ORTHOCOMPACTNESS OF INVERSE LIMITS
AND PRODUCTS

By

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Introduction. A topological space is said to be orthocompact if every open cover has an interior-preserving open refinement. B. M. Scott and H. J. K. Junnila have investigated the finite product theory for orthocompactness and have shown in [4] and [11] that several known theorems concerning normality of product spaces with compact or metric factors remain true if one replaces "paracompact" by "metacompact" and "normal" by "orthocompact" in the theorems. Scott has also proved in [12] that a finite product of locally compact linearly ordered topological space is orthocompact if and only if it is normal. In this paper we study the behavior of orthocompactness of inverse limits and infinitely many procuct spaces.

In Section 1 we present some definitions and preliminary lemmas which are used in later sections. Inverse systems are considered in Section 2. Let $\{X_\alpha, f_{\alpha\beta}\}$ be an inverse system over a directed set A and let X be the inverse limit of this system. For each $\alpha \in A$, let $f_\alpha : X \to X_\alpha$ be the projection map. First we deal with the case that X is $|A|$-paracompact and all f_α's are pseudo-open maps. Next we deal with the case that X is $|A|$-metacompact and all f_α's are closed maps. In both cases we will prove that X is orthocompact if all X_α's are orthocompact. The first case is used in Section 3 to investigate the orthocompactness of product spaces. Metacompactness as well as normality and paracompactness of the limit space X is also considered in this section.

In Section 3 we will consider product spaces. A. H. Stone [14] has proved that the product of uncountably many copies of the countable discrete space is not normal. We will show that this space is not orthocompact. The main theorem of this section is that a product of arbitrarily many ordinals is orthocompact if and only if it is normal. This is a partial generalization of Scott's result [12] which is stated above.

The last section contains some examples.

1. Preliminaries.

In this paper n, λ and κ will denote cardinal numbers and ω the first infinite ordinal. As the usual convention, an ordinal is the set of smaller ordinals and a cardinal is an initial ordinal. Whenever we regard an ordinal as a topological space, it will be assumed to have the order topology. The space κ^λ is the Tychonoff product of λ copies of the space κ. The *cofinality* of an ordinal α will be denoted by $cf(\alpha)$, and the cardinality of a set A will be denoted by $|A|$.

Throughout this paper, no separation axioms are assumed unless otherwise stated, and all maps are continuous. For a subset A of a space X, cl A (resp. int A) denotes the closure (resp. interior) of A in X. The word "iff" reads "if and only if".

DEFINITION 1.1 (Junnila [5]). Let $\mathcal{A}$ be a family of subsets of a space X. The family $\mathcal{A}$ is said to be *interior-preserving* (resp. *closure-preserving*) if we have $\text{int}(\cap\{A\,|\,A\in\mathcal{A}'\}=\cap\{\text{int}\,A\,|\,A\in\mathcal{A}'\}$ (resp. $\text{cl}(\cup\{A\,|\,A\in\mathcal{A}'\}=\cup\{\text{cl}\,A\,|\,A\in\mathcal{A}'\})$ for every subfamily $\mathcal{A}'$ of $\mathcal{A}$.

DEFINITION 1.2. A space X is λ-*orthocompact* (λ-*metacompact*, λ-*paracompact*, resp.) if every open cover of X of cardinality $\leq\lambda$ has an interior-preserving (point-finite, locally finite, resp.) open refinement. (In this paper, a refinement always means a cover.)

Of course, a space is an orthocompact (metacompact, paracompact, resp.) space iff it is λ-orthocompact (λ-metacompact, λ-paracompact, resp.) for every λ.

In the case that we deal with λ-orthocompact spaces, λ-metacompact spaces, etc., the following lemma is very useful, and we will use it without referring explicitly. The proof of it is clear.

LEMMA 1.3. *A space X is λ-orthocompact (λ-metacompact, λ-paracompact, resp.) iff for every open cover $\mathcal{U}=\{U_\xi\,|\,\xi\in\varXi\}$ of X with $|\varXi|\leq\lambda$, there is an interior-preserving (point-finite, locally finite, resp.) open cover $\mathcal{V}=\{V_\xi\,|\,\xi\in\varXi\}$ of X such that $V_\xi\subseteqq U_\xi$ for each $\xi\in\varXi$.*

DEFINITION 1.4 (Scott [11]). A space X is $\sigma(\lambda)$-*orthocompact* ($\sigma(\lambda)$-*metacompact*, $\sigma(\lambda)$-*paracompact*, resp.) if every open cover of X has a refinement of type $\cup\{\mathcal{V}_\gamma\,|\,\gamma\in\varGamma\}$ such that $|\varGamma|\leq\lambda$ and $\mathcal{V}_\gamma$ is an interior-preserving (point-finite, locally finite, resp.) family of open subsets of X for every $\gamma\in\varGamma$.

DEFINITION 1.5. A space X is $\sigma(\lambda)$-*normal* if for every open cover $\{U_1, U_2\}$ of X, there is an open cover $\{V_{i\gamma}\,|\,\gamma\in\varGamma,\ i=1, 2\}$ of X such that $|\varGamma|\leq\lambda$ and

cl $V_{i\gamma} \subseteq U_i$ for every $\gamma \in \Gamma$ and $i=1, 2$.

PROPOSITION 1.6 (Scott [11]).   *Let X be a λ-metacompact and $\sigma(\lambda)$-orthocompact (resp. $\sigma(\lambda)$-metacompact) space, then X is orthocompact (resp. metacompact).*

With a similar method of the proof of Proposition 1.6, we have the following proposition.

