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ON SOME CLASS OF INITIAL BOUNDARY VALUE PROBLEMS FOR SECOND ORDER QUASILINEAR HYPERBOLIC SYSTEMS

By

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Summary We consider some class of initial-boundary value problems for second order, quasilinear, hyperbolic systems containing the Neumann and Dirichlet problems. Using Shibata's ideas we prove the existence of an unique local, smooth solution. In the separate paper [4] we show that the presented results can be applied to elasticity and generalized thermoelasticity.

Introduction.

In recent years we can observe an interesting progress in the theory of the existence of local solutions to the initial-boundary value problems for second order, quasilinear, hyperbolic systems. The Cauchy-Dirichlet problem was investigated in the papers [3], [6], [11] and the Cauchy-Neumann problem was solved in [16], [20], [21]. In the paper [10] an abstract, semigroup approach was presented which allows for solving the both types of mentioned problems. Although the semigroup approach is very elegant, it seems that from the point of view of applications the concrete and elementary energy methods used in [20] are more adequate. Furthermore, using the energy methods one can consider the systems with coefficients depending explicitely on t and on the derivatives of the unknown function with respect to t (cf. (1.1), (1.2) below). In the present paper we demonstrate an unified approach to the mixed problems with Neumann and Dirichlet boundary conditions. We assume that some components of the unknown vector-function satisfy the Neumann boundary conditions, while the remaining ones the Dirichlet conditions. Since we do not exclude the situation in which all components satisfy the same type of boundary conditions we obtain generalization of the results presented in [3], [6] and [20]. In a consequence our theory can be applied to such problems of elastodynamics as the traction or presure problem as well as to the problem of place. On the other

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hand, our results allow for some new applications. There are, namely physically important theories for which the Neumann end Dirichlet boundary conditions are considered at the same time. For example, in generalized thermoelasticity the components of the displacement vector can satisfy the Neumann conditions and the temperature difference the Dirichlet one (or reverse). Since the present paper is relatively long we have decided to present the mentioned applications in a separate article [4] (cf. also [7], [8], [15]).

1. Formulation of the problem.

In the present paper we consider the initial-boundary value problem

$$\sum_{I,J=0}^{n} a_{IJ}(t, \cdot, D^1u(t))\partial_I\partial_J u(t)+a_\Omega(t, \cdot, D^1u(t))=f_\Omega(t) \quad \text{in } (0, T)\times\Omega, \tag{1.1}$$

$$\sum_{i=1}^{n} n_i a_i(t, \cdot, D^1u(t))+a_\Gamma(t, \cdot, D^1u(t))=f_\Gamma(t), \quad u_D(t)=0 \quad \text{on } (0, T)\times\Gamma, \tag{1.2}$$

$$u(0)=u_0,\ \partial_t u(0)=u_1 \quad \text{in } \Omega, \tag{1.3}$$

where Ω is a domain in R^n with a compact and infinitely smooth boundary Γ, $(n_1, n_2, \cdots, n_n)$ denotes the unit outer normal to Γ (for simplicity we assume that $n_i\in C_0^\infty(R^n)$, $i=1, \cdots, n$), $\partial_0=\partial_t$, ∂_i, $i=1, \cdots, n$ denote the differentiations with respect to t and x_i respectively, $u={}^t(u^1, \cdots u^m)$ is the vector-valued unknown function on $(0, T)\times\Omega$, (${}^t(\cdot)$ denotes the operation of transposition). The real matrices $a_{IJ}=(a_{IJ}^{ab})$, $I, J=0, \cdots, n$, $a, b=1, \cdots, m$ and the real vectors $a_\Omega={}^t(a_\Omega^1, \cdots, a_\Omega^m)$, $a_i={}^t(a_i^1, \cdots, a_i^m)$, $i=1, \cdots, n$, $a_\Gamma={}^t(a_\Gamma^1, \cdots, a_\Gamma^m)$ are given functions of the variables $t\in[0, T]$, $x=(x_1, \cdots, x_n)\in\Omega$ and

$$U(t)=D^1u(t)=(\partial_t u(t), \nabla_x u(t), u(t)). \tag{1.4}$$

The real vectors $f_\Omega={}^t(f_\Omega^1, \cdots, f_\Omega^m)$, $f_\Gamma={}^t(f_\Gamma^1, \cdots, f_\Gamma^m)$ are given functions defined on $[0, T]\times\Omega$ and $u_k={}^t(u_k^1, \cdots, u_k^m)$, $k=0, 1$ are defined on Ω.

REMARK 1.1. In (1.1)-(1.4) and in the sequel, the dependence of vector or matrix functions on $x=(x_1, \cdots, x_n)\in\Omega$ is allowed but usunally omited for brevity.

To describe the precise meaning of the boundary conditions (1.2) let us assume that two subsets M_D and M_N of the set $M=\{1, \cdots, m\}$ are given, such that

$$M_D\cap M_N=\emptyset \quad \text{and} \quad M_D\cup M_N=M \tag{1.5}$$

and for arbitrary vector-valued function $\phi={}^t(\phi^1, \cdots, \phi^m)$ let us define the function ϕ_D:

$$(1.6) \qquad \phi_D={}^t(\phi_D^1, \cdots, \phi_D^m), \quad \text{where} \quad \phi_D^a=\begin{cases} \phi^a & \text{if } a\in M_D \\ 0 & \text{if } a\in M_N. \end{cases}$$

We shall assume that

$$(\text{a.0}) \qquad a_i^a=a_\Gamma^a=f_\Gamma^a\equiv 0 \quad \text{if} \quad a\in M_D \quad \text{and} \quad i=1, \cdots, n.$$

Thus the second part of the boundary condition (1.2) indicates that some components of the unknown function satisfy the homogeneous Dirichlet boundary conditions while the first part of (1.2) is simply the Neumann boundary condition (roughly speaking) for remaining components. Let us note that we do not exclude the situation $M_D=\emptyset$ or $M_N=\emptyset$. Thus the usual Neumann or Dirichlet boundary conditions are allowed. In this sense our considerations generalize the results proved in [3], [6], [11], [16], [20], [21]. Note also that the inhomogeneous Dirichlet boundary conditions can be reduced to the homogeneous ones by replacing the unknown function by the difference between it and the function having appropriate traces on the boundary (cf. for example [5]).

Let us list the basic assumptions of the present paper. Let u_0^a, u_i^a and u_{n+1}^a, $i=1, \cdots, n$, $a=1, \cdots, m$ denote the independent variables corresponding to $\partial_t u^a$, $\partial_i u^a$ and u^a respectively. Let U_0 and T_0 be given positive constants. Put

$$(1.7) \qquad D(U_0)=\{U\in R^{(n+2)m}: |U|<U_0\}, \qquad U=(u_0^a, u_i^a, u_{n+1}^a).$$

First of all we shall assume that the coefficients of the system (1.1), (1.2) are smooth. To be more precise, we assume that

$$(\text{a.1})_1 \qquad \begin{gathered} a_{IJ}(t, x, U),\ a_i(t, x, U),\ a_V(t, x, U)\in B^\infty([-T_0, T_0]\times\bar{\Omega}\times D(U_0)), \\ I, J=0, 1, \cdots, n, \quad i=1, \cdots, n, \quad V\in\{\Omega, \Gamma\}, \end{gathered}$$

where B^∞ denotes the space of vector functions with continuous and bounded derivatives of arbitrary order. In the case of unbounded domain Ω we assume additionally

$$(\text{a.1})_2 \qquad a_i(t, x, 0)=a_V(t, x, 0)=0 \qquad \text{for} \quad (t, x)\in[-T_0, T_0]\times\Omega,\ i=1, \cdots, n, \quad V\in\{\Omega, \Gamma\},$$

Now, let us introduce the $m\times m$ matrices

$$(1.8) \qquad \begin{aligned} &b_{VI}=(b_{VI}^{ab}), \quad \text{where} \quad b_{VI}^{ab}=\frac{\partial a_V^a}{\partial u_I^b}, \quad I, J=0, \cdots, n+1, i=1, \cdots, n, \\ &b_{iJ}=(b_{iJ}^{ab}), \quad \text{where} \quad b_{iJ}^{ab}=\frac{\partial a_i^a}{\partial u_J^b}, \quad a, b=1, \cdots, m. \end{aligned}$$

As our second assumption we take the following one

$$
\text{(a.2)}\qquad
\begin{aligned}
&{}^{t}a_{IJ}(t, x, U)=a_{JI}(t, x, U), \quad I, J=0, \cdots, n, \\
&-b_{ij}^{ab}(t, x, U)=a_{ij}^{ab}(t, x, U) \quad \text{if} \quad a\in M_N, \\
&b=1, \cdots, m, \quad i, j=1, \cdots, n, \\
&{}^{t}b_{\Gamma i}(t, x, U)+b_{\Gamma i}(t, x, U)=0, \quad i=1, \cdots, n, \\
&\text{for arbitrary} \quad (t, x, U)\in[-T_0, T_0]\times\bar{\Omega}\times D(U_0).
\end{aligned}
$$

Next we assume that there exist positive constants δ_0, δ_1, δ_2, such that

$$
\text{(a.3)}\qquad
\begin{aligned}
&a_{00}(t, x, U)\geqq\delta_0 I \qquad \text{for} \quad (t, x, U)\in[-T_0, T_0]\times\bar{\Omega}\times D(U_0), \\
&-\sum_{ij=1}^{n}(a_{ij}(t, \cdot, U)\partial_j v, \partial_i v)+\sum_{i=1}^{n}\langle b_{\Gamma i}(t, \cdot, U)\partial_i v, v\rangle \\
&\geqq\delta_1\|v\|_1^2-\delta_2\|v\|_0^2 \qquad \text{for arbitrary} \quad t\in[-T_0, T_0], \\
&v\in H_D^2(\Omega) \quad \text{and} \quad U\in H^{\infty,1}(\bar{\Omega}, D(U_0)).
\end{aligned}
$$

Here and hereafter $H^s(G)$, $s\in R$ denotes the Sobolev space of scalar or vector functions on G with the norm $\|\cdot\|_{s,G}$. In the case $G=\Omega$ we write $\|\cdot\|_{s,G}=\|\cdot\|_s$. Similary $\|\cdot\|_{s,\Gamma}=\langle\!\langle\cdot\rangle\!\rangle_s$ denotes the norm of the Sobolev space $H^s(\Gamma)$ on the boundary Γ of Ω. The brackets $(\cdot, \cdot)$ and $\langle\cdot, \cdot\rangle$ denote the inner products of $L^2(\Omega)=H^0(\Omega)$ and $L^2(\Gamma)=H^0(\Gamma)$ respectively. The symbol $H_0^s(G)$ denotes the closure of the set $C_0^\infty(G)$ of infinitely smooth functions with compact support contained in G, with respect to the norm $\|\cdot\|_{s,G}$. In (a.3) and in the sequel we use also the following spaces

(1.9) $H_D^s(\Omega)=\{u\in H^s(\Omega)\colon u_D\in H_0^1(\Omega)\}$ if $s\geqq 1$, (cf. (1.6)), $H_D^0(\Omega)=L^2(\Omega)$,

(1.10) $H^{\infty,1}(\bar{\Omega}, D(U_0))=\{U\in L^\infty(\Omega, R^{(n+2)m})\colon |U(x)|<U_0$ for $x\in\bar{\Omega}\}$, cf. (1.7)

Our fourth assumption is of the form

$$
\text{(a.4)}\qquad \sum_{i=1}^{n} n_i(x)b_{\Gamma i}(t, x, U)=0 \qquad \text{for} \quad (t, x, U)\in[-T_0, T_0]\times\bar{\Omega}\times D(U_0)
$$

and the final one is the following

$$
\text{(a.5)}\qquad
\begin{aligned}
&S\left\{\frac{1}{2}\sum_{i=1}^{n} n_i(x)(a_{0i}+a_{i0})(t, x, U)+b_0(t, x, U)\right\}\xi\cdot\xi\geqq 0 \\
&\text{for arbitrary } (t, x, U)\in[-T_0, T_0]\times\Gamma\times D(U_0) \quad \text{and} \\
&\xi={}^t(\xi^1, \cdots, \xi^m)\in R^m \text{ such that } \xi^a=0 \text{ if } a\in M_D.
\end{aligned}
$$

In (a.5) we have posed $S\{A\}=1/2(A+{}^tA)$,

$$b_0=\sum_{i=1}^{n} n_i b_{i0}+b_{\Gamma 0}, \qquad \text{(cf. (1.8))} \tag{1.11}$$

and the dot denotes the usual inner product in R^m i.e.

$$\xi\cdot\eta=\xi^1\eta^1+\cdots+\xi^m\eta^m \qquad \text{for} \quad \xi={}^t(\xi^1,\cdots,\xi^m),\ \eta={}^t(\eta^1,\cdots,\eta^m) \tag{1.12}$$

To formulate our main result, let us introduce the following notations. For an interval J of R and Hilbert space X we denote by $C^k(J, X)$ and $\mathrm{Lip}(J, X)$ the spaces of all X-valued functions of class C^k or Lipschitz-continuous on J, respectively. If L is non-negative integer and M a real number we put

$$X^{L,M}(J, G)=\bigcap_{N=0}^{L} C^N(J, H^{L+M-N}(G)). \tag{1.13}$$

Using these notations we can state

THEOREM 1.1. *Let Ω be an open domain in R^n, $n\geqq 2$, with C^∞ and compact boundary Γ, T_0 a given postive number and K an integer $\geqq[n/2]+3$. Assume that the conditions* (a.0)-(a.5) *are valid and the data u_0, u_1, f_Ω, f_Γ satisfy the following hypotheses*

$$\begin{aligned} &u_0\in H_D^K(\Omega),\ u_1\in H_D^{K-1}(\Omega),\ f_\Omega\in X^{K-2,0}([0, T_0], \Omega),\\ &f_\Gamma\in X^{K-2,1/2}([0, T_0], \Gamma),\quad \partial_t^{K-2}f_\Omega\in \mathrm{Lip}([0, T_0], L^2(\Omega)),\\ &\qquad\qquad \partial_t^{K-2}f_\Gamma\in \mathrm{Lip}([0, T_0], H^{1/2}(\Gamma)), \end{aligned} \tag{1.14}$$

(1.15) *the compatibility condition of order $K-2$ is satisfied*

(cf. sect. 3 below)

$$(u_1(\cdot), D_x^1u_0(\cdot))\in H^\infty(\bar{\Omega}, D(U_0)), \qquad (D_x^1u=(\nabla_x u, u)). \tag{1.16}$$

Let B be a positive constant such that

$$\begin{aligned} &\|u_0\|_K+\|u_1\|_{K-1}+|f_\Omega|_{K-2,0,[0,T_0]}+|f_\Gamma|_{K-2,1/2,[0,T_0]}\\ &+\operatorname*{ess\,sup}_{t\in[0,T_0]}\|\partial_t^{K-1}f_\Omega(t)\|_0+\operatorname*{ess\,sup}_{t\in[0,T_0]}\langle\!\langle\partial_t^{K-1}f_\Gamma(t)\rangle\!\rangle_{1/2}\leqq B
\end{aligned} \tag{1.17}$$

where $|\cdot|_{K-2,0,[0,T_0]}$ and $|\cdot|_{K-2,1/2,[0,T_0]}$ are norms of $X^{K-2,0}([0, T_0], \Omega)$ and $X^{K-2,1/2}([0, T_0], \Gamma)$ respectively which will be defined in formulas (2.3), (2.4) *below. Then, there exist $T\in[0, T^0]$ and $\Lambda>0$ depending only on K and B such that the prablem* (1.1)-(1.3) *admits a unique solution*

$$u(t)\in X_D^{K,0}([0, T], \Omega), \qquad \text{(cf. (2.5) below)} \tag{1.18}$$

satisfying the conditions

$$|u|_{K,0,[0,T]}\leqq\Lambda, \tag{1.19}$$

(1.20) $\qquad D^1 u(t) \in H^{\infty,1}(\bar{\Omega}, D(U_0)) \quad for \quad t \in [0, T]$.

REMARK 1.2. Since we have assumed that $K \geqq [n/2]+3$, by Sobolev imbedding theorem we have $u(t) \in C^2([0, T] \times \bar{\Omega})$, thus the theorem 1.1 gives the existence of classical solutions to the problem (1.1)-(1.3).

The present paper is organized as follows. In Section 2 we introduce basic notations. In Section 3 we formulate the compatibility condition for (1.1)-(1.3). In Section 4 we define the iteration procedure leading to the solution of (1.1)-(1.3). In Sections 5 and 6 we present some results from the theory of linear elliptic and hyperbolic problems. These results are used in Section 7 where the convergence of our iteration scheme is proved. In the Appendix we sumarize some facts concerning the estimates of nonlinear terms.

