S$  are injective cogenerators. Hence, as a consequence of (3.6), we obtain

**THEOREM 3.7.** *Let  $Q$  be a dual-bimodule with  ${}_R Q$  and  $Q_S$  finitely generated. Then the following conditions are equivalent:*

- (1)  $Q$  defines a Morita duality.
- (2)  $Q$  is a balanced bimodule such that  ${}_R Q$  and  $Q_S$  are injective.
- (3)  $(H', H'')$  defines a duality between  ${}_R FG$  and  $FC_S$  and one between  ${}_R FC$  and  $FG_S$ .

**THEOREM 3.8.** *Let  $R$  and  $S$  be rings. Then the following conditions are equivalent:*

- (1) There exists a duality between  ${}_R FG$  and  $FG_S$ .
- (2) There exists a dual-bimodule  ${}_R Q_S$  such that  ${}_R R$  is Artinian and  ${}_R Q$  is finitely generated.
- (3) There exists a dual-bimodule  ${}_R Q_S$  such that  $S_S$  is Artinian and  $Q_S$  is finitely generated.

Moreover, if this is the case, a left  $R$ -(right  $S$ )-module is  $Q$ -reflexive if and only if it is finitely generated if and only if it is finitely cogenerated.

**PROOF.** (1) $\Rightarrow$ (2) follows from [1, Theorem 24.8].

(2) $\Rightarrow$ (1). Assume (2). Then  $Q_S$  is Noetherian and is finitely generated. Hence, by (1.14) and its right-hand version, both  ${}_R Q$  and  $Q_S$  are injective and, by (2.7) and its right-hand version,  $Q$  is a balanced bimodule. Therefore,  $Q$  defines a Morita duality by (3.7). Thus, again by [1, Theorem 24.8] there exists a duality between  ${}_R FG$  and  $FC_S$ .

Similarly we can prove the equivalence of (1) and (3). The rest of the theorem also follows from [1, Theorem 24.8].

#### 4. Examples.

**EXAMPLE 4.1.** Let  $p$  be a prime number and  $R = Z_{(p)} = \{a/b \in \mathbb{Q} \mid (a, b) = 1 \text{ and } p \nmid b\}$ , where  $\mathbb{Q}$  is the field of rational numbers. Then  $R$  is a commutative local ring with the unique maximal ideal  $Rp$  and nonzero proper ideals of  $R$  are exhausted by  $Rp^n$ ,  $n > 0$ . The quotient field of  $R$  is  $\mathbb{Q}$ .

Now let  $Q = \mathbb{Q}/R$ . Then  $Q$  is an  $(R, R)$ -bimodule and the only nonzero proper submodules of  $Q_R$  are those of the form  $p^{-n}R/R$  for some  $n > 0$ . Furthermore we have

- (1)  ${}_R Q_R$  is a dual-bimodule, since for each  $n > 0$ ,  $r_Q(Rp^n) = p^{-n}R/R$  and  $l_R(p^{-n}R/R) = Rp^n$ .
- (2)  $Q_R$  is an injective cogenerator. However, it is not finitely generated.
- (3)  ${}_R Q_R$  can not define a Morita duality. Indeed, as was pointed out in [4,

Example 6.1],  $\lambda$  is not surjective and hence  ${}_R R$  is not  $Q$ -reflexive. However, by (2.5), each  $R/Rp^n$  is  $Q$ -reflexive. The class of  $Q$ -reflexive  $R$ -modules is closed under extensions. Hence each  $Rp^n$  can not be  $Q$ -reflexive. This shows that every factor module ( $\neq R$ ) of  ${}_R R$  is  $Q$ -reflexive, but there is no nonzero left ideal of  $R$  which is  $Q$ -reflexive.

(4)  $Q_R$  is not  $Q$ -reflexive. Indeed, if  $Q_R$  is  $Q$ -reflexive, then so is  $Q^*$ . Hence, the exactness of the sequence  $0 \rightarrow R \rightarrow {}^\lambda Q^*$  implies that  ${}_R R$  must be  $Q$ -reflexive, a contradiction.