PROPOSITION 1.7.   *Let X be a λ-paracompact and $\sigma(\lambda)$-normal (resp. $\sigma(\lambda)$-paracompact) space, then X is normal (resp. paracompact).*

DEFINITION 1.8 ([3]).   A space X is *λ-bounded* if for each subset $A \subseteq X$ with $|A| \leq \lambda$, there is a compact set $C \subseteq X$ such that $A \subseteq C$.

PROPOSITION 1.9 ([3]).   *Every λ-bounded space is λ-compact.*

At the present, a space X is *λ-compact* iff each open cover of X of cardinality $\leq \lambda$ has a finite subcover.

PROPOSITION 1.10 ([3]).   *λ-boundness is productive; that is, the product of arbitrarily many λ-bounded spaces is also λ-bounded.*

PROPOSITION 1.11.   *Let α be an ordinal (with the order topology), and let λ be an infinite cardinal.   Then the following are equivalent.*
    (1) *α is λ-bounded.*
    (2) *α is λ-compact.*
    (3) *$cf(\alpha) > \lambda$ or $cf(\alpha) \leq 1$.*

The proof of Proposition 1.11 is obvious.   More generally, the equivalence of (1) and (2) for a linearly ordered topological space is proved in [3].

DEFINITION 1.12.   A map $f: X \to Y$ is called *pseudo-open* if for each point $y \in Y$ and a neighborhood U of $f^{-1}(y)$, $f(U)$ is a neighborhood of y.

It is clear that pseudo-open maps are onto maps and both open onto maps and closed onto maps are pseudo-open.

2. Inverse limits.

In this section we consider inverse systems and orthocompactness as well as metacompactness, normality and paracompactness of their limits.   There are two cases that we consider here.   One is an inverse system with pseudo-open pro-

jections from its limit space and the other is an inverse system with closed pro-
jections from its limit space.

First of all, the following propositions are useful to investigate inverse
systems over a directed set A whose limit space are $|A|$-paracompact or $|A|$-
metacompact. The first proposition is due to Mack [7], and the second is due
to Junnila [5]. Recall that a cover is *directed* if it is directed by set inclusion.

PROPOSITION 2.1. *Let λ be an infinite cardinal. A space X is λ-paracompact
iff for each directed open cover $\mathcal{U}$ of X with $|\mathcal{U}|\leq\lambda$, there exists a locally finite
open cover $\mathcal{V}$ of X such that $\{\mathrm{cl}\, V\,|\, V\in\mathcal{V}\}$ refines $\mathcal{U}$.*

PROPOSITION 2.2. *Let $\mathcal{U}$ be an interior-preserving (especially, point-finite)
open cover of a space X. Then there exists a closure-preserving closed cover
$\mathcal{E}=\{E(x)\,|\,x\in X\}$ of X such that $x\in E(x)\subseteq\mathrm{St}\,(x,\,\mathcal{U})$ for every $x\in X$, where
$\mathrm{St}\,(x,\,\mathcal{U})=\cup\{U\in\mathcal{U}\,|\,x\in U\}$.*

Now we state and prove our theorems.

LEMMA 2.3. *Let $\{X_\alpha,\, f_{\alpha\beta}\}$ be an inverse system over a directed set A, and
let X be the inverse limit of the system. Assume that all projections $f_\alpha: X\to X_\alpha$
are pseudo-open maps and X is $|A|$-paracompact, then we have the following.*
  (1) *If all X_α's are orthocompact spaces, then X is $\sigma(|A|)$-orthocompact.*
  (2) *If all X_α's are metacompact spaces, then X is $\sigma(|A|)$-metacompact.*
  (3) *If all X_α's are normal spaces, then X is $\sigma(|A|)$-normal.*
  (4) *If all X_α's are paracompact spaces, then X is $\sigma(|A|)$-paracompact.*

PROOF. If A is finite, then the theorem is obvious. Hence we can assume
that A is infinite. Let $|A|=\lambda$.

First of all, for each open subset U of X and $\alpha\in A$, let $G_\alpha(U)$ denote the
largest open subset of X_α such that $f_\alpha^{-1}(G_\alpha(U))\subseteq U$. Then it is easy to see that
  (a) $G_\alpha(U)\subseteq G_\alpha(V)$ whenever U and V are open subsets of X such that
$U\subseteq V$.

If $\alpha\leq\beta$, then $f_\beta^{-1}(f_{\alpha\beta}^{-1}(G_\alpha(U)))=(f_{\alpha\beta}f_\beta)^{-1}(G_\alpha(U))=f_\alpha^{-1}(G_\alpha(U))\subseteq U$, and hence
$f_{\alpha\beta}^{-1}(G_\alpha(U))\subseteq G_\beta(U)$ and $f_\alpha^{-1}(G_\alpha(U))\subseteq f_\beta^{-1}(G_\beta(U))$. Moreover, for each $x\in U$, we
can take $\alpha\in A$ and an open subset V of X_α such that $x\in f_\alpha^{-1}(V)\subseteq U$. Then
$V\subseteq G_\alpha(U)$, and hence $x\in f_\alpha^{-1}(G_\alpha(U))$. Thus, for every open subset U of X, we
have the following.
  (b) If $\alpha,\,\beta\in A$ and $\alpha\leq\beta$, then $f_\alpha^{-1}(G_\alpha(U))\subseteq f_\beta^{-1}(G_\beta(U))$.
  (c) $\cup\{f_\alpha^{-1}(G_\alpha(U))\,|\,\alpha\in A\}=U$.

Now we prove (1)—(4) simultaneously.