2. Notations.

For $v = {}^t(v_1, \cdots, v_k)$ where $v_1, \cdots, v_k$ are real functions and for $\alpha = (\alpha_1, \cdots, \alpha_k)$ where $\alpha_1, \cdots, \alpha_k$ are nonegative integers we put $v^\alpha = v_1^{\alpha_1} \cdots v_k^{\alpha_k}$, $|\alpha| = \alpha_1 + \cdots + \alpha_k$. We use the following notations concerning differentiations

$$\partial_x = (\partial_1, \cdots, \partial_n),\ \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_k}^{\alpha_k},\ \partial_t^j \partial_x^\alpha v = {}^t(\partial_t^j \partial_x^\alpha v_1, \cdots, \partial_t^j \partial_x^\alpha v_k),$$

$$D^L D_x^M v = (\partial_t^j \partial_x^\alpha v : j + |\alpha| \leqq L + M,\ j \leqq L),\ D^L v = D^L D_x^0 v,\ D_x^M v = D^0 D_x^M v.$$

Put

$$|v|_{\infty, L} = \sup_{x \in \Omega} |D_x^L v(x)|,$$
(2.1)
$$|v|_{\infty, L, T} = \sup\{|D^L v(t, x)| : (t, x) \in [-T, T] \times \bar{\Omega}\}$$

For time interval J and Hilbert space X, let $L^\infty(J, X)$ denote the set of all X-valued, measurable and bounded (everywhere!) functions defined on J. For $s \in R$ put $Y^{0,s}(J, G) = L^\infty(J, H^s(G))$ and for an integer $L \geqq 1$ put $Y^{L,s}(J, G) = \{u(t) \in X^{L-1,s}(J, G) : \partial_t^M u(t) \in L^\infty(J, H^{L+s-M}(G)) \cap \mathrm{Lip}(J, H^{L+s-M-1}(G))$ for $0 \leqq M \leqq L-1\}$. Note that $Y^{L,s} \subset Y^{L-M, s+M}$ and $X^{L,s} \subset X^{L-M, s+M}$ for $0 \leqq M \leqq L$. The space $Y^{L,s}(J, G)$ is endowed with the norm defined as follows

$$|v|_{0,s,J,G} = \sup_{t \in J} \|v(t)\|_{s,G} \quad \text{and if } L \text{ is an integer} \geqq 1$$
(2.2)
$$|v|_{L,s,J,G} = |v|_{0,L+s,J,G} + \sum_{M=0}^{L-1} \sup_{\substack{t, t' \in J \\ t \neq t'}} \frac{\|(\partial_t^M v)(t) - (\partial_t^M v)(t')\|_{L+s-M-1}}{|t - t'|}$$

If $v(t) \in X^{L,s}(J, G)$, then

(2.3) $$|v|_{L,s,J,G} = \sum_{M=0}^{L} \sup_{t \in J} \|\partial_t^M v(t)\|_{L+s-M, G}.$$

Hence we also use $|\cdot|_{L,s,J,G}$ as the norm of $X^{L,s}(J, G)$. In the case $G = \Omega$ we put

$$|V|_{L,s,J}=|v|_{L,s,J,\Omega} \quad \text{and} \quad \langle v\rangle_{L,s,J}=|v|_{L,s,J,\Gamma}. \tag{2.4}$$

For $Z=X$ or Y we define the subspace $Z_D^{L,s}(J, \Omega)$:

$$Z_D^{L,s}(J, \Omega)=\{u(t)\in Z^{L,s}(J, \Omega): u_D(t)\in H_0^1(\Omega)\} \quad (\text{cf. } (1.6)), \tag{2.5}$$

endowed with the norm $|\cdot|_{L,s,J}$.

REMARK 2.1. If the vector function is replaced by the scalar or matrix one we use analogous notations.

REMARK 2.2. In the paper we shall use the same letter C to denote different constants depending on the same set of arguments. $C(\cdots)$ denotes a constant depending essentially on the quantities appearing in the parentheses. By using the subscripts 1, 2, $\cdots$ we distinguish the important constants.

For a sufficiently smooth function $F(t, x, v)$ we put

$$(\partial_t^k\partial_x^\alpha d^h F)(t, x, v)(w_1, \cdots, w_h)=\frac{d^h}{d\theta_1\cdots d\theta_h}\left[(\partial_t^k\partial_x^\alpha F)\left(t, x, v+\sum_{i=1}^h \theta_i w_i\right)\right]\Big|_{\theta=0}$$

$$\text{for} \quad \theta=(\theta_1, \cdots, \theta_h), \tag{2.6}$$

$$(F)_1(t, x, v)=F(t, x, v)-F(t, x, 0)=\int_0^1 (dF)(t, x, \theta v)v\,d\theta$$

Observe that $(F)_1(t, x, 0)=0$ and $F(t, x, v)=F(t, x, 0)+(F)_1(t, x, v)$. The remaining part of this Section is devoted to some estimates of bilinear forms connected with first order linear differential operators on the boundary Γ. To be more precise let us assume that $R^i(x)$ is a $m\times m$ matrix of functions in $B^1(\bar{\Omega})$ such that

$$\sum_{i=1}^n n_i(x)R^i(x)=0 \qquad \text{for} \quad x\in\Gamma. \tag{2.7}$$

We shall describe some bilinear forms $S_1(R)[v, w]$ on $\{H_D^1(\Omega)\}^2$ and $S_2(R)[v, w]$ on $H_D^1(\Omega)\times L^2(\Omega)$ such that

$$\left\langle \sum_{i=1}^n R^i\partial_i v, w\right\rangle=S_1(R)[v, w]+S_2(R)[v, w] \quad \text{for} \quad v\in H_D^2(\Omega),\ w\in H_D^1(\Omega), \tag{2.8}$$

where $R=(R^1, \cdots, R^n)$. The bilinear forms $S_k(R)[\cdot, \cdot]$ $k=1, 2$ will be useful in the investigation of linear problems connected with our iteration procedure, cf. Sections 5, 6 (for example (5.5) or (6.6), (6.8)). To define these forms let us note that since Γ is a compact and C^∞hypersurface of R^n we may select a finite number of open sets G_l in R^n, positive numbers σ_l and C^∞ diffeomorphisms Ψ_l from G_l' onte G_l, $l=1, \cdots, p$, such that $G_l'=\{y=(y_1, \cdots, y_n)\in R^n: |y'|=|(y_1, \cdots, y_{n-1})|<\sigma_l$ and $|y_n|<\sigma_l\}$, $\Omega\cap G_l=\Psi_l(\{y\in G_l': y_n>0\})$ and $\Gamma\cap G_l=\Psi_l(\{y\in G_l': y_n=0\})$. Let $\Phi_l=(\Phi_{l1}, \cdots, \Phi_{ln})$ be the inverse maps of Ψ_l. If we put $Y_{li}^j(y)=(\partial\Phi_{lj}/\partial x_i)(\Psi_l(y))$ and $J_l(y')=|(Y_{l1}^n(y', 0), \cdots, Y_{ln}^n(y', 0))|$ we have $n_i(x)=-Y_{li}^n(y', 0)/J_l(y')$ and $d\Gamma_x=J_l(y')dy'$ for $x=\Psi_l(y', 0)\in G_l\cap\Gamma$ where $d\Gamma_x$ is the surface element of Γ. Using the assumption (2.7) we see that

$$\sum_{i=1}^n R^i(\Psi_l(y', 0))Y_{li}^n(y', 0)=0 \qquad \text{for} \quad (y', 0)\in G_l'. \tag{2.9}$$

Let $\phi_l\in C_0^\infty(G_l)$, $l=1, \cdots, p$ be the partition of unity on Γ and put $\psi_l(y)=\phi_l(\Psi_l(y))\in C_0^\infty(G_l')$. By the change of variables $x=\Psi_l(y)$ and (2.9) we obtain

$$\begin{aligned}\Big\langle \sum_{i=1}^n R^i\partial_i v, w\Big\rangle &= \sum_{i=1}^n \sum_{l=1}^p \int_{G_l\cap\Gamma}\phi_l(x)R^i(t, x)\partial_{x_i}v(x)w(x)d\Gamma_x \\ &= \sum_{l=1}^p \sum_{i=1}^n \sum_{q=1}^{n-1}\int_{R^{n-1}}\psi_l(y', 0)R^i(t, \Psi_l(y', 0))\partial_{y_q}v'(y', 0)(\partial\Phi_{lq}/\partial x_i)(y', 0) \\ &\quad \times w'(y', 0)J_l(y')dy'=\sum_{l=1}^p \sum_{q=1}^{n-1}\int_{R^{n-1}}\psi_l(y', 0)S_l^q(R, y')\partial_q'v'(y', 0) \\ &\quad \times w'(y', 0)dy', \text{ where } \partial_j'=\partial/\partial y_j,\ v'(y)=v(\Psi_l(y)),\ w'(y)=w(\Psi_l(y)) \\ &\quad \text{and} \quad S_l^q(R, y')=\sum_{i=1}^n R^i(\Psi_l(y', 0))Y_{li}^q(y', 0)J_l(y').\end{aligned} \tag{2.10}$$

If we put

$$\begin{aligned}S_1(R)[v, w]=\sum_{l=1}^p \sum_{q=1}^{n-1}\int_{R_+^n}\psi_l(y)\{S_l^q(R, y')\partial_n'v'(y)\cdot\partial_q'w'(y) \\ -S_l^q(R, y')\partial_q'v'(y)\cdot\partial_n'w'(y)\}\,dy,\end{aligned} \tag{2.11}$$

$$\begin{aligned}S_2(R)[v, w]=\sum_{l=1}^p \sum_{q=1}^{n-1} -\int_{R_+^n}\{\psi_l(y)(\partial_q'S_l^q(R, y'))\partial_n'v'(y)\cdot w'(y) \\ -(\partial_n'\psi_l(y))S_l^q(R, y')\partial_q'v'(y)\cdot w'(y)\}\,dy\end{aligned} \tag{2.12}$$

where $R_+^n=\{y=(y_1, \cdots, y_n)\in R^n: y_n>0\}$, noting the formula

$$\Big\langle \sum_{i=1}^n R^i\partial_i v, w\Big\rangle=\sum_{l=1}^p \sum_{q=1}^{n-1} -\int_{R_+^n}\partial_n'\{\psi_l(y)S_l^q(R, y')\partial_q'v'(y)\cdot w'(y)\}\,dy$$

we obtain (2.8) integrating by parts. Furthermore using Schwarz's inequality we can prove that for $v, w\in H_D^1(\Omega)$

$$|S_1(R)[v, w]|\leqq C\Big\{\sum_{i=1}^n \|R^i\|_{\infty,0}\Big\}\|v\|_1\|w\|_1, \tag{2.13}$$

$$|S_2(R)[v, w]|\leqq C\Big\{\sum_{i=1}^n \|R^i\|_{\infty,1}\Big\}\|v\|_1\|w\|_0, \tag{2.14}$$

with some constants C independent of R^i, v and w.

3. Compatibility conditions.

Let us introduce the following notations

$$\begin{aligned}(3.1)\qquad &\partial_t^M(a_{IJ}(t, D^1u))=(\partial_t^M a_{IJ})(t, D^1u)+\sum_{h=1}^{M}\Sigma^* a_{IJ\alpha^h\beta^h}^{Mh}(t, D^1u)\\ &\times(D_x^1\partial_t u)^{\alpha_1^h}\cdots(D_x^1\partial_t^h u)^{\alpha_h^h}(\partial_t^2 u)^{\beta_1^h}\cdots(\partial_t^{h+1}u)^{\beta_h^h}\\ &\equiv a_{IJ}^M(t, u, \partial_t u, \cdots, \partial_t^{M+1}u)\qquad \text{for}\quad I, J=0, \cdots, n,\quad M=0, 1, \cdots\end{aligned}$$

$$\begin{aligned}(3.2)\qquad &\partial_t^M(a_V(t, D^1u))=(\partial_t^M a_V)(t, D^1u)+\sum_{h=1}^{M}\Sigma^* a_{V\alpha^h\beta^h}^{Mh}(t, D^1u)\\ &\times(D_x^1\partial_t u)^{\alpha_1^h}\cdots(D_x^1\partial_t^h u)^{\alpha_h^h}(\partial_t^2 u)^{\beta_1^h}\cdots(\partial_t^{h+1}u)^{\beta_h^h}\\ &\equiv a_V^M(t, u, \partial_t u, \cdots, \partial_t^{M+1}u)\qquad \text{for}\quad V\in\{\Omega, \Gamma, i, \cdots, n\},\quad M=0, 1, \cdots,\end{aligned}$$

where $a_{IJ\alpha^h\beta^h}^{Mh}$, $a_{V\alpha^h\beta^h}^{Mh}$ and in a consequence a_{IJ}^M, a_V^M are uniquely determined functions of their arguments, $\alpha^h=(\alpha_1^h, \cdots, \alpha_h^h)$, $\beta^h=(\beta_1^h, \cdots, \beta_h^h)$ are sets of multiindices and the summation Σ^* is taken over all (α^h, β^h) such that $\Sigma_{s=1}^h(|\alpha_s^h|+|\beta_s^h|)s=h$.

Now we can define the vector functions u_{M+2}, $0\leqq M\leqq K-2$ by the recursive formula (u_0, u_1 are the initial values from (1.3))

$$\begin{aligned}(3.3)\qquad &a_{00}^0(0, u_0, u_1)u_{M+2}=(\partial_t^M f)(0)\\ &-\Sigma'\binom{M}{k}a_{IJ}^k(0, u_0, u_1, \cdots, u_{k+1})\partial_I^{sI}\partial_J^{sJ}u_{M+2-k-sI-sJ}\\ &-a_\Omega^M(0, u_0, \cdots, u_{M+1})\end{aligned}$$

where Σ' denotes the summation over all indices $k=0, \cdots, M$, I, $J=0, \cdots, n$ such that $(k, I, J)\neq(0, 0, 0)$ and where $sI=0$ if $I=0$ and $sI=1$ if $I\neq 0$. Let us note that due to (a.3) the matrix $a_{00}^0(0, u_0, u_1)$ is invertible and the equality (3.3) allows for determining u_{M+2} if we know $u_0, u_1, \cdots, u_{M+1}$.

Using Theorems Ap. 1, Ap. 3 from the appendix we can prove

LEMMA 3.1. *If u_0, u_1, f_Ω are the functions and B, K the constants from Theorem* 1.1, *then*

$$(3.4)\qquad u_M\in H^{K-M}(\Omega)\quad \textit{and}\quad \|u_M\|_{K-M}\leqq C_1(K, B)\qquad \textit{for}\quad 2\leqq M\leqq K.$$

We shall say that u_0, u_1, f_Ω, f_Γ satisfy the *compatibility condition of order* $K-2$ if

$$\begin{aligned}(3.5)\qquad &\sum_{i=1}^{n} n_i a_i^M(0, u_0, \cdots, u_{M+1})+a_\Gamma^M(0, u_0, \cdots, u_{M+1})=\partial_t^M f_\Gamma(0).\\ &\textit{for}\quad 0\leqq M\leqq K-2\quad \textit{and}\quad u_{MD}=0\qquad \textit{for}\quad 0\leqq M\leqq K-1\ \textit{on}\ \Gamma.\end{aligned}$$

Recall that for arbitrary ϕ, ϕ_D is defined in (1.6).

4. Iteration procedure.

Since the boundary condition (1.2) is fully nonlinear, the usual (Picard's) iteration technique is not applicable to the problem (1.1)-(1.3). It leads namely to the so called derivative loss. To omit this dificulty we use Shibata's idea presented in [16], [20], [21], which roughly speaking consist in reduction of the problem (1.1)-(1.3) to a "hyperbolic-elliptic" coupled system for unknowns u and $\partial_t u$. To describe precisely this iteration scheme let us differentiate eqs (1.1)-(1.3) once in t and put $\partial_t u=v$, $U(t)=(v(t), D_x^1 u(t))$. Using the notations introduced in (1.8), (1.11), (1.16) and (3.3) with $M=0$ we obtain

$$(4.1)\qquad \begin{aligned} &\sum_{I,J=0}^{n} a_{IJ}(t, U(t))\partial_I\partial_J v(t)+\bar{a}_\Omega(t, U(t))=\partial_t f_\Omega(t) \qquad \text{in} \quad (0, T)\times\Omega, \\ &\sum_{J=0}^{n}\Big(\sum_{i=1}^{n} n_i b_{iJ}(t, U(t))+b_{\Gamma J}(t, U(t))\Big)\partial_J v(t) \\ &+\bar{a}_\Gamma(t, U(t))=\partial_t f_\Gamma(t), \quad v_D(t)=0 \qquad \text{on} \quad (0, T)\times\Gamma, \\ &\qquad\qquad v(0)=u_1, \quad \partial_t v(0)=u_2 \qquad \text{in} \quad \Omega. \end{aligned}$$

In (4.1) we have posed

$$(4.2)\qquad \begin{aligned} &\bar{a}_\Omega(t, U(t))=(\partial_t a_\Omega)(t, U(t))+\sum_{I=0}^{n+1} b_{\Omega I}(t, U(t))\partial_I v(t) \\ &+\sum{}''\Big[(\partial_t a_{IJ})(t, U(t))+\sum_{L=0}^{n+1} a_{IJL}(t, U(t))\partial_L v(t)\Big]\partial_I^{sI}\partial_J^{sJ} u^{2-sI-sJ}(t) \\ &+\sum_{i,j=1}^{n}\Big[(\partial_t a_{ij})(t, U(t))+\sum_{L=0}^{n+1} a_{ijL}(t, U(t))\partial_L v(t)\Big]\partial_i\partial_j u(t), \end{aligned}$$

where

(4.3) $\sum''$ denotes the summation over all pairs of indices $I, J=0, \cdots, n$ such that $I=0$ or $J=0$, the functions $\partial_J u^1$, $J\neq 0$ are identified with $\partial_J v$, the functions $\partial_0^0\phi$, $\partial_{n+1}\phi$ are identified with ϕ and u^2 is identified with $\partial_t v$, furthermore $a_{IJL}=\partial a_{IJ}/\partial(\partial_L u)$ for $I, J=0, \cdots, n$, $L=0, \cdots, n+1$,

$$(4.4)\qquad \begin{aligned} \bar{a}_\Gamma(t, U(t))=&\Big(\sum_{i=1}^{n} n_i b_{i n+1}(t, U(t))+b_{\Gamma n+1}(t, U(t))\Big)v(t) \\ &+\sum_{i=1}^{n} n_i(\partial_t a_i)(t, U(t))+(\partial_t a_\Gamma)(t, U(t)) \end{aligned}$$