EXAMPLE 4.2. Using the same notations as above, let  $Q' = p^{-n}R/R$  and  $\bar{R} = R/Rp^n$ . Then  ${}_R Q'_R$  is a left dual-bimodule by (1.15), but not a right dual-bimodule. Indeed there is no lattice isomorphism between the submodule lattices of  $R_R$  and  ${}_R Q'$ .

EXAMPLE 4.3. Using the same notations as above,  ${}_R Q'_R$  can be regarded as a dual-bimodule again by (1.15).  ${}_R Q'_R$  defines a Morita duality, since  $\bar{R}$  is an Artinian ring.

EXAMPLE 4.4. Let  $Q = Q/\mathbb{Z}$ , where  $\mathbb{Z}$  is the ring of integers. Then  $Q$  is a  $(\mathbb{Z}, \mathbb{Z})$ -bimodule and every factor module of  ${}_{\mathbb{Z}}\mathbb{Z}$  and  $Q_{\mathbb{Z}}$  is  $Q$ -torsionless, since  $Q$  is a cogenerator over  $\mathbb{Z}$ . However,  $\lambda$  is not surjective and  $Q$  is not a left dual-bimodule by (1.10).

EXAMPLE 4.5. Let  $R$  be a simple Artinian ring and take  ${}_R Q = R^N$ , where  $N$  denotes the set of positive integers. Then  ${}_R Q$  is not finitely cogenerated and hence by (1.16)  ${}_R Q_S$  with  $S = \text{End}({}_R Q)$  is a left dual-bimodule but not a right dual-bimodule by (1.8).

EXAMPLE 4.6. Let  $R$  be the ring of  $2 \times 2$  upper triangular matrices over a field and let  $Q = {}_R R_R$ . Then

- (1)  $Q$  is not a left dual-bimodule, since  $\text{soc}({}_R Q) \neq \text{soc}(Q_R)$ .
- (2) Every finitely generated  $Q$ -torsionless left  $R$ -module is  $Q$ -reflexive, since  $R$  is left and right Artinian and hereditary and every  $Q$ -torsionless left  $R$ -module is projective.
- (3)  $Q_R$  is not (quasi-)injective.
- (4)  $\underline{M} = \{ {}_R M \mid M \text{ is finitely generated projective} \}$ .
- (5)  $\underline{N} = \{ N_R \mid N \text{ is finitely generated projective} \}$ .

It is clear that each  $N_R \in \underline{N}$  is finitely generated projective. Conversely, let  $N_R$  be a finitely generated projective  $R$ -module. Then  $R^m \rightarrow N_R \rightarrow 0$  is split exact

for some  $m > 0$ . Hence  $0 \rightarrow N_R \rightarrow {}^\alpha R^m$  is also split exact. Thus,  $R^m/\alpha(N)$  is finitely generated projective and is finitely cogenerated  $Q$ -reflexive. There exists an  $k > 0$  such that  $0 \rightarrow N_R \rightarrow {}^\alpha R^m \rightarrow R^k$  is exact.

(6) Though  $Q_R$  is not quasi-injective, the pair  $(H', H'')$  defines a duality between  $\underline{M}$  and  $\underline{N}$ , as is well-known.

### Acknowledgement

The authors wish to thank to the referee for his valuable advices.

### References

- [1] F.W. Anderson and K.R. Fuller, Rings and Categories of Modules, Springer-Verlag, New York Heidelberg Berlin. 1973.
- [2] G. Azumaya, A duality theory for injective modules, Amer. J. Math. **81** (1959), 249-278.
- [3] J.L. Garcia Hernandez and J.L. Gomez Pardo, Closed submodules of free modules over the endomorphism ring of a quasi-injective module, Comm. Algebra **16** (1988), 115-137.
- [4] C.R. Hajarnavis and N.C. Norton, On dual rings and their modules, J. Algebra **93** (1985), 253-266.
- [5] Y. Kurata, Some remarks on quasi-Frobenius modules, Osaka Math. J. **10** (1958), 213-220.
- [6] Y. Kurata and H. Katayama, On a generalization of QF-3' rings, Osaka J. Math. **13** (1976). 407-418.
- [7] T. Onodera, Koendlich erzeugte Moduln und Kogeneratoren, Hokkaido Math. J. **2** (1973), 69-83.

Department of Mathematics  
Yamaguchi University  
Yoshida, Yamaguchi 753