Let $\mathcal{U}=\{U_\xi|\xi\in\varXi\}$ be an open cover of X. (In the case of (3), assume that $|\varXi|=2$.) For each $\alpha\in A$, let $V_\alpha=\cup\{G_\alpha(U_\xi)|\xi\in\varXi\}$. They by virtue of (b) and (c), $\mathcal{V}=\{f_\alpha^{-1}(V_\alpha)|\alpha\in A\}$ is a directed open cover of X. Since X is λ-paracompact it follows from Proposition 2.1 that there is a locally finite open cover $\mathcal{W}$ of X such that $\{\mathrm{cl}\,W|W\in\mathcal{W}\}$ refines $\mathcal{V}$. For each $\alpha\in A$, let $W_\alpha=\cup\{W\in\mathcal{W}|\,\mathrm{cl}\,W\subseteq f_\alpha^{-1}(V_\alpha)\}$. Since $\mathcal{W}$ is as above, it is easy to see the following.

(d) W_α is an open subset of X such that $\mathrm{cl}\,W_\alpha\subseteq f_\alpha^{-1}(V_\alpha)$.

(e) If α, $\beta\in A$ and $\alpha\leq\beta$, then $W_\alpha\subseteq W_\beta$.

(f) $\cup\{W_\alpha|\alpha\in A\}=X$.

For each $x\in X$, we can take $\alpha\in A$ such that $x\in W_\alpha$. By virtue of (c), $W_\alpha=\cup\{f_\beta^{-1}(G_\beta(W_\alpha))|\,\beta\in A\}$. Hence $x\in f_\beta^{-1}(G_\beta(W_\alpha))$ for some $\beta\in A$. Let $\gamma\in A$, $\gamma>\alpha$, β. Then by virtue of (a), (b) and (e), $x\in f_\beta^{-1}(G_\beta(W_\alpha))\subseteq f_\gamma^{-1}(G_\gamma(W_\alpha))\subseteq f_\gamma^{-1}(G_\gamma(W_\gamma))$. Thus we have

(g) $\cup\{f_\alpha^{-1}(G_\alpha(W_\alpha))|\alpha\in A\}=X$.

Moreover we can see that

(h) $\mathrm{cl}\,G_\alpha(W_\alpha)\subseteq V_\alpha$ for every $\alpha\in A$.

In fact, for each $y\in X_\alpha\backslash V_\alpha$, $f_\alpha^{-1}(y)\subseteq f_\alpha^{-1}(X_\alpha\backslash V_\alpha)=X\backslash f_\alpha^{-1}(V_\alpha)\subseteq X\backslash\mathrm{cl}\,W_\alpha$. Since f_α is pseudo-open, $f_\alpha(X\backslash\mathrm{cl}\,W_\alpha)$ is a neighborhood of y. It is easy to see that $f_\alpha(X\backslash\mathrm{cl}\,W_\alpha)\cap G_\alpha(W_\alpha)=\phi$. Hence $y\in X_\alpha\backslash\mathrm{cl}\,G_\alpha(W_\alpha)$, and we have $\mathrm{cl}\,G_\alpha(W_\alpha)\subseteq V_\alpha$.

In the case of (1), (2) or (4), by virtue of (h), there is an interior-preserving (point-finite, locally finite, resp.) family $\{W_{\alpha\xi}|\xi\in\varXi\}$ of open subsets of X_α such that $\mathrm{cl}\,G_\alpha(W_\alpha)\subseteq\cup\{W_{\alpha\xi}|\xi\in\varXi\}$ and $W_{\alpha\xi}\subseteq G_\alpha(U_\xi)$ for every $\xi\in\varXi$, and every $\alpha\in A$. Then the following are easily verified.

(i) $\{f_\alpha^{-1}(W_{\alpha\xi})|\xi\in\varXi\}$ is an interior-preserving (point-finite, locally finite, resp.) family of open subsets of X for each $\alpha\in A$.

(j) $f_\alpha^{-1}(W_{\alpha\xi})\subseteq U_\xi$ for every $\xi\in\varXi$ and $\alpha\in A$.

(k) $\cup\{f_\alpha^{-1}(W_{\alpha\xi})|\xi\in\varXi,\ \alpha\in A\}=X$.

Checking up conditions in Definition 1.4 we have shown that X is a $\sigma(\lambda)$-orthocompact ($\sigma(\lambda)$-metacompact, $\sigma(\lambda)$-paracompact, resp.) space, by virtue of (i), (j) and (k).

In the case of (3), let $\varXi=\{\xi_1,\ \xi_2\}$. For each $\alpha\in A$, since X_α is normal, we can find open subsets $W_{\alpha\xi_1}$ and $W_{\alpha\xi_2}$ of X_α such that $\mathrm{cl}\,G_\alpha(W_\alpha)\backslash G_\alpha(U_{\xi_2})\subseteq W_{\alpha\xi_1}\subseteq\mathrm{cl}\,W_{\alpha\xi_1}\subseteq G_\alpha(U_{\xi_1})$ and $\mathrm{cl}\,G_\alpha(W_\alpha)\backslash W_{\alpha\xi_1}\subseteq W_{\alpha\xi_2}\subseteq\mathrm{cl}\,W_{\alpha\xi_2}\subseteq G_\alpha(U_{\xi_2})$. Then it is easy to see that $\mathrm{cl}\,G_\alpha(W_\alpha)\subseteq W_{\alpha\xi_1}\cup W_{\alpha\xi_2}$.

Moreover, for each $x\in\mathrm{cl}\,f_\alpha^{-1}(W_{\alpha\xi_1})$, $f_\alpha(x)\in f_\alpha(\mathrm{cl}\,f_\alpha^{-1}(W_{\alpha\xi_1}))\subseteq\mathrm{cl}\,(f_\alpha(f_\alpha^{-1}(W_{\alpha\xi_1})))\subseteq\mathrm{cl}\,W_{\alpha\xi_1}\subseteq G_\alpha(U_{\xi_1})$, and hence we have $x\in f_\alpha^{-1}(G_\alpha(U_{\xi_1})\subseteq U_{\xi_1}$. Thus $\mathrm{cl}\,f_\alpha^{-1}(W_{\alpha\xi_1})\subseteq U_{\xi_1}$. Similarly, $\mathrm{cl}\,f_\alpha^{-1}(W_{\alpha\xi_2})\subseteq U_{\xi_2}$. Hence the famiy $\{f_\alpha^{-1}(W_{\alpha\xi_i})|\alpha\in A,\ i=1, 2\}$

satisfies the conditions in Definition 1.5. Thus X is a $\sigma(\lambda)$-normal space. The proof is complete.