With the use of the new notations, the original problem (1.1)-(1.2) can be written as follows

$$\begin{aligned}
&\Sigma'' a_{IJ}(t, U(t))\partial_I^{sI}\partial_J^{sJ}u^{2-sI-sJ}(t)+\sum_{i,j=1}^{n} a_{ij}(t, U(t))\partial_i\partial_j u(t) \\
&+a_\Omega(t, U(t))+\lambda u(t)=f_\Omega(t)+\lambda\Big(u_0+\int_0^t v(s)ds\Big) \qquad \text{in } \Omega \qquad (4.5)\\
&\sum_{i=1}^{n} n_i a_i(t, U(t))+a_\Gamma(t, U(t))=f_\Gamma(t), \qquad u_D(t)=0 \qquad \text{on } \Gamma
\end{aligned}$$

for $t\in[0, T]$, where λ is a constant determined in Theorem 5.3 below. Since (4.5) is still fully nonlinear with respect to $u(t)$ we shall reduce it to an equivalent problem as follows. Let $u^0(t)$ be a function in $X_D^{K-2,2}(R, \Omega)$ such that

$$\partial_t^M u^0(0)=u_M \text{ in } \Omega \qquad \text{for} \quad 0\leqq M\leqq K-2, \tag{4.6}$$

$$\|D^{K-2}u^0(t)\|_2\leqq C_2(K, B) \qquad \text{for} \quad t\in R. \tag{4.7}$$

The existence of u^0 which satisfies (4.6), (4.7) is proved in Theorem Ap. 5b. Put $u(t)=u^0(t)+w(t)$ and $U^0(t)=(v(t), D_x^1 u^0(t))$. Noting that $U^0(0)=(u_1, D_x^1 u_0)$ we can rewrite (4.5) as an equation for unknown $w(t)$. In this purpose let us put $U(\theta)=(v(t), D_x^1(u^0(t)+\theta w(t)))$ for $0\leqq\theta\leqq 1$, and adopt the formula (2.6) with $k=0$, $\alpha=0$, $h=1, 2$, $v=D_x^1 u^0(t)$ and w_1, $w_2=D_x^1 w(t)$. Using the notations introduced in (4.3) and Taylor formula we obtain for $I=0$ or $J=0$

$$\begin{aligned}
&a_{IJ}(t, U(t))\partial_I^{sI}\partial_J^{sJ}u^{2-sI-sJ}(t)=a_{IJ}(t, U^0(t))\partial_I^{sI}\partial_J^{sJ}u^{2-sI-sJ}(t)\\
&+d a_{IJ}(0, U^0(0))D_x^1 w(t)\partial_I^{sI}\partial_J^{sJ}u_{2-sI-sJ}\\
&+d a_{IJ}(0, U^0(0))D_x^1 w(t)\partial_I^{sI}\partial_J^{sJ}(u^{2-sI-sJ}(t)-u_{2-sI-sJ}) \qquad (4.8)\\
&+[d a_{IJ}(t, U^0(t))-d a_{IJ}(0, U_0(0))]D_x^1 w(t)\partial_I^{sI}\partial_J^{sJ}u^{2-sI-sJ}(t)\\
&+\int_0^1 d^2 a_{IJ}(t, U(\theta))(D_x^1 w(t), D_x^1 w(t))\partial_I^{sI}\partial_J^{sJ}u^{2-sI-sJ}(t)d\theta,
\end{aligned}$$

for $i, j=1, \cdots, n$

$$\begin{aligned}
&a_{ij}(t, U(t))\partial_i\partial_j(u^0(t)+w(t))=a_{ij}(t, U^0(t))\partial_i\partial_j u^0(t)\\
&+a_{ij}(0, U^0(0))\partial_i\partial_j w(t)+[a_{ij}(t, U^0(t))-a_{ij}(0, U^0(0))]\partial_i\partial_j w(t)\\
&+d a_{ij}(0, U^0(0))D_x^1 w(t)\partial_i\partial_j u_0+d a_{ij}(0, U^0(0))D_x^1 w(t)\partial_i\partial_j(u^0(t)-u_0) \qquad (4.9)\\
&+[d a_{ij}(t, U^0(t))-d a_{ij}(0, U^0(0))]D_x^1 w(t)\partial_i\partial_j u^0(t)\\
&+d a_{ij}(t, U^0(t))D_x^1 w(t)\partial_i\partial_j w(t)\\
&+\int_0^1 d^2 a_{ij}(t, U(\theta))(D_x^1 w(t), D_x^1 w(t))\partial_i\partial_j(u^0(t)+w(t))d\theta
\end{aligned}$$

and for $V\in\{\Omega, \Gamma, 1, \cdots, n\}$

$$\begin{aligned}(4.10)\qquad a_V(t, U(t)) &= a_V(t, U^0(t)) + d a_V(0, U^0(0)) D_x^1 w(t)\\ &+ [d a_V(t, U^0(t)) - d a_V(0, U^0(0))] D_x^1 w(t)\\ &+ \int_0^1 d^2 a_V(t, U(\theta))(D_x^1 w(t), D_x^1 w(t)) d\theta .\end{aligned}$$

Combining the relations (4.8)-(4.10) we can check that the problem (4.5) can be written in the form

$$(4.11)\qquad p_{\Omega\lambda}[w(t)] = g_\Omega(t) \text{ in } \Omega,\ p_{\Gamma\lambda}[w(t)] = g_\Gamma(t),\ w_D(t) = 0 \text{ on } \Gamma,$$

where

$$\begin{aligned}(4.12)\qquad p_{\Omega\lambda}[w] &= \sum_{i,j=1}^{n} a_{ij}(0, U^0(0)) \partial_i \partial_j w + \sum_{l=1}^{n+1} a_l^*(0, U^0(0), u_2) \partial_l w + \lambda w\\ p_{\Gamma\lambda}[w] &= \sum_{l=1}^{n+1} \Big(\sum_{i=1}^{n} n_i b_{il}(0, U^0(0)) + b_{\Gamma l}(0, U^0(0)) \Big) \partial_l w, \quad \text{(cf. (1.8))}\end{aligned}$$

and for $l=1, \cdots, n+1$

$$\begin{aligned}(4.13)\qquad a_l^*(0, U^0(0), u_2) \partial_l w = & \sum_{I,J=0}^{n} a_{IJl}(0, U^0(0)) \partial_l w \partial_I^{sI} \partial_J^{sJ} u_{2-sI-sJ}\\ & + b_{\Omega l}(0, U^0(0)) \partial_l w, \qquad \text{(cf. (1.8), (4.3))},\end{aligned}$$

$$(4.14)\qquad g_V(t) = G_{V1}(t, v(t)) + \sum_{k=2}^{3} G_{Vk}(t, v(t), w(t)), \qquad V \in \{\Omega, \Gamma\}$$

where the terms G_{Vk}, $k=1, 2, 3$, $V \in \{\Omega, \Gamma\}$ are defined as follows

$$\begin{aligned}(4.15)\qquad G_{\Omega 1}(t, v(t)) = & f_\Omega(t) - \sum_{I,J=0}^{n} a_{IJ}(t, U^0(t)) \partial_I^{sI} \partial_J^{sJ} u^{2-sI-sJ}(t)\\ & - a_\Omega(t, U^0(t)) + \lambda \int_0^t (v(s) - \partial_s u^0(s)) ds ,\end{aligned}$$

$$\begin{aligned}(4.16)\qquad & G_{\Omega 2}(t, v(t), w(t)) =\\ & - \sum_{I,J=0}^{n} d a_{IJ}(0, U^0(0)) D_x^1 w(t) \partial_I^{sI} \partial_J^{sJ} (u^{2-sI-sJ}(t) - u_{2-sI-sJ})\\ & - \sum_{I,J=0}^{n} [d a_{IJ}(t, U^0(t)) - d a_{IJ}(0, U^0(0))] D_x^1 w(t) \partial_I^{sI} \partial_J^{sJ} u^{2-sI-sJ}(t)\\ & - [d a_\Omega(t, U^0(t)) - d a_\Omega(0, U^0(0))] D_x^1 w(t)\\ & - \sum_{i,j=1}^{n} [a_{ij}(t, U^0(t)) - a_{ij}(0, U^0(0))] \partial_i \partial_j w(t) ,\\ & G_{\Omega 3}(t, v(t), w(t)) =\\ & - \Sigma'' \int_0^1 d^2 a_{IJ}(t, U(\theta))(D_x^1 w(t), D_x^1 w(t)) \partial_I^{sI} \partial_J^{sJ} u^{2-sI-sJ}(t) d\theta\end{aligned}$$

$$-\sum_{i,j=1}^{n}\int_0^1 d^2 a_{ij}(t,\ U(\theta))(D_x^1 w(t),\ D_x^1 w(t))\partial_i\partial_j(u^0(t)+w(t))d\theta \tag{4.17}$$

$$-\int_0^1 d^2 a_\Omega(t,\ U(\theta))(D_x^1 w(t),\ D_x^1 w(t))d\theta$$

$$-\sum_{ij=1}^{n} d a_{ij}(t,\ U^0(t))D_x^1 w(t)\partial_i\partial_j w(t), \qquad \text{(cf. (4.3))},$$

$$G_{\Gamma 1}(t,\ v(t))=f_\Gamma(t)-\sum_{i=1}^{n} n_i a_i(t,\ U^0(t))-a_\Gamma(t,\ U^0(t)), \tag{4.18}$$

$$\begin{aligned} G_{\Gamma 2}(t,\ v(t),\ w(t))=&-\sum_{i=1}^{n} n_i[d a_i(t,\ U^0(t))-d a_i(0,\ U^0(0))]D_x^1 w(t)\\ &-[d a_\Gamma(t,\ U^0(t))-d a_\Gamma(0,\ U^0(0))]D_x^1 w(t), \end{aligned} \tag{4.19}$$

$$G_{\Gamma 3}(t,\ v(t),\ w(t))=-\sum_{i=1}^{n} n_i\int_0^1 d^2 a_i(t,\ U(\theta))(D_x^1 w(t),\ D_x^1 w(t))d\theta \tag{4.20}$$

$$-\int_0^1 d^2 a_\Gamma(t,\ U(\theta))(D_x^1 w(t),\ D_x^1 w(t))d\theta .$$

The problems (4.1), (4.11) form a coupled "hyperbolic-elliptic" system with unknowns v and w. To solve this system we shall use the method of succesive approximations. To this end, let us introduce the functional spaces Z and Z_c. By definition Z is the set of all pairs $(v(t),\ w(t))$ such that

$$(v(t),\ w(t))\in Y_D^{K-1,0}([0,\ T],\ \Omega)\times Y_D^{K-2,2}([0,\ T],\ \Omega), \qquad \text{(cf. (2.5))}, \tag{4.21}$$

$$\partial_t^M w(0)=0, \quad 0\leqq M\leqq K-3, \quad \partial_t^M v(0)=u_{M+1}, \quad 0\leqq M\leqq K-2, \tag{4.22}$$

$$|v|_{K-1,0,[0,T]}\leqq \Lambda_H, \ |w|_{K-2,2,[0,T]}\leqq \Lambda_E, \ |w|_{K-3,2,[0,T]}\leqq \varepsilon_E, \tag{4.23}$$

$$(v(t),\ D_x^1 u^0(t)) \text{ and } (v(t),\ D_x^1(u_0(t)+w(t)))\in H^{\infty,1}(\bar{\Omega},\ D(U_1)) \quad \text{for } t\in[0,\ T]. \tag{4.24}$$

Here and hereafter T, Λ_H, Λ_E, ε_E are constants determined below, which depend only on K and B essentially and U_1 is a constant $\in(0,\ U_0)$ also determined below. We shall assume that

$$0<T<\min(1,\ T_0) \text{ and } 0<\varepsilon_E<1. \tag{4.25}$$

Analogousely we define the space Z_c as the set of pairs $(v(t),\ w(t))\in Z$ such that

$$(v(t),\ w(t))\in X_D^{K-1,0}([0,\ T],\ \Omega)\times X_D^{K-2,2}([0,\ T],\ \Omega), \tag{4.26}$$

$$\partial_t^M w(0)=0, \ 0\leqq M\leqq K-2, \ \partial_t^M v(0)=u_{M+1}, \ 0\leqq M\leqq K-1. \tag{4.27}$$

The iteration scheme used in this paper can be described as follows. Let $(v^1(t),\ w^1(t))$ be an arbitrary element of Z_c (cf. (7.1)). For $p\geqq 2$ and $(v^{p-1}(t),$

$w^{p-1}(t))\in Z_c$ we define $v^p(t)$ as a solution of the following linear "hyperbolic" problem

$$\sum_{I,J=0}^{n} a_{IJ}(t, U^{p-1}(t))\partial_I\partial_J v^p(t)=\partial_t f_\Omega(t)-\bar{a}_\Omega(t, U^{p-1}(t)) \quad \text{in } (0, T)\times\Omega,$$

$$\tag{4.28} \begin{aligned} &\sum_{J=0}^{n}\Big(\sum_{i=1}^{n} n_i b_{iJ}(t, U^{p-1}(t))+b_{\Gamma J}(t, U^{p-1}(t))\Big)\partial_J v^p(t) \qquad \text{on } (0, T)\times\Gamma, \\ &=\partial_t f_\Gamma(t)-\bar{a}_\Gamma(t, U^{p-1}(t)), \qquad v_D^p(t)=0 \\ &\qquad v^p(0)=u_1, \qquad \partial_t v^p(0)=u_2 \qquad \text{in } \Omega, \end{aligned}$$

where

$$\tag{4.29} U^{p-1}(t)=(v^{p-1}(t), D_x^1(u^0(t)+w^{p-1}(t)))$$

and the function $w^p(t)$ is defined as a solution of the linear "elliptic" problem

$$\tag{4.30} \begin{aligned} &p_{\Omega\lambda}[w^p(t)]=g_\Omega^p(t) && \text{in } \Omega \\ &p_{\Gamma\lambda}[w^p(t)]=g_\Gamma^p(t), \quad w_D^p(t)=0 && \text{on } \Gamma \end{aligned} \qquad \text{for } t\in[0, T],$$

where

$$\tag{4.31} g_V^p(t)=G_{V1}(t, v^p(t))+\sum_{k=2}^{3} G_{Vk}(t, v^p(t), w^{p-1}(t)), \qquad V\in\{\Omega, \Gamma\}.$$

It is clear that to prove the convergence of the presented iteration procedure we have to investigate the linear problems corresponding to (4.28) and (4.30).

5. Auxiilary theorems from the theory of linear elliptic problems.

In the present section we consider the boundary-value problem

$$\tag{5.1} q_{\Omega\lambda}[w]=h_\Omega \text{ in } \Omega, \qquad q_{\Gamma\lambda}[w]=h_\Gamma, \qquad w_D=0 \text{ on } \Gamma,$$

where w_D is defined by the use of formula (1.6) and where

$$\tag{5.2} q_{\Omega\lambda}[w]=\sum_{i,j=1}^{n} q_{ij}^\Omega\partial_i\partial_j w+\sum_{l=1}^{n+1} q_l^\Omega\partial_l w+\lambda w,$$

$$\tag{5.3} q_{\Gamma\lambda}[w]=\sum_{l=1}^{n+1}\Big(\sum_{i=1}^{n} n_i q_{il}^\Gamma+q_l^\Gamma\Big)\partial_l w.$$

We assume that the $m\times m$ matrices $q_{ij}^V=(q_{ij}^{Vab})$, $q_l^V=(q_l^{Vab})$, $q_{il}^\Gamma=(q_{il}^{\Gamma ab})$ and m-vectors $h_V={}^t(h_V^1, \cdots, h_V^m)$ are functions of $x\in\Omega$. Here and in the sequel we shall assume that

$$\tag{5.4} i, j=1, \cdots, n, \quad l=1, \cdots, n+1, \quad a, b=1, \cdots, m, \quad V\in\{\Omega, \Gamma\}, \quad \partial_{n+1}\phi\equiv\phi.$$

Taking into account the relations (5.4) we may list shortly our assumptions concerning the operators $q_{\Omega\lambda}$, $q_{\Gamma\lambda}$:

(a.5.0) $\quad q_{ij}^{\Gamma ab}=q_l^{\Gamma ab}=h_\Gamma^a\equiv 0$ if $a\in M_D$, (cf. (1.5)),

$$q_{ij}^{\Omega}=q_{ij}^{\Omega\infty}+q_{ij}^{\Omega s},\quad q_{il}^{\Gamma}=q_{il}^{\Gamma\infty}+q_{il}^{\Gamma s},\quad q_l^{\Gamma}=q_l^{\Gamma\infty}+q_l^{\Gamma s},$$

(a.5.1) $\quad q_{ij}^{\Omega\infty},\ q_{il}^{\Gamma\infty},\ q_l^{\Gamma\infty}\in B^{K-1}(\bar{\Omega}),\quad q_l^{\Omega\infty}\in B^{K-2}(\bar{\Omega}),$

$$q_{ij}^{\Omega s},\ q_{il}^{\Gamma s},\ q_l^{\Gamma s}\in H^{K-1}(\Omega),\quad q_l^{\Omega s}\in H^{K-2}(\Omega),$$

(a.5.2) $\quad {}^t q_{ij}^{\Omega}=q_{ji}^{\Omega},\quad {}^t q_i^{\Gamma}+q_i^{\Gamma}=0,\quad -q_{ij}^{\Gamma ab}=q_{ij}^{\Omega ab}$ if $a\in M_N$,

there exist constants $\delta_1,\ \delta_2>0$ such that

(a.5.3) $$-\sum_{i,j=1}^{n}(q_{ij}^{\Omega}\partial_j w,\ \partial_i w)+\sum_{i=1}^{n}\langle q_i^{\Gamma}\partial_i w,\ w\rangle\geqq\delta_1\|w\|_1^2-\delta_2\|w\|_0^2$$

for arbitrary $w\in H_D^2(\Omega)$, (cf. (1.9)),

(a.5.4) $$\sum_{i=1}^{n}n_i q_i^{\Gamma}=0\quad\text{on } \Gamma.$$

To investigate the problem (5.1) we shall adopt the well known method of coercive bilinear forms (cf. for example [2], Sect. 8). First let us discuss the uniqueness of solutions in $H_D^2(\Omega)$ and the existence of weak solutions in $H_D^1(\Omega)$. Assuming $w\in H_D^2(\Omega)$, multiplying (5.1) by $v\in H_D^1(\Omega)$ and integrating by parts we obtain

(5.5)
$$\begin{aligned}(q_{\Omega\lambda}[w],\ v)=&-\sum_{i,j=1}^{n}(q_{ij}^{\Omega}\partial_j w,\ \partial_i v)+\sum_{i,j=1}^{n}\langle n_i q_{ij}^{\Omega}\partial_j w,\ v\rangle\\ &-\sum_{i,j=1}^{n}(\partial_i(q_{ij}^{\Omega})\partial_j w,\ v)+\sum_{l=1}^{n+1}(q_l^{\Omega}\partial_l w,\ v)+\lambda(w,\ v),\\ \langle q_{\Gamma\lambda}[w],\ v\rangle=&\sum_{i,j=1}^{n}\langle n_i q_{ij}^{\Gamma}\partial_j w,\ v\rangle+\sum_{k=1}^{2}S_k(q^{\Gamma})[w,\ v]\\ &+\Big\langle\Big(\sum_{i=1}^{n}n_i q_{in+1}^{\Gamma}+q_{n+1}^{\Gamma}\Big)w,\ v\Big\rangle\qquad(\text{cf. (2.8) with } R=q^{\Gamma}=(q_1^{\Gamma},\ \cdots,\ q_n^{\Gamma})).\end{aligned}$$

In a consequence

(5.6) $$(q_{\Omega\lambda}[w],\ v)+\langle q_{\Gamma\lambda}[w],\ v\rangle=q_\lambda[w,\ v],$$

where

(5.7)
$$\begin{aligned}q_\lambda[w,\ v]=&-\sum_{i,j=1}^{n}(q_{ij}^{\Omega}\partial_j w,\ \partial_i v)+\sum_{k=1}^{2}S_k(q^{\Gamma})[w,\ v]+\lambda(w,\ v)\\ &-\sum_{i,j=1}^{n}(\partial_i(q_{ij}^{\Omega})\partial_j w,\ v)+\sum_{l=1}^{n+1}(q_l^{\Omega}\partial_l w,\ v)+\Big\langle\Big(\sum_{i=1}^{n}n_i q_{in+1}^{\Gamma}+q_{n+1}^{\Gamma}\Big)w,\ v\Big\rangle.\end{aligned}$$

Applying the trace theory (cf. Theorem Ap. 4b) one can prove that there exist a constant $C(\delta_1,\ \|\sum_{i=1}^{n}n_i q_{in+1}^{\Gamma}+q_{n+1}^{\Gamma}\|_{\infty,0},\ \Gamma)$ for which

$$\langle q^{\Gamma}_{n+1}w,\ w\rangle\leqq\frac{1}{4}\delta_1\|w\|_1^2+C\Big(\delta_1,\ \Big\|\sum_{i=1}^{n}n_i q^{\Gamma}_{in+1}+q^{\Gamma}_{n+1}\Big\|_{\infty,0},\ \Gamma\Big)\|w\|_0^2. \tag{5.8}$$

Assuming that γ_∞ is a constant such that

$$\sum_{i,j,k=1}^{n}\|\partial_k q^{\Omega}_{ij}\|_{\infty,0}+\sum_{l=1}^{n+1}\|q^{\Omega}_l\|_{\infty,0}+\Big\|\sum_{i=1}^{n}n_i q^{\Gamma}_{in+1}+q^{\Gamma}_{n+1}\Big\|_{\infty,0}\leqq\gamma_\infty \tag{5.9}$$

using Schwarz's inequality and (a.5.3) we can show that

$$\begin{aligned}(q_{\Omega\lambda}[w],\ w)+\langle q_{\Gamma\lambda}[w],\ w\rangle&\geqq\frac{1}{2}\delta_1\|w\|_1^2\\ &+\Big(\lambda-\delta_2-C\Big(\delta_1,\ \Big\|\sum_{i=1}^{n}q^{\Gamma}_{in+1}+q^{\Gamma}_{n+1}\Big\|_{\infty,0},\ \Gamma\Big)-\delta_1^{-1}\gamma_\infty^2\Big)\|w\|_0^2\\ &\geqq\frac{1}{2}\delta_1\|w\|_1^2+(\lambda-\delta_2-\mu_0)\|w\|_0^2,\end{aligned} \tag{5.10}$$

for some constant $\mu_0=\mu_0(\delta_1,\ \gamma_\infty,\ \Gamma)$ for which

$$\mu_0\geqq C\Big(\delta_1,\ \Big\|\sum_{i=1}^{n}n_i q^{\Gamma}_{in+1}+q^{\Gamma}_{n+1}\Big\|_{\infty,0},\ \Gamma\Big)+\delta_1^{-1}\gamma_\infty^2. \tag{5.11}$$

If we choose $\lambda>0$ so that

$$\lambda>\mu_0+\delta_2, \tag{5.12}$$

then the uniqueness of solutions from the space $H^2_D(\Omega)$ holds. Furthermore from the Schwarz's inequality, the inequality (5.10) and the density of the space $H^2_D(\Omega)$ in $H^1_D(\Omega)$ it follows that the bilinear form $q_\lambda[w,\ v]$ is continuous and coercive on $H^1_D(\Omega)\times H^1_D(\Omega)$ if (5.12) is satisfied. Thus, applying the Lax-Milgram theorem (cf. for example [2], p. 99) we can show the existence of weak solutions of the problem (5.1) in the space $H^1_D(\Omega)$. Usual methods of considerations (cf. [2], Sect. 9, [13], [19]) lead to the regularity theorem.

THEOREM 5.1. *Assume that* (a.5.0)-(a.5.4) *are valid. Let L be an integer such that $2\leqq L\leqq K$. Let γ_K be a constant such that*

$$\begin{aligned}&\sum_{i,j=1}^{n}(\|q^{\Omega\infty}_{ij}\|_{\infty,K-1}+\|q^{\Omega s}_{ij}\|_{K-1})+\sum_{l=1}^{n+1}\Big(\sum_{i=1}^{n}(\|q^{\Gamma\infty}_{il}\|_{\infty,K-1}+\|q^{\Gamma s}_{il}\|_{K-1})\\ &+(\|q^{\Omega\infty}_{l}\|_{\infty,K-2}+\|q^{\Omega s}_{l}\|_{K-2})+(\|q^{\Gamma\infty}_{l}\|_{\infty,K-1}+\|q^{\Gamma s}_{l}\|_{K-1})\Big)\leqq\gamma_K.\end{aligned} \tag{5.13}$$

Then, there exist $\lambda_0>0$ depending only on γ_K, δ_1, δ_2 and Γ essentially, such that for $\lambda\geqq\lambda_0$ and given $h_\Omega\in H^{L-2}(\Omega)$, $h_\Gamma\in H^{L-3/2}(\Gamma)$, the problem (5.1) *admits a unique solution $w\in H^L_D(\Omega)$ satisfying the estimate*

$$\|w\|_L\leqq C(K,\ \gamma_K,\ \Gamma,\ \delta_1,\ \delta_2,\ n,\ m,\ \lambda)\{\|h_\Omega\|_{L-2}+\|h_\Gamma\|_{L-3/2}\}. \tag{5.14}$$

If the data depend additionally on $t \in [0, T]$ we have the following.

THEOREM 5.2. *Let* (a.5.0)-(a.5.4) *be valid,* λ *be the same as in Theorem* 5.1, $T>0$ *and* $J=[0, T]$. *If* $h_\Omega(t) \in X^{K-2,0}(J, \Omega)$ *and* $h_\Gamma(t) \in X^{K-2,1/2}(J, \Gamma)$ *then there exists a unique* $w(t) \in X_D^{K-2,2}(J, \Omega)$ *which is a solution of the problem*

$$(5.15) \qquad q_{\Omega\lambda}[w(t)]=h_\Omega(t) \text{ in } \Omega,\ q_{\Gamma\lambda}[w(t)]=h_\Gamma(t),\ w_D(t)=0 \text{ on } \Gamma \text{ for } t \in J.$$

The theorems 5.1, 5.2 can be proved in a similar way to the corresponding theorems in the paper [19]. The details will be given in the author's separate paper.

The main theorem of the present section we can formulate as follows.

THEOREM 5.3. *Let* (a.0)-(a.5) *be valid and* u_0, u_1, $f_\Omega(t)$, $f_\Gamma(t)$ *be the same as in Theorem* 1.1.

(i) *Let* $(v(t), w(t)) \in Z_c$ *and* $p_{\Omega\lambda}$, $g_\Omega(t)$, $p_{\Gamma\lambda}$, $g_\Gamma(t)$ *be the same as in* (4.12)-(4.20). *Then there exists a* λ *depending only on* K *and* B *such that there exists a unique* $z(t) \in X_D^{K-2,2}([0, T], \Omega)$ *satisfying the conditions*

$$(5.16) \qquad \begin{gathered} p_{\Omega\lambda}[z(t)]=g_\Omega(t) \text{ in } \Omega, \quad p_{\Gamma\lambda}[z(t)]=g_\Gamma(t), \quad z_D(t)=0 \quad \text{on } \Gamma, \\ \text{for every} \quad t \in [0, T], \end{gathered}$$

$$(5.17) \qquad \partial_t^M z(0)=0 \qquad \text{for} \quad 0 \leq M \leq K-2,$$

$$(5.18) \qquad |z|_{K-2,2,[0,T]} \leq \Lambda_E, \qquad |z|_{K-3,2,[0,T]} \leq \varepsilon_E,$$

for some T, Λ_E, ε_E *depending only on* K, B, Λ_H.

(ii) *Let* $(v(t), w(t)) \in Z$. *Then there exists a* T *depending only on* K, B, Λ_H, Λ_E *such that for the present* λ *the inequality*

$$(5.19) \qquad \begin{aligned} &-\sum_{i,j=1}^{n}(a_{ij}(t, U(t))\partial_j z, \partial_i z)+\sum_{l=1}^{n}\langle b_{\Gamma l}(t, U(t))\partial_l z, z\rangle+\lambda\|z\|_0^2 \\ &+\Big\langle\Big\{\sum_{i=1}^{n} n_i b_{in+1}(t, U(t))+b_{\Gamma n+1}(t, U(t))\Big\}z, z\Big\rangle \\ &+\sum_{l=1}^{n+1}(b_{\Omega l}(t, U(t))\partial_l z, z)-\sum_{i,j=1}^{n}(\partial_i a_{ij}\partial_j z, z) \geq \frac{1}{2}\delta_1\|z\|_1^2 \end{aligned}$$

is valid for $t \in [0, T]$ *and* $z \in H_D^2(\Omega)$ *(recall that* $U(t)=(v(t), D_x^1(u(t)+w(t)))$*).*

In the proof of Theorem 5.3 we use the following

LEMMA 5.4. *Assume that* (a.0)-(a.5) *are valid and that* u_0, u_1 *are the same as in the Theorem* 1.1. *For indices satisfying* (5.4) *let us put* $U^0=(u_1, D_x^1 u_0)$ *and*

$$
\begin{aligned}
& q_{ij}^{\Omega\infty}=a_{ij}(0,0), && q_{ij}^{\Omega s}=(a_{ij})_1(0,U^0), && (\text{cf. } (2.6)),\\
& q_{il}^{\Gamma\infty}=b_{il}(0,0), && q_{il}^{\Gamma s}=(b_{il})_1(0,U^0), &&\\
(5.20)\qquad & q_l^{\Omega\infty}=b_{\Omega l}(0,0), && q_l^{\Omega s}=a_l^*(0,U^0,u_2), && (\text{cf. } (4.13)),\\
& q_l^{\Gamma\infty}=b_{\Gamma l}(0,0), && q_l^{\Gamma s}=(b_{\Gamma l})_1(0,U^0), &&\\
& q_{ij}^{\Omega}=q_{ij}^{\Omega\infty}+q_{ij}^{\Omega s}, && q_{il}^{\Gamma}=q_{il}^{\Gamma\infty}+q_{il}^{\Gamma s}, && q_l^V=q_l^{V\infty}+q_l^{Vs}.
\end{aligned}
$$

Then, the present q_{ij}^{Ω}, q_{il}^{Γ}, q_l^V, *satisfy* (a.5.0)-(a.5.4). *Furthermore, the following inequality is valid*

$$
\begin{aligned}
(5.21)\qquad & \sum_{i=1}^{n}\Big\{\sum_{j=1}^{n}(\|q_{ij}^{\Omega\infty}\|_{\infty,K-1}+\|q_{ij}^{\Omega s}\|_{K-1})+\sum_{l=1}^{n+1}(\|q_{il}^{\Gamma\infty}\|_{\infty,K-1}+\|q_{il}^{\Gamma s}\|_{K-1})\Big\}\\
&+\sum_{l=1}^{n+1}(\|q_l^{\Omega\infty}\|_{\infty,K-2}+\|q_l^{\Omega s}\|_{K-2}+\|q_l^{\Gamma\infty}\|_{\infty,K-1}+\|q_l^{\Gamma s}\|_{K-1})\leqq C_3(K,B).
\end{aligned}
$$

PROOF OF LEMMA 5.4. Since the initial data u_0, u_1 belong to the space $H^{\infty,1}(\bar{\Omega}, D(U_0))$ (cf. (1.7), (1.10), (1.16)) we can see that (a.k) implies (a.5.k), $k=0, 2, 3, 4$. Applying Theorem Ap. 3, (Ap. 1), and accounting the relations (1.16), (1.17), (3.4) we obtain (a.5.1), (5.21).

In the investigation of the right-hand side of the equation (5.16) we shall use the following.

LEMMA 5.5. *Assume that* (a.1) *is valid. Let* u_0, u_1, $f_\Omega(t)$, $f_\Gamma(t)$ *be the same as in Theorem* 1.1. *Let* $(v(t), w(t))\in Z_c$ *and* $g_\Omega(t)$, $g_\Gamma(t)$ *be the same as in* (4.14). *Then the following two assertions are valid:*

$$
(5.22)\qquad (\partial_t^M g_\Omega)(0)=0 \text{ on } \Omega,\quad (\partial_t^M g_\Gamma)(0)=0 \text{ on } \Gamma,\quad \text{for } 0\leqq M\leqq K-2
$$

$$
g_\Omega(t)\in X^{K-2,0}([0,T],\Omega),\qquad g_\Gamma(t)\in X^{K-2,1/2}([0,T],\Gamma) \text{ and}
$$

$$
\begin{aligned}
(5.23)\qquad & |g_\Omega|_{K-2,0,[0,T]}+|g_\Gamma|_{K-2,1/2,[0,T]}\leqq C_1(K,B,\Lambda_H)\\
&+C_2(K,B,\Lambda_H)T\Lambda_E+C_3(K,B,\Lambda_H)T\Lambda_E^2+C_1(K,B,\Lambda_H,\Lambda_E)\varepsilon_E.
\end{aligned}
$$

PROOF OF (5.22). From (4.22), (4.6) and (3.3) it follows the equality

$$
\partial_t^M\Big(f_\Omega(t)-\sum_{I,J=0}^{n}a_{IJ}(t,U^0(t))\partial_I^{s_I}\partial_J^{s_J}u^{2-s_I-s_J}(t)-a_\Omega(t,U^0(t))\Big)(0)=0
$$

on Ω for $0\leqq M\leqq K-2$. It is also clear that $\Big(\partial_t^M\int_0^t(v(s)-\partial_s u^0(s))ds\Big)(0)=0$ for the same M. Thus $(\partial_t^M G_{\Omega 1})(0)=0$ on Ω for $0\leqq M\leqq K-2$. Analogousely, from (4.18), (4.22) and (3.5) it follows that $(\partial_t^M G_{\Gamma 1})(0)=0$ on Γ for $0\leqq M\leqq K-2$. The definitions (4.16), (4.17) and (4.19), (4.20) together with the relations (4.22) give $(\partial_t^M G_{Vk})(0)=0$, $0\leqq M\leqq K-2$, $V\in\{\Omega,\Gamma\}$, $k=2, 3$. In consequence, (5.22) holds true.

PROOF OF (5.23). First let us note that due to (1.14), (1.17), (4.7), (4.23), applying Theorem Ap. 3 and (Ap. 1) we obtain

$$\begin{aligned}&G_{\Omega 1}(t)\in X^{K-2,0}([0, T], \Omega), \qquad G_{\Gamma 1}(t)\in X^{K-2,1/2}([0, T], \Gamma)\\&|G_{\Omega 1}(t)|_{K-2,0,[0,T]}+|G_{\Gamma 1}(t)|_{K-2,1/2,[0,T]}\leqq C(K, B, \Lambda_H).\end{aligned}\tag{5.24}$$

Applying an analogy of (Ap. 2) with $G(t, u(t))=\partial_I^{sI}\partial_J^{sJ}u^{2-sI-sJ}(t)$, $v(t)=d a_{IJ}(0, U^0(0))D_x^1 w(t)$ and next (Ap. 1), and using (3.4), (4.23) we get

$$\begin{aligned}&\Big|-\sum_{I,J=0}^{n} d a_{IJ}(0, U^0(0))D_x^1 w(t)\partial_I^{sI}\partial_J^{sJ}(u^{2-sI-sJ}(t)-u_{2-sI-sJ})\Big|_{K-2,0,[0,T]}\\&\leqq \sum_{I,J=0}^{n} C(K, B, \Lambda_H)\{T|d a_{IJ}(0, U^0(0))D_x^1 w(t)|_{K-2,1,[0,T]}\\&\quad+|d a_{IJ}(0, U^0(0))D_x^1 w(t)|_{K-3,1,[0,T]}\}\leqq C(K, B, \Lambda_H)\{T|w(t)|_{K-2,2,[0,T]}\\&\quad+|w(t)|_{K-3,2,[0,T]}\}\leqq C(K, B, \Lambda_H)\{T\Lambda_E+\varepsilon_E\}.\end{aligned}\tag{5.25}$$

Similary, using (Ap. 2) with $G(t, u(t))=d a_{IJ}(t, U^0(t))$ and Theorem Ap. 2 we have

$$\begin{aligned}&\Big|-\sum_{I,J=0}^{n}[d a_{IJ}(t, U^0(t))-d a_{IJ}(0, U^0(0))]D_x^1 w(t)\partial_I^{sI}\partial_J^{sJ}u^{2-sI-sJ}(t)\Big|_{K-2,0,[0,T]}\\&\leqq C(K, B, \Lambda_H)\{T|D_x^1 w(t)|_{K-2,1,[0,T]}+|D_x^1 w(t)|_{K-3,1,[0,T]}\}\\&\quad\times\sum_{I,J=0}^{n}|\partial_I^{sI}\partial_J^{sJ}u^{2-sI-sJ}(t)|_{K-2,0,[0,T]}\leqq C(K, B, \Lambda_H)\{T\Lambda_E+\varepsilon_E\}.\end{aligned}\tag{5.26}$$