From Propositions 1.6, 1.7 and Lemma 2.3, we get

THEOREM 2.4. *Uuder the same conditions as in Lemma 2.3, we have the following.*

(1) *If all X_α's are orthocompact spaces, then so is X.*

(2) *If all X_α's are metacompact spaces, then so is X.*

(3) *If all X_α's are normal spaces, then so is X.*

(4) *If all X_α's are paracompact spaces, then so is X.*

COROLLARY 2.5. *Let $\{X_i | i \in I\}$ be a family of spaces, and let $X = \Pi \{X_i | i \in I\}$ be $|I|$-paracompact. Then X is orthocompact (metacompact, normal, paracompact, resp.) iff for each finite subset I' of I, $\Pi \{X_i | i \in I'\}$ is orthocompact (metacompact, normal, paracompact, resp.).*

PROOF. If I is finite, there is nothing to prove. Hence we can assume that I is infinite. Let A be the set of all finite subsets of I, then A is a directed set by set inclusion. For each $\alpha \in A$, let $Y_\alpha = \Pi \{X_i | i \in \alpha\}$. And for each pair $\alpha, \beta \in A$ with $\alpha \subseteq \beta$, let $f_{\alpha\beta} : Y_\beta \to Y_\alpha$ be the natural projection map, that is, $f_{\alpha\beta}((x_i)_{i \in \beta}) = (x_i)_{i \in \alpha}$ for every $(x_i)_{i \in \beta} \in Y_\beta$. Then it is well-known that $\{Y_\alpha, f_{\alpha\beta}\}$ is an inverse system over A and the limit of this system is homeomorphic to X, moreover the projection maps from the limit space can be viewed as the natural projection maps from X, hence they are open onto, *a fortiori* pseudo-open. Since $|A| = |I|$, we can apply Theorem 2.4 to obtain Corollary 2.5. The proof is complete.

COROLLARY 2.6. *Let $\{X_n, f_{nm}\}$ be an inverse sequence over ω, and let X be the inverse limit of the sequence. Suppose all f_{nm}'s are open onto maps and X is countably paracompact (that is, ω-paracompact). Then the statements of Theorem 2.4 are also true.*

PROOF. If all f_{nm}'s are open onto maps, then all projections $f_n : X \to X_n$ are also open onto maps. Hence by virtue of Theorem 2.4, the proof is complete.

REMARK. The statements (3) and (4) of Corollary 2.6 are proved in Nagami [9].

LEMMA 2.7. *Let $\{X_\alpha, f_{\alpha\beta}\}$ be an inverse system over a directed set A, and*

let X be the inverse limit of the system. Assume that all projections $f_\alpha : X \to X_\alpha$ are closed maps and X is $|A|$-metacompact, then we have the following.

(1) *If all X_α's are orthocompact spaces, then X is $\sigma(|A|)$-orthocompact.*

(2) *If all X_α's are metacompact spaces, then X is $\sigma(|A|)$-metacompact.*

(3) *If all X_α's are normal spaces, then X is $\sigma(|A|)$-normal.*

(4) *If all X_α's are paracompact spaces, then X is $\sigma(|A|)$-paracompact.*

PROOF. If A is finite, then the theorem is obvious. Hence we can assume that A is infinite. Let $|A| = \lambda$. We prove (1)—(4) simultaneously. The proof is quite similar to that of Lemma 2.3, hence we use the same notations that are defined in the proof of Lemma 2.3.

Let $\mathcal{U} = \{U_\xi | \xi \in \varXi\}$ be an open cover of X. (In the case of (3), assume that $|\varXi| = 2$.) Then, as in the proof proof of Lemma 2.3, $\{f_\alpha^{-1}(V_\alpha) | \alpha \in A\}$ is a directed open cover of X, where $V_\alpha = \cup \{G_\alpha(U_\xi) | \xi \in \varXi\}$ for every $\alpha \in A$. Since X is λ-metacompact, there is a point-finite open cover $\mathcal{W} = \{W_\alpha | \alpha \in A\}$ of X such that $W_\alpha \subseteq f_\alpha^{-1}(V_\alpha)$ for every $\alpha \in A$. By virtue of Proposition 2.2, we can find a closure-preserving closed cover $\{E(x) | x \in X\}$ such that

(a) $x \in E(x) \subseteq \mathrm{St}(x, \mathcal{W})$ for every $x \in X$.

Let $\varPhi$ be the set of all finite subsets of A, then it is clear that

(b) $|\varPhi| = \lambda$.

Since A is a directed set, for each $B \in \varPhi$, we can take $\alpha(B) \in A$ such that $\alpha \leq \alpha(B)$ for every $\alpha \in B$. Moreover, for each $B \in \varPhi$, let $E(B)$ denote the union of all $E(x)$ such that $\{\alpha \in A | x \in W_\alpha\} = B$, then

(c) $\{E(B) | B \in \varPhi\}$ is a closed cover of X, and

(d) $E(B) \subseteq f_{\alpha(B)}^{-1}(V_{\alpha(B)})$.

Indeed, (c) is obvious since $\mathcal{W}$ is a point-finite cover of X and by virtue of (a), and (d) is implied by the following relations:

$$E(B) \subseteq \cup \{W_\alpha | \alpha \in B\} \subseteq \{f_\alpha^{-1}(V_\alpha) | \alpha \in B\} \subseteq f_{\alpha(B)}^{-1}(V_{\alpha(B)}).$$

By hypothesis, $f_{\alpha(B)}$ is a closed map. Hence $f_{\alpha(B)}(E(B))$ is closed in $X_{\alpha(B)}$, and $\{G_{\alpha(B)}(U_\xi) | \xi \in \varXi\}$ is a family of open subsets of $X_{\alpha(B)}$ which covers $f_{\alpha(B)}(E(B))$, by virtue of (d). Thus, as in the proof of Lemma 2.3, for each $B \in \varPhi$, there exists an interior-preserving (point-finite, —, locally finite, resp.) family $\{W_{B\xi} | \xi \in \varXi\}$ of open subsets of $X_{\alpha(B)}$ such that $f_{\alpha(B)}(E(B)) \subseteq \cup \{W_{B\xi} | \xi \in \varXi\}$ and $W_{B\xi} \subseteq G_{\alpha(B)}(U_\xi)$ for every $\xi \in \varXi$. (In the above and below, "—" reads "without any property".) Moreover, in the case of (3), we can take $W_{B\xi}$ such that $\mathrm{cl}\, W_{B\xi} \subseteq G_{\alpha(B)}(U_\xi)$. Then it is easy to see the following.

(e) $\{f_{\alpha(B)}^{-1}(W_{B\xi}) | \xi \in \varXi\}$ is an interior-preserving (point-finite, —, locally finite, resp.) family of open subsets of X.

(f) $f_{\alpha(B)}^{-1}(W_{B\xi}) \subseteq U_\xi$ for every $\xi \in \mathcal{Z}$.

(g) $\cup \{ f_{\alpha(B)}^{-1}(W_{B\xi}) \mid \xi \in \mathcal{Z} \} = X$.

Moreover, in the case of (3), we have

(f')   cl $f_{\alpha(B)}^{-1}(W_{B\xi}) \subseteq U_\xi$ for every $\xi \in \mathcal{Z}$.

In fact, since cl $W_{B\xi} \subseteq G_{\alpha(B)}(U_\xi)$, we have $f_{\alpha(B)}(\mathrm{cl}\, f_{\alpha(B)}^{-1}(W_{B\xi})) \subseteq \mathrm{cl}\, (f_{\alpha(B)}(f_{\alpha(B)}^{-1}(W_{B\xi})))$ $\subseteq \mathrm{cl}\, W_{B\xi} \subseteq G_{\alpha(B)}(U_\xi)$.   Hence cl $f_{\alpha(B)}^{-1}(W_{B\xi}) \subseteq f_{\alpha(B)}^{-1}(G_{\alpha(B)}(U_\xi)) \subseteq U_\xi$.

By virtue of (b), (e), (f) and (g) (or (b), (e), (f') and (g)), $\mathcal{U}$ has an appropriate open refinement mentioned in Definition 1.4 (or Definition 1.5).   Hence the proof is complete.

From Propositions 1.6 and 1.7 and Lemma 2.7, we get

COROLLARY 2.8 (Katuta [6]).   *Let* $\{ X_\alpha, f_{\alpha\beta} \}$ *be an inverse system over a directed set* A, *and let* X *be the inverse limit of the system.   Assume that all projections* $f_\alpha : X \to X_\alpha$ *are closed maps and* X *is* $|A|$-*paracompact, then we have the following.*

(a)   *If all* X_α's *are normal spaces, then so is* X.

(b)   *If all* X_α's *are paracompact spaces, then so is* X.

THEOREM 2.9.   *Let* $\{ X_\alpha, f_{\alpha\beta} \}$ *be an inverse system over a directed set* A, *and let* X *be the inverse limit of the system.   Assume that all projections* $f_\alpha : X \to X_\alpha$ *are closed maps and* X *is* $|A|$-*metacompact, then we have the following.*

(a)   *If all* X_α's *are orthocompact spaces, then so is* X.

(b)   *If all* X_α's *are metacompact spaces, then so is* X.

REMARK.   (1) Since normality and paracompactness are closed hereditary properties, in Corollary 2.8, we can assume that all projections are closed onto maps.   Thus Corollary 2.8 is also from Theorem 2.4, since closed onto maps are pseudo-open.

(2)   Note that Theorem 2.9 is of the form which is obtained from Corollary 2.8 by replacing "paracompact" by "metacompact" and "normal" by "orthocompact".

3.   Products.

In this section we investigate the product theory for orthocompactness.

THEOREM 3.1.   *For a space* X, *the following are equivalent.*

(1) X *is* λ-*metacompact.*

(2) $X \times Y$ *is* λ-*metacompact for every compact space* Y *of weight at most* λ.

(3) $X \times Y$ *is* λ-*orthocompact for every compact space* Y *of weight at most* λ.

(4) $X \times I^\lambda$ is λ-orthocompact, where I is the closed unit interval.

(5) $X \times 2^\lambda$ is λ-orthocompact.

(6) $X \times A(\lambda)$ is λ-orthocompact, where $A(\lambda)$ is the space of one-point compactification of the discrete space of cardinality λ.

PROOF. Since the product of a λ-metacompact space with a compact space is also λ-metacompact, the implication (1)$\rightarrow$(2) is obvious. The implications (2)$\rightarrow$(3) $\rightarrow$(4)$\rightarrow$(5) are clear. Since $A(\lambda)$ can be embedded in 2^λ as a closed subset, (5)$\rightarrow$ (6) is also clear.