In the same manner we obtain

$$\begin{aligned}&|-[d a_\Omega(t, U^0(t))-d a_\Omega(0, U^0(0))]D_x^1 w(t)|_{K-2,0,[0,T]}\\&\leqq C(K, B, |U^0|_{K-2,1,[0,T]})\{T|D_x^1 w(t)|_{K-2,1,[0,T]}+|D_x^1 w(t)|_{K-3,1,[0,T]}\}\\&\leqq C(K, B, \Lambda_H)\{T\Lambda_E+\varepsilon_E\},\end{aligned}\tag{5.27}$$

$$\begin{aligned}&\Big|-\sum_{i,j=1}^{n}[a_{ij}(t, U^0(t))-a_{ij}(0, U^0(0))]\partial_i\partial_j w(t)|_{K-2,0,[0,T]}\\&\leqq C(K, B, \Lambda_H)\sum_{i,j=1}^{n}\{T|\partial_i\partial_j w(t)|_{K-2,0,[0,T]}+|\partial_i\partial_j w(t)|_{K-3,0,[0,T]}\}\\&\leqq C(K, B, \Lambda_H)\{T\Lambda_E+\varepsilon_E\}.\end{aligned}\tag{5.28}$$

From the estimates (5.25)-(5.28) and the definition (4.16) it follows the relations:

$$\begin{aligned}&G_{\Omega 2}(t, v(t), w(t))\in X^{K-2,0}([0, T], \Omega),\\&|G_{\Omega 2}(t, v(t), w(t))|_{K-2,0,[0,T]}\leqq C(K, B, \Lambda_H)\{T\Lambda_E+\varepsilon_E\}.\end{aligned}\tag{5.29}$$

Now, let us estimate all terms of $G_{\Omega 3}$. Applying Theorem Ap. 2, the estimate (Ap. 4) and using relations (4.7), (4.23) we get

$$
\begin{aligned}
&\Big|-\Sigma''\int_0^1 d^2 a_{IJ}(t, U(\theta))(D_x^1 w(t), D_x^1 w(t))\partial_I^{sI}\partial_J^{sJ} u^{2-sI-sJ}(t)d\theta \\
&\quad -\sum_{i,j=1}^n \int_0^1 d^2 a_{ij}(t, U(\theta))(D_x^1 w(t), D_x^1 w(t))\partial_i\partial_j(u^0(t)+w(t))d\theta\Big|_{K-2,0,[0,T]} \\
(5.30)\quad &\leqq \Sigma''\int_0^1 |d^2 a_{IJ}(t, U(\theta))(D_x^1 w(t), D_x^1 w(t))|_{K-2,0,[0,T]}d\theta \\
&\quad \times |\partial_I^{sI}\partial_J^{sJ} u^{2-sI-sJ}(t)|_{K-2,0,[0,T]} \\
&\quad +\sum_{i,j=1}^n \int_0^1 |d^2 a_{ij}(t, U(\theta))(D_x^1 w(t), D_x^1 w(t))|_{K-2,0,[0,T]}d\theta \\
&\quad \times |\partial_i\partial_j(u^0(t)+w(t))|_{K-2,0,[0,T]} \\
&\leqq C(K, B, \Lambda_H, \Lambda_E)|D_x^1 w(t)|_{K-2,0,[0,T]}|D_x^1 w(t)|_{K-3,1,[0,T]} \\
&\leqq C(K, B, \Lambda_H, \Lambda_E)\varepsilon_E
\end{aligned}
$$

and similary

$$
\begin{aligned}
&\Big|-\int_0^1 d^2 a_\Omega(t, U(\theta))(D_x^1 w(t), D_x^1 w(t))d\theta\Big|_{K-2,0,[0,T]} \\
(5.31)\quad &\leqq C(K, B, \Lambda_H, \Lambda_E)|D_x^1 w(t)|_{K-2,0,[0,T]}|D_x^1 w(t)|_{K-3,1,[0,T]} \\
&\leqq C(K, B, \Lambda_H, \Lambda_E)\varepsilon_E.
\end{aligned}
$$

Applying Theorem Ap. 2, (Ap. 2), (Ap. 1) and (4.7), (4.23) we obtain

$$
\begin{aligned}
&\Big|-\sum_{i,j=1}^n d a_{ij}(t, U^0(t))D_x^1 w(t)\partial_i\partial_j w(t)\Big|_{K-2,0,[0,T]} \\
&\leqq C(K)\sum_{i,j=1}^n \{T|d a_{ij}(t, U^0(t))D_x^1 w(t)|_{K-2,1,[0,T]} \\
(5.32)\quad &\quad +|d a_{ij}(t, U^0(t))D_x^1 w(t)|_{K-3,1,[0,T]}\}\,|\partial_i\partial_j w(t)|_{K-2,0,[0,T]} \\
&\leqq C(K, B, \Lambda_H)\{T|D_x^1 w(t)|_{K-2,1,[0,T]}+|D_x^1 w(t)|_{K-3,1,[0,T]}\}\,|w|_{K-2,2,[0,T]} \\
&\leqq C(K, B, \Lambda_H)T\Lambda_E^2+C(K, B, \Lambda_H, \Lambda_E)\varepsilon_E.
\end{aligned}
$$

Thus from the definition (4.17) and relations (5.30)-(5.32) it follows that

$$
(5.33)\quad
\begin{aligned}
&G_{\Omega 3}(t, v(t), w(t))\in X^{K-2,0}([0, T], \Omega), \\
&|G_{\Omega 3}(t, v(t), w(t))|_{K-2,0,[0,T]}\leqq C(K, B, \Lambda_H)T\Lambda_E^2+C(K, B, \Lambda_H, \Lambda_E)\varepsilon_E.
\end{aligned}
$$

Now, let us estimate the boundary terms $G_{\Gamma k}$, $k=2, 3$. Due to the Theorem Ap. 4a we have

$$|G_{\Gamma k}(t, v(t), w(t))|_{K-2,1/2,[0,T]} \leqq C(K, \Gamma)|G_{\Gamma k}(t, v(t), w(t))|_{K-2,1,[0,T]}.$$

Applying (4.19), (Ap. 2), (4.7), (4.23) we obtain $G_{\Gamma 2}(t, v(t), w(t)) \in X^{K-2,1}([0, T], \Omega)$ and

$$\begin{aligned} &|G_{\Gamma 2}(t, v(t), w(t))|_{K-2,1,[0,T]} \\ &\leqq C(K, \Gamma)\left\{\left|\sum_{i=1}^{n}[d a_i(t, U^0(t)) - d a_i(0, U^0(0))]D_x^1 w(t)\right|_{K-2,1,[0,T]}\right. \\ &\qquad \left. + |[d a_\Gamma(t, U^0(t)) - d a_\Gamma(0, U^0(0))]D_x^1 w(t)|_{K-2,1,[0,T]}\right\} \\ &\leqq C(K, B, \Lambda_H)\{T|w(t)|_{K-2,2,[0,T]} + |w(t)|_{K-3,2,[0,T]}\} \\ &\leqq C(K, B, \Lambda_H)\{T\Lambda_E + \varepsilon_E\}. \end{aligned} \tag{5.34}$$

Applying (4.20), (4.23) and (Ap. 4) we get also $G_{\Gamma 3}(t, v(t), w(t)) \in X^{K-2,1}([0, T], \Omega)$ and

$$\begin{aligned} &|G_{\Gamma 3}(t, v(t), w(t))|_{K-2,1,[0,T]} \\ &\leqq C(K, B, \Lambda_H, \Lambda_E)|w(t)|_{K-2,2,[0,T]}|w(t)|_{K-3,2,[0,T]} \leqq C(K, B, \Lambda_H, \Lambda_E)\varepsilon_E. \end{aligned} \tag{5.35}$$

From (5.24), (5.29), (5.33)-(5.35) we obtain (5.23).

Proof of Theorem 5.3. First we prove (5.19). Put $U(t)=(v(t), D_x^1(u^0(t)+w(t)))$ for $(v(t), w(t)) \in Z$. By (Ap. 9) we have

$$\begin{aligned} &\sum_{i,j,k=1}^{n}\|\partial_k a_{ij}(t, U(t))\|_{\infty,0} + \sum_{l=1}^{n+1}\|b_{\Omega l}(t, U(t))\|_{\infty,0} \\ &\qquad + \left\|\left(\sum_{i=1}^{n} n_i b_{in+1} + b_{\Gamma n+1}\right)(t, U(t))\right\|_{\infty,0} \leqq C_1\{1+T|U|_{K-2,1,[0,T]}\} \\ &\leqq C_1\{1+T(C_2(K, B)+\Lambda_H+\Lambda_E)\} \qquad \text{for} \quad t \in [0, T]. \end{aligned} \tag{5.36}$$

Choose $T>0$ so that

$$T(C_2(K, B)+\Lambda_H+\Lambda_E) \leqq 1. \tag{5.37}$$

If we put $q_l^\Omega = b_{\Omega l}$, $q_{ij}^\Omega = a_{ij}$, $q_{in+1}^\Gamma = b_{in+1}$, $q_{n+1}^\Gamma = b_{\Gamma n+1}$, then the estimate (5.9) is valid with $\gamma_\infty = 2C_1$. Hence we can choose the constant μ_0 from (5.11) independent of K, B, Λ_H, Λ_E, ε_E and T. If $\lambda \geqq \mu_0 + \delta_2$ then by (5.10) we have (5.19). In the sequel μ_0 will allways denote the constant determined in the just prescribed way.

Now we prove the first part of Theorem 5.3. Let q_{il}^Γ, q_{ij}^Ω, q_l^V, $j, i=1, \cdots, n$, $l=1, \cdots, n+1$ be the same as in (5.20). By (5.21) we may put $\gamma_K = C_3(K, B)$ (cf. (5.13)). In the view of lemma 5.4 we can apply theorems 5.1, 5.2. Thus we can chose $\lambda \geqq \mu_0 + \delta_2$ depending only on K, B such that there exist unique solution $z(t) \in X_D^{K-2,2}([0, T], \Omega)$ satisfying (5.16) for $t \in [0, T]$. By (5.22) we have

$$p_{\Omega\lambda}[\partial_t^M z(0)]=0 \text{ in } \Omega, \quad p_\Gamma[\partial_t^M z(0)]=0, \quad \partial_t^M z_D(0)=0 \text{ on } \Gamma, \quad \text{for } 0 \leqq M \leqq K-2.$$

Hence by (5.10) and (5.12) we obtain (5.17).

Finally we prove the estimate (5.18). Differentiating (5.16) M-times in t and applying (5.14) with $L=K-M$ we get

$$\|\partial_t^M z(t)\|_{K-M} \leqq C_4(K, B)\{\|\partial_t^M g_\Omega(t)\|_{K-2-M} + \langle\!\langle \partial_t^M g_\Gamma(t) \rangle\!\rangle_{K-3/2-M}\} \tag{5.38}$$

for $t \in [0, T]$ and $0 \leqq M \leqq K-2$, where we have used the fact that the present γ_K and λ depend on K and B only. Combining (5.23) with (5.38) we have

$$\begin{aligned} |z|_{K-2,2,[0,T]} \leqq C_4(K, B)\{&C_1(K, B, \Lambda_H) + C_2(K, B, \Lambda_H) T \Lambda_E \\ &+ C_3(K, B, \Lambda_H) T \Lambda_E^2 + C_1(K, B, \Lambda_H, \Lambda_E)\varepsilon_E\}\,. \end{aligned} \tag{5.39}$$

If we choose Λ_E, ε_E and T so that

$$\Lambda_E = C_4(K, B)\left\{\sum_{k=1}^{3} C_k(K, B, \Lambda_H) + 1\right\}, \tag{5.40}$$

$$C_1(K, B, \Lambda_H, \Lambda_E)\varepsilon_E \leqq 1\,, \tag{5.41}$$

$$T\Lambda_E,\ T\Lambda_E^2 \leqq \varepsilon_E \leqq 1\,, \tag{5.42}$$

then we obtain

$$|z|_{K-2,2,[0,T]} \leqq \Lambda_E\,. \tag{5.43}$$

Furthermore, since $\partial_t^M z(t) = \int_0^t \partial_s^{M+1} z(s)\,ds$ for $0 \leqq M \leqq K-3$, we have

$$\|\partial_t^M z(t)\|_{K-1-M} \leqq \int_0^t \|\partial_s^{M+1} z(s)\|_{K-1-M}\,ds \tag{5.44}$$

From (5.44) it follows that

$$|z|_{K-3,2,[0,T]} \leqq T|z|_{K-2,2,[0,T]} \leqq T\Lambda_E \leqq \varepsilon_E\,. \tag{5.45}$$

Thus (5.18) is proved and the proof of Theorem 5.3 is complete.

6. Auxiliary theorems from the theory of linear hyperbolic problems.

Let us consider the problem

$$R_\Omega(t)[v(t)] \equiv \sum_{I,J=0}^{n} R_{IJ}^\Omega(t)\partial_I\partial_J v(t) = h_\Omega(t) \quad \text{in} \quad (0, T)\times\Omega\,, \tag{6.1}$$

$$R_\Gamma(t)[v(t)] \equiv \sum_{J=0}^{n}\left(\sum_{i=1}^{n} n_i R_{iJ}^\Gamma(t) + R_J^\Gamma(t)\right)\partial_J v(t) = h_\Gamma(t), \qquad v_D(t)=0 \tag{6.2}$$

$$\text{on} \quad (0, T)\times\Gamma, \qquad (\text{cf. } (1.6)),$$

(6.3) $$v(0)=v_0, \qquad (\partial_t v)(0)=v_1 \quad \text{in } \Omega,$$

where $R^{\Omega}_{IJ}=(R^{\Omega ab}_{IJ})$, $R^{\Gamma}_{iJ}=(R^{\Gamma ab}_{iJ})$, $R^{\Gamma}_{J}=(R^{\Gamma ab}_{J})$ are $m\times m$ matrices depending on t and x (cf. Remark 1.1, Sect. 1). In the present Section we assume that the indices satisfy the relations

(6.4) $$I,\ J=0, \cdots, n, \quad i,\ j=1, \cdots, n, \quad a,\ b=1, \cdots, m, \quad V\in\{\Omega, \Gamma\}.$$

The functions

(6.5) $$h_V={}^t(h^1_V, \cdots, h^m_V), \qquad v_k={}^t(v^1_k, \cdots, v^m_k), \qquad k=0, 1$$

are given vector functions and $v={}^t(v^1, \cdots, v^m)$ is the unknown one.

We assume that for all indices satisfying the relations (6.4) and arbitrary $t\in[-T_1, T_1]$, $T_1\in(0, T_0]$,

(a.6.0) $$R^{\Gamma ab}_{iJ}(t)=R^{\Gamma ab}_{J}(t)=h^a_{\Gamma}(t)=0 \quad \text{if } a\in M_D \quad (\text{cf. } (1.5)),$$

$$R^{\Omega}_{IJ}=R^{\Omega\infty}_{IJ}+R^{\Omega s}_{IJ}, \quad R^{\Gamma}_{iJ}=R^{\Gamma\infty}_{iJ}+R^{\Gamma s}_{iJ}, \quad R^{\Gamma}_{J}=R^{\Gamma\infty}_{J}+R^{\Gamma s}_{J} \quad \text{where}$$

(a.6.1) $$R^{\Omega\infty}_{IJ},\ R^{\Gamma\infty}_{iJ},\ R^{\Gamma\infty}_{J}\in B^{K-1}([-T_1, T_1], \bar{\Omega})$$

$$R^{\Omega s}_{IJ},\ R^{\Gamma s}_{iJ},\ R^{\Gamma s}_{J}\in Y^{K-2,1}([-T_1, T_1], \Omega)$$

(a.6.2) $$R^{\Omega}_{IJ}={}^tR^{\Omega}_{JI}, \quad {}^tR^{\Gamma}_{i}+R^{\Gamma}_{i}=0, \quad -R^{\Gamma ab}_{ij}=R^{\Omega ab}_{ij} \quad \text{if } a\in M_N \quad (\text{cf. } (1.5))$$

$$R^{\Omega}_{00}(t)\geqq\delta_0 I \qquad (I \text{ denotes the } m\times m \text{ unit matrix}),$$

(a.6.3) $$-\sum_{i,j=1}^{n}(R^{\Omega}_{ij}(t)\partial_j w, \partial_i w)+\sum_{i=1}^{n}\langle R^{\Gamma}_{i}(t)\partial_i w, w\rangle\geqq\delta_1\|w\|^2_1-\delta_2\|w\|^2_0$$

for some positive constants δ_0, δ_1, δ_2 and arbitrary $w\in H^2_D(\Omega)$,

(a.6.4) $$\sum_{i=1}^{n}n_i(x)R^{\Gamma}_{i}(t, x)=0 \qquad \text{for} \quad x\in\Gamma,$$

(a.6.5) $$S\Big\{\sum_{i=1}^{n}n_i(x)(R_{0i}+R_{i0})(t, x)+2R_0(t, x)\Big\}\xi\cdot\xi\geqq0 \qquad \text{for} \quad x\in\Gamma \quad \text{and}$$

for arbitrary $\xi={}^t(\xi^1, \cdots, \xi^m)$ such that $\xi^a=0$ if $a\in M_D$,

where $\quad R_0=\sum_{i=1}^{n}n_iR^{\Gamma}_{i0}+R^{\Gamma}_{0}.$

Let us define the energy norm

(6.6) $$E(R(t))[v(t)]=\|(R^{\Omega}_{00}(t))^{1/2}v(t)\|^2_0-\sum_{i,j=1}^{n}(R^{\Omega}_{ij}(t)\partial_j v(t), \partial_i v(t)) +S_1(R(t))[v(t), v(t)]+d\|v(t)\|^2_0,$$

where $S_1(R(t))$ is the bilinear form defined in the formula (2.11) with $R=$

$(R_1^\Gamma, \cdots, R_n^\Gamma)$ and d is a constant determined in the following way. Let $S_2(R(t))$ be defined by (2.12) with $R=(R_1^\Gamma, \cdots, R_n^\Gamma)$ and let $M(K, T_1)$ be a constant such that

$$\begin{aligned}&\sum_{I,J=0}^{n}(|R_{IJ}^{\Omega\infty}|_{\infty,K-1,T_1}+|R_{IJ}^{\Omega s}|_{K-2,1,[-T_1,T_1]})\\ &+\sum_{J=0}^{n}\Big\{\sum_{i=1}^{n}(|R_{iJ}^{\Gamma\infty}|_{\infty,K-1,T_1}+|R_{iJ}^{\Gamma s}|_{K-2,1,[-T_1,T_1]})\\ &+|R_J^{\Gamma\infty}|_{\infty,K-1,T_1}+|R_J^{\Gamma s}|_{K-2,1,[-T_1,T_1]}\Big\}\leqq M(K, T_1).\end{aligned} \tag{6.7}$$

Using (2.8) and (a.6.3) we can prove

$$\begin{aligned}&E(R(t))[v(t)]+S_2(R(t))[v(t), v(t)]\\ &\geqq\delta_0\|\partial_t v(t)\|_0^2+\delta_1\|v(t)\|_0^2+(d-\delta_2)\|v(t)\|_0^2.\end{aligned} \tag{6.8}$$

Thus, by (2.14) and (6.7) we have

$$\begin{aligned}E(R(t))[v(t)]&\geqq\delta_0\|\partial_t v(t)\|_0^2+\delta_1\|v(t)\|_0^2+(d-\delta_2)\|v(t)\|_0^2\\ &-CM(K, T_1)\|v(t)\|_1\|v(t)\|_0\geqq\delta_0\|\partial_t v(t)\|_0^2+\delta_1\|v(t)\|_1^2\\ &+(d-\delta_2)\|v(t)\|_0^2-(\delta_1/2)\|v(t)\|_1^2-((CM(K, T_1))^2/(2\delta_1))\|v(t)\|_0^2.\end{aligned} \tag{6.9}$$

If we take

$$d=\delta_2+(CM(K, T_1))^2/(2\delta_1), \tag{6.10}$$

then we obtain

$$E(R(t))[v(t)]\geqq\delta_0\|\partial_t v(t)\|_0^2+(\delta_1/2)\|v(t)\|_1^2 \quad \text{for} \quad v(t)\in H_D^2(\Omega). \tag{6.11}$$

This is the manner of choosing the constant d.

Now, we describe the compatibility conditions for the problem (6.1)-(6.3). Let $v_{M+2}=v_{M+2}(x)$, $0\leqq M\leqq K-3$ be defined by the recursive formula (we use the same notations as in (3.3))

$$R_{00}^{\Omega}(0)v_{M+2}=(\partial_t^M h_\Omega)(0)-\sum{}'\binom{M}{k}(\partial_t^k R_{IJ}^{\Omega})(0)\partial_I^{sI}\partial_J^{sJ}v_{M+2-k-sI-sJ}. \tag{6.12}$$

We shall say that v_0, v_1, $h_\Omega(t)$, $h_\Gamma(t)$ satisfy *the compatibility condition of order* $K-3$ for (6.1)-(6.3) if

$$\begin{aligned}&\sum_{k=0}^{M}\binom{M}{k}\Big\{\sum_{J=0}^{n}\Big[\sum_{i=1}^{n}n_i(\partial_t^k R_{iJ}^\Gamma)(0)+(\partial_t^k R_J^\Gamma)(0)\Big]\partial_J^{sJ}v_{M+1-k-sJ}\Big\}=(\partial_t^M h_\Gamma)(0),\\ &\qquad \text{for} \quad 0\leqq M\leqq K-3, \qquad v_{MD}(0)=0 \quad \text{for} \quad 0\leqq M\leqq K-2, \quad \text{on } \Gamma.\end{aligned} \tag{6.13}$$

The solvability of the problem (6.1)-(6.3) is described in the following.

THEOREM 6.1. *Assume that* (a.6.0)-(a.6.5) *are valid and* $T \in (0, T_1)$.

(i) *Let*

$$v_0 \in H_D^{K-1}(\Omega), \quad v_1 \in H_D^{K-2}(\Omega), \quad h_\Omega(t) \in X^{K-3,0}([0, T], \Omega),$$

$$h_\Gamma(t) \in X^{K-3,1/2}([0, T], \Gamma), \quad \partial_t^{K-3} h_\Omega(t) \in \mathrm{Lip}([0, T], L^2(\Omega)), \tag{6.14}$$

$$\partial_t^{K-3} h_\Gamma(t) \in \mathrm{Lip}([0, T], H^{1/2}(\Gamma)),$$

(6.15) v_0, v_1, $h_\Omega(t)$, $h_\Gamma(t)$ *satisfy the compatibility condition of order* $K-3$ *for* (6.1)-(6.3),

then (6.1)-(6.3) *admits a solution* $v \in X_D^{K-1,0}([0, T], \Omega)$ *with the property*

$$\partial_t^M v(0) = v_M \qquad \textit{for} \quad 2 \leqq M \leqq K-1. \tag{6.16}$$

(ii) *Let* $v \in X_D^{2,0}([0, T], \Omega)$, $h_\Omega(t) = R_\Omega(t)[v(t)]$, $h_\Gamma(t) = R_\Gamma(t)[v(t)]$, *then*

$$\|D^1 v(t)\|_0^2 \leqq C \left\{ \|(D^1 v)(0)\|_0^2 + \int_0^t (\|h_\Omega(s)\|_0^2 + \langle\!\langle h_\Gamma(s)\rangle\!\rangle_{1/2}^2) ds \right\}$$
$$\textit{for} \quad t \in [0, T],\ C = C(T_1, M(K, T_1), \delta_0, \delta_1, \delta_2, n, m, \Gamma). \tag{6.17}$$

(iii) *If* $v \in X_D^{K-1,0}([0, T], \Omega)$ *and* h_Ω, h_Γ *satisfy* (6.14) *then*

$$E(R(t))[\partial_t^{K-2} v(t)] \leqq e^{Ct} \{ (E(R(t))[\partial_t^{K-2} v(t)])|_{t=0} + C t^{1/2} F(t) \} \tag{6.18}$$

for $t \in [0, T]$, *where* $C = C(T_1, M(K, T_1), \delta_0, \delta_1, \delta_2, n, m, \Gamma)$ *and*

$$F(t) = \|(D^{K-1} v)(0)\|_0^2 + |h_\Omega|_{K-3,0,[0,t]}^2 + \langle h_\Gamma \rangle_{K-3,1/2,[0,t]}^2$$
$$+ \operatorname*{ess\,sup}_{0 \leqq s \leqq t} \|\partial_s^{K-2} h_\Omega(s)\|_0^2 + \operatorname*{ess\,sup}_{0 \leqq s \leqq t} \langle\!\langle \partial_s^{K-2} h_\Gamma(s) \rangle\!\rangle_{1/2}^2. \tag{6.19}$$

REMARK 6.1. Theorem 6.1 can be proved exactly in the same way as the corresponding theorem in Shibata's paper [19]. The details will be given in the separate author's paper. Let us remark only that in Shibata's approach it is essential that the coefficients of the operators $R_\Omega(t)$, $R_\Gamma(t)$ are defined for arbitrary $t \in [-T_1, T_1] \supset [0, T]$ and the assumptions (a.6.0)-(a.6.5) are satisfied for all such t.

The next theorem describes the properties of the linear hyperbolic problems of the type (4.28), which are used in our iteration scheme.

THEOREM 6.2. *Assume that* (a.0)-(a.5) *are valid and* u_0, u_1, f_Ω, f_Γ *are the same as in Theorem* 1.1. *Let* $(v(t), w(t)) \in Z$ *and* $U(t) = (v(t), D_x^1(u^0(t) + w(t)))$. *Let us consider the linear problem*

$$\sum_{I,J=0}^{n} a_{IJ}(t, U(t)) \partial_I \partial_J z(t) = \partial_t f_\Omega(t) - \bar{a}_\Omega(t, U(t)) \qquad \textit{in } (0, T) \times \Omega, \tag{6.20}$$

$$\sum_{J=0}^{n}\Big(\sum_{i=1}^{n} n_i b_{iJ}(t, U(t))+b_{\Gamma J}(t, U(t))\Big)\partial_J z(t)=\partial_t f_\Gamma(t)-\bar{a}_\Gamma(t, U(t)),\tag{6.21}$$

$$z_D(t)=0 \quad on \quad (0, T)\times\Gamma,$$

$$z(0)=u_1, \qquad (\partial_t z)(0)=u_2 \qquad in\ \Omega,\tag{6.22}$$

where the notations (4.2), (4.4) *are used. Then.*

(i) *There exists a* $T_1\in(0, T_0]$ *depending only on* K, B, Λ_H, Λ_E, *such that for any* $T\in(0, T_1)$ *the problem* (6.20)-(6.22) *admits a unique solution* $z(t)\in X_D^{K-1,0}([0, T], \Omega)$ *with the property*

$$(\partial_t^M z)(0)=u_{M+1} \qquad for \quad 0\leqq M\leqq K-1.\tag{6.23}$$

(ii) *If* $z_k(t)\in X_D^{2,0}([0, T], \Omega)$, $k=1, 2$ *satisfy* (6.20)-(6.22) *then* $z_1(t)=z_2(t)$ *for* $t\in[0, T]$.

(iii) *Let* $(v(t), w(t))\in Z_c$. *Then there exist* T *and* Λ_H *depending only on* K *and* B *such that the solution* $z(t)$ *of* (6.20)-(6.22) *satisfies the estimate*

$$|z|_{K-1,0,[0,T]}\leqq\Lambda_H.\tag{6.24}$$

We shall prove Theorem 6.2 using Theorem 6.1. In this purpose we have to extend the operators from (6.20)-(6.22) to a wider interval (cf. Remark 6.1). From the theorem Ap. 6 it follows the existence of functions $V(t)\in Y^{K-1,0}(R, \Omega)$, $W(t)\in Y^{K-2,2}(R, \Omega)$ such that

$$v(t)=V(t), \quad w(t)=W(t) \qquad for \quad t\in[0, T],$$

$$\begin{aligned}|V|_{K-1,0,R}&\leqq C(K)\Big\{|v|_{K-1,0,[0,T]}+\sum_{L=0}^{K-2}\|(\partial_t^L v)(0)\|_{K-1-L}\Big\}\\&\leqq C(K)\{\Lambda_H+C_1(K, B)\},\\|W|_{K-2,2,R}&\leqq C(K)\Big\{|w|_{K-2,2,[0,T]}+\sum_{L=0}^{K-3}\|(\partial_t^L w)(0)\|_{K-L}\Big\}\\&\leqq C(K)\Lambda_E,\end{aligned}\tag{6.25}$$

where the relations (3.4), (4.22), (4.23) are used. Since the function $(V(t), D_x^1(u^0(t)+W(t)))$ must be substituted into nonlinear functions defined on $\{U: |U|<U_0\}$, let us choose $T_1>0$ depending only on K, B, Λ_H, Λ_E, such that

$$\|(V(t), D_x^1(u^0(t)+W(t)))\|_{\infty,1}\leqq U_2+(T_1)^\varepsilon C_2(K, B, \Lambda_H, \Lambda_E)<U_1\tag{6.26}$$

for $t\in[-T_1, T_1]$, where U_1, U_2 are the same as in (7.4), (7.6) below, (cf. the argument leading to (7.4), (7.6)). Let us also introduce the following notations

$$U(t)=(v(t), D_x^1(u^0(t)+w(t))), \qquad U'(t)=(V(t), D_x^1(u^0(t)+W(t))),$$

$$R_{IJ}^\Omega(t)=a_{IJ}(t, U'(t)), \qquad R_{iJ}^\Gamma(t)=b_{iJ}(t, U'(t)),$$

$$R_J^\Gamma(t)=b_{\Gamma J}(t,\, U'(t)), \quad \text{with indices as in (6.4)}, \tag{6.27}$$

$$v_0=u_1,\ v_1=u_2,\ h_\Omega(t)=\partial_t f_\Omega(t)-\bar{a}_\Omega(t,\, U(t)),$$

$$h_\Gamma(t)=\partial_t f_\Gamma(t)-\bar{a}_\Gamma(t,\, U(t)).$$

One can check that the coefficients (6.27) satisfy the hypotheses (a.6.0)-(a.6.5). More precisely, we have the following

LEMMA 6.3. *Assume that* (a.0)-(a.5) *are valid and* u_0, u_1, f_Ω, f_Γ *are the same as in Theorem* 1.1. *Let* $(v(t),\, w(t))\in Z$ *and let* $R_{IJ}^\Omega(t)$, $R_{iJ}^\Gamma(t)$, $R_J^\Gamma(t)$ *be defined by* (6.27). *Then the present* $R_{IJ}^\Omega(t)$, $R_{iJ}^\Gamma(t)$, $R_J^\Gamma(t)$ *satisfy* (a.6.0) *and* (a.6.2)-(a.6.5). *Furthermore, if we put (for indices as in* (6.4))

$$\begin{aligned} &R_{IJ}^{\Omega\infty}(t)=a_{IJ}(t,\, 0), && R_{IJ}^{\Omega s}(t)=(a_{IJ})_1(t,\, U'(t)),\\ &R_{iJ}^{\Gamma\infty}(t)=b_{iJ}(t,\, 0), && R_{iJ}^{\Gamma s}(t)=(b_{iJ})_1(t,\, U'(t)),\\ &R_J^{\Gamma\infty}(t)=b_{\Gamma J}(t,\, 0), && R_J^{\Gamma s}(t)=(b_{\Gamma J})_1(t,\, U'(t)), \end{aligned} \tag{6.28}$$

then (a.6.1) *is valid and*

$$\begin{aligned} &\sum_{I,J=0}^{n} |R_{IJ}^{\Omega\infty}|_{\infty,K-1,T_1}+|R_{IJ}^{\Omega s}|_{K-2,1,[-T_1,T_1]}\\ &+\sum_{J=0}^{n}\Big\{\sum_{i=1}^{n}(|R_{iJ}^{\Gamma\infty}|_{\infty,K-1,T_1}+|R_{iJ}^{\Gamma s}|_{K-2,1,[-T_1,T_1]})\\ &+|R_J^{\Gamma\infty}|_{\infty,K-1,T_1}+|R_J^{\Gamma s}|_{K-2,1,[-T_1,T_1]}\Big\}\leqq C_3(K,\, B,\, \Lambda_H,\, \Lambda_E) \end{aligned} \tag{6.29}$$

PROOF. Since (6.26) is valid (a.6.k) follows from (a.k) for $k=0, 2, 3, 4, 5$. Applying Theorem Ap. 3 to (6.28) we obtain (6.29) and (a.6.1).

Now we shall show that the data v_0, v_1, h_Ω, h_Γ, defined in (6.27) satisfy the hypotheses of Theorem 6.1.

LEMMA 6.4. *Let the assumption* (a.1) *be valid and* u_0, u_1, f_Ω, f_Γ, *be the same as in Theorem* 1.1. *If* $(v(t),\, w(t))\in Z$ *and* v_0, v_1, h_Ω, h_Γ *are defined by* (6.27) *then* $v_0\in H_D^{K-1}(\Omega)$, $v_1\in H_D^{K-2}(\Omega)$, $h_\Omega(t)\in X^{K-3,0}([0,\, T],\, \Omega)$, $h_\Gamma(t)\in X^{K-3,1/2}([0,\, T],\, \Gamma)$ *and* (6.14), (6.15) *are valid. Furthermore, if* v_M *is defined by* (6.12) *then*

$$v_M=u_{M+1} \quad \textit{for} \quad 2\leqq M\leqq K-1. \tag{6.30}$$

PROOF OF LEMMA 6.4. By (1.14) and Lemma 3.1 we have $v_0=u_1\in H_D^{K-1}(\Omega)$ and $v_1=u_2\in H_D^{K-2}(\Omega)$. From (1.14) it follows also that to obtain the needed regularity of h_Ω, h_Γ it is sufficient to prove that (cf. (4.2)-(4.4))

$$\bar{a}_\Omega(t,\, U(t))\in Y^{K-2,0}([0,\, T],\, \Omega),\ \bar{a}_\Gamma(t,\, U(t))\in Y^{K-2,1}([0,\, T],\, \Omega). \tag{6.31}$$

Since $(v(t), w(t))\in Z$ we have $U(t)\in Y^{K-2,1}([0, T], \Omega)$. Applying Theorem Ap. 3 we obtain $(\partial_t a_\Omega)(t, U(t))\in Y^{K-2,1}([0, T], \Omega)$. Applying (Ap. 1) and the relations $\partial_I v(t), \partial_i\partial_j(u^0(t)+w(t))\in Y^{K-2,0}([0, T], \Omega)$ we can check that the first relation (6.31) is satisfied. The second part follows from the relation $v(t)\in Y^{K-2,1}([0, T], \Omega)$ if we use again Theorem Ap. 3 and (Ap. 1).

Now we shall prove (6.30) and (6.15). We have

$$\begin{aligned}
&\partial_t\Big(\sum_{I,J=0}^{n} a_{IJ}(t, D^1u^0(t))\partial_I\partial_J u^0(t)+a_\Omega(t, D^1u^0(t))\Big)\\
&=\sum_{I,J=0}^{n} \bar{R}^{\Omega}_{IJ}(t)\partial_I\partial_J\partial_t u^0(t)+\bar{a}_\Omega(t, D^1u^0(t)),\\
&\partial_t\Big(\sum_{i=1}^{n} n_i a_i(t, D^1u^0(t))+a_\Gamma(t, D^1u^0(t))\Big)\\
&=\sum_{J=0}^{n}\Big(\sum_{i=1}^{n} n_i\bar{R}^{\Gamma}_{iJ}(t)+\bar{R}^{\Gamma}_{J}(t)\Big)\partial_J\partial_t u^0(t)+\bar{a}_\Gamma(t, D^1u^0(t)),
\end{aligned}\tag{6.33}$$

where $\bar{R}^{\Omega}_{IJ}(t)=a_{IJ}(t, D^1u^0(t))$, $\bar{R}^{\Gamma}_{iJ}(t)=b_{iJ}(t, D^1u^0(t))$, $\bar{R}^{\Gamma}_{J}(t)=b_{\Gamma J}(t, D^1u^0(t))$. Using the fact that from (4.22) and (4.6) for $0\leqq M\leqq K-3$ the following equalities follow:

$$\partial_t^M U'(0)=\partial_t^M U(0)=\partial_t^M(v, D_x^1(u^0+w))(0)=(u_{M+1}, D_x^1 u_M),\tag{6.34}$$

we have

$$\begin{aligned}
&(\partial_t^M R^{\Omega}_{IJ})(0)=(\partial_t^M\bar{R}^{\Omega}_{IJ})(0), \qquad (\partial_t^M R^{\Gamma}_{iJ})(0)=(\partial_t^M\bar{R}^{\Gamma}_{iJ})(0),\\
&\partial_t^M(a_V(t, U(t))|_{t=0}=\partial_t^M(a_V(t, D^1u^0(t))|_{t=0},
\end{aligned}\tag{6.35}$$

for indices satisfying (6.4). If we compare the equation (6.12) where $h_\Omega=\partial_t f_\Omega-\bar{a}_\Omega$ with the equation (3.3) written in the following way

$$\begin{aligned}
&a_{00}(0, D^1u^0(0))u_{M+3}=(\partial_t^{M+1}f_\Omega)(0)-\partial_t^{M+1}(a_\Omega(t, D^1u^0(t)))|_{t=0}\\
&-\Sigma'\binom{M+1}{k}\partial_t^k(a_{IJ}(t, D^1u^0(t)))|_{t=0}\partial_I^{sI}\partial_J^{sJ}u_{M+3-k-sI-sJ},\ 1\leqq M+1\leqq K-2
\end{aligned}\tag{3.3$'$}$$

and using (6.33), (6.35) we obtain (6.30).