To prove (6)$\rightarrow$(1), we can assume that $A(\lambda) = \{a_\alpha \mid \alpha \leq \lambda\}$ and a_λ is the only non-isolated point of $A(\lambda)$. Let $\mathcal{U} = \{U_\alpha \mid \alpha < \lambda\}$ be an open cover of X. Then it is easy to see that $\{U_\alpha \times (A(\lambda) \setminus \{a_\alpha\}) \mid \alpha < \lambda\} \cup \{X \times (A(\lambda) \setminus \{a_\lambda\})\}$ is an open cover of $X \times A(\lambda)$. Since $X \times A(\lambda)$ is λ-orthocompact, there is an interior-preserving open cover $\{V_\alpha \mid \alpha \leq \lambda\}$ of $X \times A(\lambda)$ such that $V_\alpha \subseteq U_\alpha \times (A(\lambda) \setminus \{a_\alpha\})$ for each $\alpha < \lambda$ and $V_\lambda \subseteq X \times (A(\lambda) \setminus \{a_\lambda\})$. For each $\alpha < \lambda$, let W_α be the set of all $x \in X$ such that $(x, a_\lambda) \in V_\alpha$. Then we can easily show that $\mathcal{W} = \{W_\alpha \mid \alpha < \lambda\}$ is an open cover of X and $W_\alpha \subseteq U_\alpha$ for every $\alpha < \lambda$. Moreover, since $\{V_\alpha \mid \alpha \leq \lambda\}$ is interior-preserving and $(x, a_\lambda) \in \cap \{W_\alpha \times \{a_\lambda\} \mid x \in W_\alpha, \alpha < \lambda\} \subseteq \cap \{V_\alpha \mid x \in W_\alpha, \alpha < \lambda\} \subseteq \cap \{X \times (A(\lambda) \setminus \{a_\alpha\}) \mid x \in W_\alpha, \alpha < \lambda\} = X \times (A(\lambda) \setminus \{a_\alpha \mid x \in W_\alpha, \alpha < \lambda\})$, $\{\alpha < \lambda \mid x \in W_\alpha\}$ must be finite for every $x \in X$. Hence $\mathcal{W}$ is a point-finite open refinement of $\mathcal{U}$. Thus X is λ-metacompact, and the proof is complete.

REMARK. The equivalence of (1)—(5) in Theorem 3.1 is essentially proved in [11]. Since there is a space X such that $X \times A(\lambda)$ is normal but X is not λ-paracompact (Example 4.3), the equivalence of (1) and (6) itself seems to be of interest.

As an immediate application of Theorem 3.1, we will state two theorems concerning orthocompactness of the product of uncountably many copies of a space, which are compaired to the following propositions concerning normality of the product of uncountably many copies of a space.

Let N denote the space of all positive integers with the discrete topology, and ω_1 the first uncountable ordinal.

PROPOSITION 3.2 (Stone [14]). N^{ω_1} is not normal.

PROPOSITION 3.3 (Noble [10]). For a T_2-space X, the following are equivalent.

(1) X^λ is normal for every λ.

(2) X is compact.

THEOREM 3.4. N^{ω_1} is not orthocompact.

PROOF. Assume that N^{ω_1} is orthocompact. Since $N^{\omega_1} \times N^{\omega_1}$ is homeomorphic to N^{ω_1} and N^{ω_1} has a closed subspace homeomorphic to 2^{ω_1}. Hence it follows from Theorem 3.1 that N^{ω_1} is ω_1-metacompact. Since N^{ω_1} is of weight ω_1, it is metacompact. It is well-known that N^{ω_1} is separable (e. g. [2] P111), and that every separable matacompact space is Lindelöf. Hence N^{ω_1} is a Lindelöf regular space, which is a contradiction since N^{ω_1} is not normal by Proposition 3.2. Thus N^{ω_1} cannot be orthocompact.

THEOREM 3.5. Let X be a T_1-space of weight γ. Then the following are equivalent.
(1) X^λ is orthocompact for every λ.
(2) X^μ is orthocompact, where $\mu = \text{Max}\{\omega_1, \gamma\}$.
(3) X is compact.

PROOF. (3)→(1)→(2) are obvious.
(2)→(3): Since X^{ω_1} is orthocompact, it follows from Theorem 3.4 that X cannot contain a closed subspace homeomorphic to N. Hence X is countably compact. Since X^γ is orthocompact, it follows from Theorem 3.1 that X is metacompact. As is well-known that every countably compact and metacompact space is compact (e. g. [2] P400), X is compact. The proof is complete.

Concerning orthocompactness of finite products, Scott has first shown in [11] that a finite product of ordinals is orthocompact iff it is normal, and later he generalized his result as follows. Note that each ordinal is a locally compact linearly ordered topological space.

PROPOSITION 3.6 (Scott [12]). A finite product of locally compact linearly ordered topological spaces is orthocompact iff it is normal.

In the rest of this section, we will show, together with Conover's result [1], that a product of arbitrarily many ordinals is orthocompact iff it is normal. This is a partial generalization of Proposition 3.6.

From the result of Scott in [11], we have the following.

PROPOSITION 3.7. Let α and β be ordinals such that $cf(\alpha) \leq cf(\beta)$ and $\alpha \times \beta$ is orthocompact. Then one of the following is satisfied:
(a) $cf(\alpha) \leq \omega$ and $cf(\beta) \leq \omega$;
(b) $cf(\alpha) \leq \omega$, $cf(\beta) > \omega$ and $\alpha < cf(\beta)$;
(c) $cf(\alpha) = \alpha > \omega$ and $\alpha = \beta$.

Recall that an ordinal α is *regular* iff $cf(\alpha)=\alpha$, and a subset of α is called *stationary* iff it intersects every closed unbounded subset of α. The following lemma is known as the "Pressing Down Lemma". For a proof of this lemma, see [11].