Differentiating both sides of the second part of (6.33) M-times with respect to t, putting $t=0$ and using (6.34), (6.35). (6.30) and (3.1), (3.5), we can check that (6.13) is valid and in the consequence (6.15) is true. The proof of Lemma 6.4 is finished.

PROOF OF THEOREM 6.2. Using lemmas 6.3 and 6.4 we can check that Theorem 6.2 (i) follows from Theorem 6.1 (i) for $T\in(0, T_1)$. Similary Theorem 6.2 (ii) follows from Theorem 6.1 (ii). To prove Theorem 6.2 (iii) we first check that the following estimate is valid

$$|z|^2_{K-1,0,[0,T]} \leqq C_5(K, B) + T^{\varepsilon}_2 C_4(K, B, \Lambda_H, \Lambda_E)$$
(6.36)
$$+ T^{\varepsilon} C_5(K, B, \Lambda_H, \Lambda_E)|z|^2_{K-1,0,[0,T]}, \quad 0<\varepsilon<\left[\frac{n}{2}\right]+1-\frac{n}{2}.$$

If we obtain (6.36), then choosing T and Λ_H so that

(6.37)
$$T^{\varepsilon} C_4(K, B, \Lambda_H, \Lambda_E) \leqq 1, \qquad T^{\varepsilon} C_5(K, B, \Lambda_H, \Lambda_E) \leqq \frac{1}{2},$$
$$(\Lambda_H)^2 \geqq 2\{C_5(K, B)+1\},$$

we get (6.24). In the proof of (6.36) we shall assume that $(v(t), w(t)) \in Z_c$. Let us note that the constant $M(K, T_1)$ from the estimate (6.7) is in the present case equal to the constant $C_3(K, B, \Lambda_H, \Lambda_E)$ from the estimate (6.29) and that T_1 depends only on $K, B, \Lambda_H, \Lambda_E$ (cf. (6.26)).

Applying the energy inequality (6.18) to the problem (6.20)-(6.22) we obtain

$$E(R(t))[\partial_t^{K-2} z(t)] \leqq (\exp C_6 t)\{E(R(t))[\partial_t^{K-2} z(t)]|_{t=0}$$
(6.38)
$$+ C_7 T^{1/2}(|\bar{a}_\Omega(t, U(t))|_{K-2,0,[0,T]} + |\bar{a}_\Gamma(t, U(t))|_{K-2,1,[0,T]} + B^2\}$$

where $C_l = C_l(K, B, \Lambda_H, \Lambda_E)$, $l=6, 7$ and where Theorem Ap. 4a is used. Repeating the argument leading to (6.31) we can prove that

(6.39) $\quad |\bar{a}_\Omega(t, U(t))|_{K-2,0,[0,T]} + |\bar{a}_\Gamma(t, U(t))|_{K-2,1,[0,T]} \leqq C_8(K, B, \Lambda_H, \Lambda_E).$

In (6.39) and in the sequel we use the fact that

(6.40) $\quad |U|_{K-2,1,[0,T]} \leqq C_2(K, B) + \Lambda_H + \Lambda_E, \qquad$ (cf. (4.7), (4.23)).

If we substitute (6.39) into (6.38) and use (6.6), (6.8) we get

$$\delta_0 \|\partial_t^{K-1} z(t)\|_0^2 + \delta_1 \|\partial_t^{K-2} z(t)\|_1^2 \leqq (\exp C_6 t)\|(R_{00}(0))^{1/2}(\partial_t^{K-1} z)(0)\|_0^2$$
(6.41)
$$+ \sum_{k=1}^{3} I_k(t) + \delta_2 \|\partial_t^{K-2} z(t)\|_0^2 + (\exp C_6 T) C_7 T^{1/2}\{C_8 + B^2\},$$

where

$$I_1(t) = d\{(\exp C_6 t)\|\partial_t^{K-2} z(0)\|_0^2 - \|\partial_t^{K-2} z(t)\|_0^2\},$$
$$I_2(t) = S_2(R(t))[\partial_t^{K-2} z(t), \partial_t^{K-2} z(t)],$$
(6.42)
$$I_3(t) = (\exp C_6 t)\Big\{-\sum_{i,j=1}^{n} (R_{ij}^{\Omega}(0)\partial_j \partial_t^{K-2} z(0), \partial_i \partial_t^{K-2} z(0))$$
$$+ S_1(R(0))[\partial_t^{K-2} z(0), \partial_t^{K-2} z(0)],$$
$$d = \delta_2 + (CM(K, T_1))^2/(2\delta_1) = C_9(K, B, \Lambda_H, \Lambda_E), \quad \text{cf. (6.10)}.$$

We shall estimate all terms of the right hand side of (6.41). First let us note that

$$(6.43)\qquad \begin{aligned}&(\exp C_6 t)\|(R^{Q}_{00}(0))^{1/2}(\partial_t^{K-1}z)(0)\|_0^2\leqq(1+C_6T(\exp C_6T))\\&\times\|(R^{Q}_{00}(0))^{1/2}(\partial_t^{K-1}z)(0)\|_0^2\leqq M_1(K,B)+TM_2(K,B,\Lambda_H,\Lambda_E)\end{aligned}$$

Here and hereafter we use the letter M_1 (resp. M_2) to denote various constants depending only on K, B, (resp. K, B, Λ_H, Λ_E). Using the inequality

$$(6.44)\qquad \begin{aligned}&|\,\|(D^{K-2}z)(t)\|_0^2-\|(D^{K-2}z)(0)\|_0^2|\leqq\left|\int_0^t\frac{d}{ds}\|(D^{K-2})(s)\|_0^2ds\right|\\&\leqq T|z|^2_{K-1,0,[0,T]}\end{aligned}$$

and the estimate

(6.45) $\qquad \|(D^{K-1}z)(0)\|_0^2\leqq C_6(K,B),\qquad$ cf. (6.23) and Lemma 3.1, we have

$$(6.46)\qquad \begin{aligned}&I_1(t)=d\,\{(\exp C_6t-1)\|\partial_t^{K-2}z(0)\|_0^2-(\|\partial_t^{K-2}z(t)\|_0^2-\|\partial_t^{K-2}z(0)\|_0^2)\}\\&\leqq C_9\{C_6T(\exp C_6T)\|\partial_t^{K-2}z(0)\|_0^2+T|z|^2_{K-1,0,[0,T]}\}.\end{aligned}$$

If we choose T so that

$$(6(47)\qquad \begin{aligned}&C_6(K,B,\Lambda_H,\Lambda_E)C_9(K,B,\Lambda_H,\Lambda_E)T\leqq1,\\&C_6(K,B,\Lambda_H,\Lambda_E)T\leqq1,\end{aligned}$$

then we obtain

$$(6.48)\qquad I_1(t)\leqq M_1+M_2T|z|^2_{K-1,0,[0,T]}.$$

From (6.44) and (6.45) it follows also the inequality

$$(6.49)\qquad |z|^2_{K-2,0,[0,T]}\leqq M_1+T|z|^2_{K-1,0,[0,T]}.$$

To evaluate I_2 let us note that $R_i^{\Gamma}(t)=b_{\Gamma i}(t,U(t))$ for $t\in[0,T]$, cf. (6.27) and that the inequality (2.14) with $R^i=R_i^{\Gamma}$ holds true. Since $\|R_i^{\Gamma}(t)-R_i^{\Gamma}(0)\|_{\infty,1}\leqq M_2\{T+T^{\varepsilon}\}$ as follows from Theorem Ap. 7 and since $\|R_i^{\Gamma}(0)\|_{\infty,1}=\|b_{\Gamma i}(0,u_1,D_x^1u_0)\|_{\infty,1}\leqq M_1$, we have $\|R_i^{\Gamma}(t)\|_{\infty,1}\leqq M_1+M_2T^{\varepsilon}$, (note that $0<T<1$, cf. (4.25)). Substituting this estimate into (2.14) and using (6.49) we obtain

$$(6.50)\qquad I_2(t)\leqq(\delta_1/2)\|\partial_t^{K-2}z(t)\|_1^2+M_1+T^{2\varepsilon}M_2|z|^2_{K-1,0,[0,T]}+T^{2\varepsilon}M_2$$

with ε as in (6.36), (we may assume additionally that $\varepsilon<1/2$).

To evaluate the term I_3 it is sufficient to note that $R_{ij}^{Q}(0)=a_{ij}(0,u_1,D_x^1u_0)$, $R_i^{\Gamma}(0)=b_{\Gamma i}(0,u_1,D_x^1u_0)$ and to apply (1.16), (2.13), (6.45), (6.47). In consequence, we obtain

$$(6.51)\qquad I_3(t)\leqq M_1.$$

Combining (6.41), (6.43), (6.48), (6.50) and (6.51) we get

$$(6.52)\qquad \delta_0\|\partial_t^{K-1}z(t)\|_0^2+(\delta_1/2)\|\partial_t^{K-2}z(t)\|_1^2\leqq M_1+T^{\varepsilon}M_2+T^{\varepsilon}M_2|z|^2_{K-1,0,[0,T]}.$$

Now we shall evaluate $\|\partial_t^M z(t)\|_{K-1-M}$ for $0\leqq M\leqq K-3$, using the elliptic estimate (5.14). In this purpose let us rewrite (6.20), (6.21) in the form

$$\sum_{i,j=1}^{n} a_{ij}(0, u_1, D_x^1 u_0)\partial_i\partial_j z(t)+\mu z(t)=\partial_t f_\Omega(t)+\mu z(t)+H_\Omega(t) \quad \text{in } \Omega, \tag{6.53}$$

$$\begin{aligned}&\sum_{l=1}^{n}\Big(\sum_{i=1}^{n} n_i b_{il}(0, u_1, D_x^1 u_0)+b_{\Gamma l}(0, u_1, D_x^1 u_0)\Big)\partial_l z(t)=\partial_t f_\Gamma(t)+H_\Gamma(t),\\ &z_D(t)=0 \qquad\qquad \text{on } \Gamma,\end{aligned} \tag{6.54}$$

for $t\in[0, T]$, where μ is a constant determined below and

$$\begin{aligned}&H_\Omega(t)=-\bar{a}_\Omega(t, U(t))-H_{\Omega 1}(t)-H_{\Omega 2}(t),\\ &H_{\Omega 1}(t)={\sum}'' a_{IJ}(0, u_1, D_x^1 u_0)\partial_I\partial_J z(t), \qquad \text{cf. (4.3)},\\ &H_{\Omega 2}(t)=\sum_{I,J=0}^{n}(a_{IJ}(t, U(t))-a_{IJ}(0, U(0)))\partial_I\partial_J z(t),\\ &H_\Gamma(t)=-\bar{a}_\Gamma(t, U(t))-H_{\Gamma 1}(t)-H_{\Gamma 2}(t),\\ &H_{\Gamma 1}(t)=b_0(0, u_1, D_x^1 u_0)\partial_t z(t), \qquad \text{cf. (1.11)},\\ &H_{\Gamma 2}(t)=\sum_{J=0}^{n}\Big\{\sum_{i=1}^{n} n_i(b_{iJ}(t, U(t))-b_{iJ}(0, U(0)))\\ &\qquad +(b_{\Gamma J}(t, U(t))-b_{\Gamma J}(0, U(0)))\Big\}\partial_J z(t).\end{aligned} \tag{6.55}$$

If we define q_{ij}^V and q_i^V as in (5.20) and if we put $q_l^\Omega=q_{in+1}^\Gamma=q_{n+1}^\Gamma=0$ for indices satisfying (5.4), then we can see that (6.53), (6.54) has the form (5.1). Using Theorem 5.1 and Lemma 5.4 one cah check that (5.38) is valid in the present case with K replaced by $K-1$ and $g_\Omega(t)=\partial_t f_\Omega(t)+\mu z(t)+H_\Omega(t)$, $g_\Gamma(t)=\partial_t f_\Gamma(t)+H_\Gamma(t)$. Thus we have

$$\begin{aligned}\|\partial_t^M z(t)\|_{K-1-M}\leqq M_1\{&\|\partial_t^{M+1} f_\Omega(t)\|_{K-3-M}+\langle\!\langle\partial_t^{M+1} f_\Gamma(t)\rangle\!\rangle_{K-5/2-M}\\ &+\|\partial_t^M z(t)\|_{K-3-M}+\|\partial_t^M \bar{a}_\Omega(t, U(t))\|_{K-3-M}+\|\partial_t^M \bar{a}_\Gamma(t, U(t))\|_{K-2-M}\\ &+\sum_{k=1}^{2}(\|\partial_t^M H_{\Omega k}(t)\|_{K-3-M}+\|\partial_t^M H_{\Gamma k}(t)\|_{K-2-M})\}.\end{aligned} \tag{6.56}$$

We shall show that the following estimates are true:

$$\begin{aligned}&\|\partial_t^M \bar{a}_\Omega(t, U(t))\|_{K-3-M}\leqq M_1+TM_2,\\ &\|\partial_t^M a_\Gamma(t, U(t))\|_{K-2-M}\leqq M_1+TM_2,\end{aligned} \tag{6.57}$$

$$\begin{aligned}&\|\partial_t^M H_{\Omega 1}(t)\|_{K-3-M}\leqq M_1\{\|\partial_t^{M+1} z(t)\|_{K-2-M}+\|\partial_t^{M+2} z(t)\|_{K-3-M}\},\\ &\|\partial_t^M H_{\Gamma 1}(t)\|_{K-2-M}\leqq M_1\|\partial_t^{M+1} z(t)\|_{K-2-M},\end{aligned} \tag{6.58}$$

$$(6.59)\qquad \begin{aligned}&\|\partial_t^M H_{\Omega 2}(t)\|_{K-3-M}\leqq M_1+TM_2|z|_{K-1,0,[0,T]},\\ &\|\partial_t^M H_{\Gamma 2}(t)\|_{K-2-M}\leqq M_1+TM_2|z|_{K-1,0,[0,T]}.\end{aligned}$$

To prove (6.57) let us remark that $\|\partial_t^M \bar{a}_\Omega(t, U(t))\|_{K-3-M}\leqq\|\partial_t^M \bar{a}_\Omega(0, U(0))\|_{K-3-M}+\int_0^t\|\partial_s^{M+1}\bar{a}_\Omega(s, U(s))\|_{K-3-M}ds$. Using (6.39), Theorem Ap. 3 and the relation (Ap. 1) we can obtain the first part of (6.57). The second one can be proved in a similar way. If I or $J=0$, using Theorems Ap. 1, Ap. 3, we obtain

$$(6.60)\qquad \begin{aligned}&\|(a_{IJ})_1(0, u_1, D_x^1 u_0)\partial_I\partial_J\partial_t^M z(t)\|_{K-3-M}\leqq\|(a_{IJ})_1(0, u_1, D_x^1 u_0)\|_{K-2}\\ &\quad\times\|\partial_I\partial_J\partial_t^M z(t)\|_{K-3-M+\delta I+\delta J}\leqq M_1\|\partial_t^{M+1}z(t)\|_{K-2-M}+M_1\|\partial_t^{M+2}z(t)\|_{K-3-M}\end{aligned}$$

and similary

$$(6.61)\qquad \begin{aligned}\|a_{IJ}(0, 0)\partial_I\partial_J\partial_t^M z(t)\|_{K-3-M}\leqq M_1\|\partial_t^{M+1}z(t)\|_{K-2-M}\\ +M_1\|\partial_t^{M+2}z(t)\|_{K-3-M}.\end{aligned}$$

Thus the first part of (6.58) is proved. Similary we prove the second one. Using (Ap. 3A), (Ap. 3B) and the relations (6.40), (1.17) we have

$$(6.62)\qquad \begin{aligned}&\|\partial_t^M[(a_{IJ}(t, U(t))-a_{IJ}(0, U(0)))\partial_I\partial_J z(t)]\|_{K-3-M}\\ &\leqq|(a_{IJ}(t, U(t))-a_{IJ}(0, U(0)))\partial_I\partial_J z(t)|_{K-3,0,[0,T]}\\ &\leqq M_1+M_2T|z|_{K-1,0,[0,T]}\end{aligned}$$

and similary using (Ap. 2)

$$(6.63)\qquad \begin{aligned}&\|\partial_t^M[n_i(b_{iJ}(t, U(t))-b_{iJ}(0, U(0)))\partial_J z(t)]\|_{K-2-M}\\ &\leqq M_1+M_2T|z|_{K-1,0,[0,T]}.\end{aligned}$$

In an analogous manner the remaining terms of $H_{\Gamma 2}(t)$ can be estimated. Thus all relations (6.57)-(6.59) hold true. Substituting (6.57)-(6.59) into (6.56) and using (6.40) we obtain

$$(6.64)\qquad \begin{aligned}&\|\partial_t^M z(t)\|_{K-1-M}\leqq M_1+M_1\{\|\partial_t^{M+2}z(t)\|_{K-3-M}+\|\partial_t^{M+1}z(t)\|_{K-2-M}\}\\ &+TM_2+TM_2|z|_{K-1,0,[0,T]}\qquad\text{for}\quad 0\leqq M\leqq K-3.\end{aligned}$$

Repeated application of (6.64) gives

$$(6.65)\qquad \begin{aligned}&\sum_{M=0}^{K-1}\|\partial_t^M z(t)\|_{K-1-M}^2\leqq M_1+M_1\{\|\partial_t^{K-1}z(t)\|_0^2+\|\partial_t^{K-2}z(t)\|_1^2\}\\ &+T^2M_2+T^2M_2|z|_{K-1,0,[0,T]}^2\qquad\text{for}\quad 0\leqq M\leqq K-3.\end{aligned}$$

Substituting (6.52) into (6.65) we obtain

$$\sum_{M=0}^{K-1} |\partial_t^M z|_{0,K-1-M,[0,T]}^2 \leqq M_1 + T^\varepsilon M_2 |z|_{K-1,0,[0,T]}^2 + T^\varepsilon M_2. \tag{6.66}$$

Since in the present case $z(t) \in X^{K-1,0}([0, T], \Omega)$, then the left hand side of (6.66) is equal to the square of the norm $|z|_{K-1,0,[0,T]}$, (cf. (2.3)). If we note that $M_1 = C(K, B)$, $M_2 = C(K, B, \Lambda_H, \Lambda_E)$, then we can see that (6.66) implies (6.36) and in a consequence (6.24). The proof of Theorem 6.2 is complete.