LEMMA. 3.8. *Let κ be an uncountable regular ordinal, and let S be a stationary subset of κ. If $f: S \to \kappa$ be a function such that $f(\alpha) < \alpha$ for every $\alpha \in S$, then there is $\alpha \in \kappa$ such that $f^{-1}(\alpha)$ is stationary in κ.*

PROPOSITION 3.9. *Let κ be an uncountable regular ordinal, then the space κ is not κ-metacompact.*

PROOF. The open cover $\{[0, \alpha] \mid \alpha < \kappa\}$ of κ has no point-finite refinement. In fact, let $\mathcal{U}$ be any open refinement of this cover, then there is a function $f: \kappa \setminus \{0\} \to \kappa$ such that $f(\alpha) < \alpha$ for each $\alpha \in \kappa \setminus \{0\}$ and $\{\{0\}\} \cup \{(f(\alpha), \alpha] \mid \alpha \in \kappa \setminus \{0\}\}$ is a refinement of $\mathcal{U}$. By virtue of Lemma 3.8, there is $\alpha \in \kappa$ such that $|f^{-1}(\alpha)| = \kappa$. For each $\beta \in f^{-1}(\alpha)$, we can take $U_\beta \in \mathcal{U}$ and $\alpha(\beta) < \kappa$ such that $(f(\beta), \beta] = (\alpha, \beta] \subseteq U_\beta \subseteq [0, \alpha(\beta)]$. Since κ is regular, we can find $T \subseteq f^{-1}(\alpha)$ such that $|\{U_\beta \mid \beta \in T\}| = \kappa$. It is obvious that $\alpha+1 \in \cap \{U_\beta \mid \beta \in T\}$. Thus $\mathcal{U}$ cannot be point-finite at $\alpha+1$. The proof is complete.

From Propositions 3.1 and 3.9, we get

PROPOSITION 3.10. *Let κ be an uncountable regular ordinal, then $\kappa \times 2^\kappa$ is not orthocompact.*

The following lemmas are well-known.

LEMMA 3.11. *Let $f: X \to Y$ be a closed onto map such that $f^{-1}(y)$ is λ-compact for every $y \in Y$. If Y is λ-paracompact, then so is X.*

LEMMA 3.12. *Let X be a λ-compact space and let Y be a space of character $\leq \lambda$, then the projection from $X \times Y$ onto Y is a closed map.*

Recall that a map $f: X \to Y$ is *perfect* iff it is a closed map such that $f^{-1}(y)$ is compact for every $y \in Y$. A space X is a *paracompact M-space* in the sence of Morita [8] iff there are a metric space M and a perfect map from X onto M.

PROPOSITION 3.13. *Let X be a λ-compact space and let Y be a paracompact M-space. Then $X \times Y$ is λ-paracompact.*

PROOF. There are a metric space M and a perfect onto map $f: Y \to M$. Let $g: X \times Y \to X \times M$ be the map defined by $g(x, y)=(x, f(y))$ for every $(x, y) \in X \times Y$. Let $p: X \times M \to M$ be the projection map. Then p is a closed map by Lemma 3.12, and $p^{-1}(z)$ is λ-compact for every $z \in M$. By virtue of Lemma 3.11, $X \times M$ is λ-paracompact. Since g is a perfect map, it follows from Lemma 3.11 that $X \times Y$ is λ-paracompact. The proof is complete.

Now we prove our main theorem in this section.

THEOREM 3.14.   *Let* $\{\alpha_i | i \in I\}$ *be a collection of non-zero ordinals with* $|I| \geqq 2$. *Let* $I_1 = \{i \in I | cf(\alpha_i)=1\}$, $I_2 = \{i \in I | cf(\alpha_i)=\omega\}$ *and* $I_3 = \{i \in I | cf(\alpha_i) > \omega\}$. *Then the following are equivalent.*

(1) $\Pi\{\alpha_i | i \in I\}$ *is orthocompact.*

(2) $\Pi\{\alpha_i | i \in I\}$ *is normal.*

(3) $|I_2| \leqq \omega$ *and one of the following is satisfied:*

   (a) $I_3 = \phi$;

   (b) $I_3 = \{i_0\}$, $|I_1| < cf(\alpha_{i_0})$ *and* $\alpha_i < cf(\alpha_{i_0})$ *for every* $i \in I_1 \cup I_2$;

   (c) $|I_3| \geqq 2$ *and there exists an uncountable regular ordinal* κ *such that* $\alpha_i = \kappa$ *for every* $i \in I_3$, $\alpha_i < \kappa$ *for every* $i \in I_1 \cup I_2$ *and* $|I_1 \cup I_2| < \kappa$.

PROOF. The equivalence $(2) \rightleftarrows (3)$ is proved by Conover (Theorem 3 of [1]). Hence we only prove the equivalence $(1) \rightleftarrows (3)$.

First of all, let $X = \Pi\{\alpha_i | i \in I\}$ and $X_k = \Pi\{\alpha_i | i \in I_k\}$ for $k=1, 2, 3$. Since $I = I_1 \cup I_2 \cup I_3$, $X = X_1 \times X_2 \times X_3$.

Now we prove $(1) \rightarrow (3)$. Assume that X is orthocompact. Now that for each $i \in I_1 \cup I_2$ and $j \in I_3$, $\alpha_i \times \alpha_j$ is orthocompact and $cf(\alpha_i) \leqq \omega < cf(\alpha_j)$. Hence it follows from (b) of Proposition 3.7 that $\alpha_i < cf(\alpha_j)$ for each $i \in I_1 \cup I_2$ and $j \in I_3$. Similarly, from (c) of Proposition 3.7, it follows that if $|I_3| \geqq 2$ then there is a regular ordinal κ such that $\alpha_j = \kappa$ for every $j \in I_3$. Thus it is sufficient to prove that $|I_2| \leqq \omega$ and $|I_1 \cup I_2| < cf(\alpha_j)$ for every $j \in I_3$. For each $i \in I_2$, α_i contains a closed subspace which is homeomorphic to N. If $|I_2| > \omega$, then X_2 and hence X contains a closed subspace homeomorphic to N^{ω_1} which is a contradiction by virtue of Theorem 3.4. Hence $|I_2| \leqq \omega$. If $|I_1 \cup I_2| \geqq cf(\alpha_j)$ for some $j \in I_3$, then we are led to a contradiction as follows. Let $cf(\alpha_j)=\kappa$. Then κ is an uncountable regular ordinal and α_j contains a closed subspace homeomorphic to κ. Since $|I_1 \cup I_2| \geqq \kappa$, $X_1 \times X_2$ contains a closed subspace homeomorphic to 2^κ. Thus X contains a closed subspace homeomorphic to $\kappa \times 2^\kappa$, which is a contradiction by virtue of Proposition 3.10. Hence, as above, the implication $(1) \rightarrow (3)$ is proved.

Next, we prove $(3) \rightarrow (1)$. Assume that (3) holds. Since the implication (3)

$\rightarrow$(2) is true, X is normal. Hence, from Proposition 3.6, $\Pi\{\alpha_i \mid i \in I'\}$ is ortho-compact for every finite subset I' of I. We can assume that I is an infinite set and let $|I| = \lambda$. By virtue of Corollary 2.5, it is sufficient to show that X is λ-paracompact. First, note that for each $i \in I_2$, α_i is a locally compact Lindelöf regular space and hence it is a paracompact M-space. Since $|I_2| \leqq \omega$, X_2 is also a paracompact M-space. Moreover, since X_1 is a compact T_2-space $X_1 \times X_2$ is a paracompact M-space.

In the case (a) of (3), there is nothing to prove. In the case (b), since it can be easily seen that $\lambda < cf(\alpha_{i_0})$, $X_3 = \alpha_{i_0}$ is λ-compact by Proposition 1.11. In the case (c), since $\lambda < \kappa = cf(\kappa)$, it follows from Proposition 1.11 that $\alpha_i = \kappa$ is λ-bounded for every $i \in I_3$. Hence by virtue of Propositions 1.9 and 1.10, X_3 is λ-compact. Thus in any case of (a), (b) and (c), we have shown that X_3 is λ-compact. By virtue of Proposition 3.13, $X = (X_1 \times X_2) \times X_3$ is λ-paracompact. The proof is complete.

In the proof above we have also obtained

COROLLARY 3.15. *Let* $\{\alpha_n \mid n < \omega\}$ *be a countable family of ordinals. Then* $\Pi\{\alpha_n \mid n < \omega\}$ *is countably paracompact.*

COROLLARY 3.16. *Let* $\{\alpha_n \mid n < \omega\}$ *be a countable family of ordinals. Then the following are equivalent.*
(1) $\Pi\{\alpha_n \mid n < \omega\}$ *is orthocompact.*
(2) $\Pi\{\alpha_n \mid n \in A\}$ *is orthocompact for every finite subset* $A \subseteqq \omega$.
(3) $\Pi\{\alpha_n \mid n < \omega\}$ *is normal.*
(4) $\Pi\{\alpha_n \mid n \in A\}$ *is normal for every finite subset* $A \subseteqq \omega$.

COROLLARY 3.17. *Let* κ *be an uncountable regular ordinal, and let* λ *be a cardinal. Then the following are equivalent.*
(1) κ^{λ} *is orthocompact.*
(2) κ^{λ} *is normal.*
(3) $\lambda < \kappa$.

REMARK. The equivalence (2)$\rightleftarrows$(3) in Corollary 3.17 is proved by Conover [1].

4. Examples.

EXAMPLE 4.1. Let Y be an orthocompact space which is not countably meta-compact, such a space exists, e. g. Scott [13]. For each $n < \omega$, let $X_n = Y \times 2^n$. And for each $n \leqq m$, let $f_{nm} : X_m \rightarrow X_n$ be the natural projection map. Then $\{X_n, f_{nm}\}$

is an inverse system over ω, and the inverse limit X of this system is homeomorphic to $Y \times 2^\omega$. It is easy to see that all X_n are orthocompact spaces and all projections $f_n : X \to X_n$ are closed and open onto maps. But by virtue of Theorem 3.1 X is not orthocompact, since Y is not countably metacompact.

Thus for (a) of Thorem 2.9 the condition that X is $|A|$-metacompact cannot be dropped.

EXAMPLE 4.2. Let Y be a Dowker space, (that is, a normal space which is not countably paracompact.) Then, as in Example 4.1, we can construct an inverse system of normal spaces and open and closed bonding maps, whose limit space is not normal.

EXAMPLE 4.3. Let $\omega(X)$ be the weight of a space X. Although that if $X \times A(\omega(X))$ is orthocompact then X is metacompact by Theorem 3.1, the normality of $X \times A(\omega(X))$ need not imply that X is paracompact. For example, $\omega_1 \times A(\omega_1)$ is a normal space but ω_1 is not paracompact. $\omega_1 \times A(\omega_1)$ is also a simple example of a (collectionwise) normal space which is not orthocompact.

More generally, for each pair of cardinals λ and κ such that $\omega < cf(\kappa) \leqq \lambda$, the product space $\kappa \times A(\lambda)$ is collectionwise normal and not orthocompact.

EXAMPLE 4.4. Let $\{\alpha_i | i \in I\}$ be a family such that $|I| = \omega_1$ and $\alpha_i = \omega_1$ for every $i \in I$. Then, by virtue of Corollary 3.17, $\varPi\{\alpha_i | i \in A\}$ is orthocompact and normal for every countable subset A of I, but $\varPi\{\alpha_i | i \in I\}$ is neither orthocompact nor normal. Hence Corollary 3.16 cannot be generalized to an uncountable family of ordinals.

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