7. The convergence of the iteration procedure.

To show that the iteration procedure defined with the use of (4.28)-(4.31) is convergent we shall prove that there exist constants Λ_H, Λ_E, ε_E and T such that the following conditions are satisfied.

(7.1) The set Z_c of the pairs of functions $(v(t), w(t))$ satisfying the conditions (4.23), (4.24), (4.26), (4.27) is not empty

$$(v^p(t), w^p(t)) \in Z_c \quad \text{for} \quad p = 1, 2, \cdots, \tag{7.2}$$

$$\begin{aligned} &|v^p - v^{p-1}|_{1,0,[0,T]} + |w^p - w^{p-1}|_{0,2,[0,T]} \\ &\leqq \frac{1}{2}\{|v^{p-1} - v^{p-2}|_{1,0,[0,T]} + |w^{p-1} - w^{p-2}|_{0,2,[0,T]}\}. \end{aligned} \tag{7.3}$$

First we prove (7.1). From the assumption (1.14) we have $(u_1, D_x^1 u_0) \in H_D^{K-1}(\Omega)$ with $K-1 \geqq [n/2]+2 > n/2+1$. From the Sobolev imbedding theorem it follows that $|D_x^1(u_1(x), D_x^1 u_0(x))| \to 0$ as $|x| \to 0$. Thus from the asusmption (1.16) it follows the existence of a positive constant $U_2 < U_0$ such that

$$\|(u_1, D_x^1 u_0)\|_{\infty,1} \leqq U_2. \tag{7.4}$$

Let $(v(t), w(t))$ satisfy the conditions (4.22), (4.23) and $U(t) = (v(t), D_x^1(u^0(t) + w(t)))$. Applying Theorem Ap. 7 with $F(t, x, U) = U$ and the relations (7.4), (6.40) we obtain

$$\begin{aligned} \|U(t)\|_{\infty,1} &\leqq \|(u_1, D_x^1 u_0)\|_{\infty,1} + CT^\varepsilon |U|_{K-2,1,[0,T]} \\ &\leqq U_2 + CT^\varepsilon (C_2(K, B) + \Lambda_H + \Lambda_E),\ t \in [0, T] \end{aligned} \tag{7.5}$$

with some $\varepsilon \in (0, [n/2]+1-n/2)$. Let U_1 be a constant such that $U_2 < U_1 < U_0$ and choose T so that

$$U_2 + CT^\varepsilon (C_2(K, B) + \Lambda_H + \Lambda_E) < U_1. \tag{7.6}$$

For such T we have the second part of the relation (4.24). The first one can be proved in an analogous way. Since $|\partial_t u^0|_{K-1,0,[0,T]} \leqq C_2(K, B)$, cf. (4.7), if Λ_H is chosen so that

$$C_2(K, B) \leqq \Lambda_H, \tag{7.7}$$

then $(\partial_t u^0(t), 0) \in Z_c$. Thus the proof of (7.1) is complete.

Now let us review the way of determining the constants Λ_H, Λ_E, ε_E and T. First we choose Λ_H so that (7.7) and $(6.37)_2$ are valid. It is clear that Λ_H depends on K and B only. Second, let Λ_E be chosen so that (5.40) holds true. Thus, Λ_E depends only on K, B. Third, we choose T_1 so that (6.26) is valid. T_1 depends only on K and B. Fourth ε_E and T are chosen so that $0<T<T_1$ and (4.25), (5.37), (5.41), (5.42), $(6.37)_1$, (6.47), (7.6) hold true. Since Λ_H, Λ_E depend only on K, B, ε_E, T have also this property.

Using Theorems 5.3, 6.2, we can show that if $(v^{p-1}(t), w^{p-1}(t)) \in Z_c$ then $(v^p(t), w^p(t)) \in Z_c$. Thus (7.2) is proved. It remains to check that the presented iteration procedure satisfies the condition (7.3). Let us introduce the following notations: $v^{p,p-1}(t)=v^p(t)-v^{p-1}(t)$, $w^{p,p-1}(t)=w^p(t)-w^{p-1}(t)$, $U^p(t)=(v^p(t), D_x^1(u^0(t) +w^p(t)))$. In the first step of the proof of (7.3) we shall show that

$$|v^{p,p-1}|_{1,0,[0,T]} \leqq MT\{|v^{p-1,p-2}|_{1,0,[0,T]}+|w^{p-1,p-2}|_{0,2,[0,T]}\}. \tag{7.8}$$

Here and in the sequel M denotes various constants depending on K, B, Λ_H, Λ_E. Since Λ_H, Λ_E depend on K and B only, M also depends on K, B only.

Subtracting side by side the equations (4.28) taken for p and $p-1$, we obtain

$$\sum_{I,J=0}^{n} a_{IJ}(t, U^{p-1}(t))\partial_I\partial_J v^{p,p-1}(t)=h_\Omega^p(t) \quad \text{in } (0, T)\times\Omega, \tag{7.9}$$

$$\begin{gathered}\sum_{J=0}^{n}\Big(\sum_{i=1}^{n} n_i b_{iJ}(t, U^{p-1}(t))+b_{\Gamma J}(t, U^{p-1}(t))\Big)\partial_J v^{p,p-1}(t)=h_\Gamma^p(t), \\ v_D^{p,p-1}(t)=0 \quad \text{on } (0, T)\times\Gamma,\end{gathered} \tag{7.10}$$

$$v^{p,p-1}(0)=0, \qquad \partial_t v^{p,p-1}(0)=0 \quad \text{in } \Omega, \tag{7.11}$$

where

$$\begin{aligned} h_\Omega^p(t)=&-(\bar{a}_\Omega(t, U^{p-1}(t))-\bar{a}_\Omega(t, U^{p-2}(t))) \\ &-\sum_{I,J=0}^{n}(a_{IJ}(t, U^{p-1}(t))-a_{IJ}(t, U^{p-2}(t)))\partial_I\partial_J v^{p-1}(t), \\ h_\Gamma^p(t)=&-(\bar{a}_\Gamma(t, U^{p-1}(t))-\bar{a}_\Gamma(t, U^{p-2}(t))) \\ &-\sum_{J=0}^{n}\Big\{\sum_{i=1}^{n} n_i(b_{iJ}(t, U^{p-1}(t))-b_{iJ}(t, U^{p-2}(t))) \\ &+b_{\Gamma J}(t, U^{p-1}(t))-b_{\Gamma J}(t, U^{p-2}(t))\Big\}\partial_J v^{p-1}(t). \end{aligned} \tag{7.12}$$

Let us extend the coefficients of the operators in (7.9), (7.10) to the interval

$[-T_1, T_1]$ as in the proof of theorem 6.2, cf. (6.25). Applying the energy estimate (6.17) we obtain

$$|v^{p,p-1}|_{1,0,[0,T]} \leqq MT\{|h^p_\Omega|_{0,0,[0,T]}+|h^p_\Gamma|_{0,1,[0,T]}\}. \tag{7.13}$$

Using (Ap. 1) and (Ap. 5) we can prove

$$\begin{aligned}&\|\bar{a}_V(t, U^{p-1}(t))-\bar{a}_V(t, U^{p-2}(t))\|_{J(V)}\\&\qquad\leqq M\{|v^{p-1,p-2}|_{1,0,[0,T]}+|w^{p-1,p-2}|_{0,2,[0,T]}\}\end{aligned} \tag{7.14}$$

where

$$J(V)=\begin{cases}1 & \text{if } V=\Gamma,\\ 0 & \text{if } V=\Omega.\end{cases} \tag{7.15}$$

Similary, using (Ap. 6) we can check that the norms:

$$\|(a_{IJ}(t, U^{p-1}(t))\partial_I\partial_J v^{p-1}(t)-a_{IJ}(t, U^{p-2}(t))\partial_I\partial_J v^{p-1}(t)\|_0,$$

$$\|b_{\Gamma I}(t, U^{p-1}(t))\partial_I v^{p-1}(t)-b_{\Gamma I}(t, U^{p-2}(t))\partial_I v^{p-1}(t)\|_1,$$

$$\|n_i(b_{iJ}(t, U^{p-1}(t))\partial_J v^{p-1}(t)-b_{iJ}(t, U^{p-2}(t))\partial_J v^{p-1}(t)\|_1$$

can be estimated by the right hand side of (7.14). In consequence, we have

$$|h^p_\Omega|_{0,0,[0,T]}+|h^p_\Gamma|_{0,1,[0,T]} \leqq M\{|v^{p-1,p-2}|_{1,0,[0,T]}+|w^{p-1,p-2}|_{0,2,[0,T]}\}. \tag{7.16}$$

Combining (7.13) and (7.16), we get (7.8).

In the second step of the proof of the relation (7.3) we shall show that

$$|w^{p,p-1}|_{0,2,[0,T]} \leqq M\{|v^{p,p-1}|_{1,0,[0,T]}+(T+\varepsilon_E)|w^{p-1,p-2}|_{0,2,[0,T]}\} \tag{7.17}$$

In this purpose let us subtract side by side the relations (4.30), (4.31) taken for p and $p-1$. We obtain $w^{p,p-1}_D(t)=0$ on Γ and

$$\begin{aligned}&p_{V\lambda}[w^{p,p-1}(t)]=[G_{V1}(t, v^p(t))-G_{V1}(t, v^{p-1}(t))]\\&\quad+\sum_{k=2}^{3}[G_{Vk}(t, v^p(t), w^{p-1}(t))-G_{Vk}(t, v^{p-1}(t), w^{p-2}(t))] \quad \text{on } V\end{aligned} \tag{7.18}$$

Using the elliptic estimate (5.14) with $L=2$ we get

$$\|w^{p,p-1}(t)\|_2 \leqq M\sum_{k=1}^{3}\{I_{\Omega k}(t)+I_{\Gamma k}(t)\}, \tag{7.19}$$

where

$$\begin{aligned}&I_{V1}(t)=\|G_{V1}(t, v^p(t))-G_{V1}(t, v^{p-1}(t))\|_{J(V)}, \quad \text{cf. (7.15)},\\&I_{Vk}(t)=\|G_{Vk}(t, v^p(t), w^{p-1}(t))-G_{Vk}(t, v^{p-1}(t), w^{p-2}(t))\|_{J(V)}, \ k=2, 3.\end{aligned} \tag{7.20}$$

Let us note that if we get the estimates

$$I_{V1}(t) \leqq M|v^{p,p-1}(t)|_{0,1,[0,T]}, \tag{7.21}$$

$$I_{V2}(t) \leqq M\{T|w^{p-1,p-2}(t)|_{0,2,[0,T]}+\varepsilon_E|v^{p,p-1}(t)|_{0,1,[0,T]}\} \tag{7.22}$$

$$I_{V3}(t) \leqq M\varepsilon_E\{|w^{p-1,p-2}(t)|_{0,2,[0,T]}+|v^{p,p-1}(t)|_{1,0,[0,T]}\}, \tag{7.23}$$

then substituting (7.21)-(7.23) into (7.19) we obtain (7.17).

One can check that (7.21) follows from (Ap. 5). To prove (7.22) let us estimate separately all terms of I_{V2}, cf. (7.20), (4.16), (4.19). Using (Ap. 10) we obtain

$$\begin{aligned}
&\Big\| \sum_{I,J=0}^{n} d a_{IJ}(0, U^0(0))[D_x^1 w^{p-1}(t)\partial_I^{sI}\partial_J^{sJ}(u_p^{2-sI-sJ}(t)-u_{2-sI-sJ}) \\
&\quad - D_x^1 w^{p-2}(t)\partial_I^{sI}\partial_J^{sJ}(u_{p-1}^{2-sI-sJ}(t)-u_{2-sI-sJ})]\Big\|_0 \leqq \\
&C(K, B)(|D_x^1 w^{p-1,p-2}(t)|_{0,1,[0,T]}T \sum_{I,J=0}^{n} |\partial_I^{sI}\partial_J^{sJ}(u_p^{2-sI-sJ}(t) \\
&\quad - u_{2-sI-sJ})|_{K-2,0,[0,T]}+|D_x^1 w^{p-2}(t)|_{K-3,1,[0,T]} \\
&\quad \times \sum_{I,J=0}^{n} |\partial_I^{sI}\partial_J^{sJ}(u_p^{2-sI-sJ}(t)-u_{p-1}^{2-sI-sJ}(t)|_{0,0,[0,T]}) \\
&\leqq M_1\{T|w^{p-1,p-2}(t)|_{0,2,[0,T]}+\varepsilon_E|v^{p,p-1}(t)|_{1,0,[0,T]}\},
\end{aligned} \tag{7.24}$$

where

$$u_p^2=\partial_t v^p, \qquad \partial_i\partial_0 u_p^1=\partial_0\partial_i u_p^1=\partial_i v^p, \qquad u_p^0=u^0. \tag{7.25}$$

Applying (Ap. 7B), we get

$$\begin{aligned}
&\Big\| \sum_{I,J=0}^{n} [d a_{IJ}(t, v^p(t), D_x^1 u^0(t))-d a_{IJ}(0, U^0(0))]D_x^1 w^{p-1}(t)\partial_I^{sI}\partial_J^{sJ} u_p^{2-sI-sJ}(t) \\
&\quad -[d a_{IJ}(t, v^{p-1}(t), D_x^1 u^0(t))-d a_{IJ}(0, U^0(0))]D_x^1 w^{p-2}(t)\partial_I^{sI}\partial_J^{sJ} u_{p-1}^{2-sI-sJ}(t)\Big\|_0 \\
&\leqq M\Big\{|D_x^1 w^{p-1}(t)|_{0,K-2,[0,T]} \sum_{I,J=0}^{n} |\partial_I^{sI}\partial_J^{sJ}(u_p^{2-sI-sJ}(t)-u_{p-1}^{2-sI-sJ}(t))|_{0,0,[0,T]} \\
&\quad +T|D_x^1 w^{p-1,p-2}(t)|_{0,1,[0,T]}+|D_x^1 w^{p-2}(t)|_{0,K-2,[0,T]}|v^{p,p-1}(t)|_{0,1,[0,T]}\Big\} \\
&\leqq M_1\{|v^{p,p-1}(t)|_{1,0,[0,T]}\varepsilon_E+T|w^{p-1,p-2}(t)|_{0,2,[0,T]}\}
\end{aligned} \tag{7.26}$$

Using (Ap. 7A), we can check that

$$\begin{aligned}
&\|[d a_\Omega(t, v^p(t), D_x^1 u^0(t))-d a_\Omega(0, U^0(0))]D_x^1 w^{p-1}(t) \\
&\quad -[d a_\Omega(t, v^{p-1}(t), D_x^1 u^0(t))-d a_\Omega(0, U^0(0))]D_x^1 w^{p-2}(t)\|_0 \\
&\leqq M_1\{T|D_x^1 w^{p-1,p-2}(t)|_{0,1,[0,T]}+|D_x^1 w^{p-2}(t)|_{0,K-2,[0,T]}
\end{aligned} \tag{7.27}$$

$$\times |v^{p,p-1}(t)|_{0,1,[0,T]}\} \leqq M_1\{T|w^{p-1,p-2}(t)|_{0,2,[0,T]} + \varepsilon_E |v^{p,p-1}(t)|_{0,1,[0,T]}\}.$$

In the same way, we obtain the analogous estimates for $I_{\Gamma 2}(t)$. Applying (Ap. 7B) we obtain

$$(7.28)\quad \begin{aligned} &\Big\| \sum_{i,j=1}^{n} \{(a_{ij}(t, v^p(t), D_x^1 u^0(t)) - a_{ij}(0, U^0(0)))\partial_i\partial_j w^{p-1}(t) \\ &\quad -(a_{ij}((t, v^{p-1}(t), D_x^1 u^0(t)) - a_{ij}(0, U^0(0)))\partial_i\partial_j w^{p-2}(t)\} \Big\|_0 \\ &\leqq M \sum_{i,j=1}^{n} \{T|\partial_i\partial_j w^{p-1,p-2}(t)|_{0,0,[0,T]} + |v^{p,p-1}(t)|_{1,0,[0,T]} \\ &\quad \times |\partial_i\partial_j w^{p-2}(t)|_{K-3,0,[0,T]}\} \leqq M_1\{T|w^{p-1,p-2}(t)|_{0,2,[0,T]} \\ &\quad +\varepsilon_E |v^{p,p-1}(t)|_{1,0,[0,T]}\}. \end{aligned}$$

Combining (7.24)-(7.28), we get (7.22).

From the relation: $\|U^p(\theta)\|_{\infty,1} \leqq C\|U^p(\theta)\|_{K-1} \leqq M$, where the notation $U^p(\theta) = (v^p(t), D_x^1(u^0(t) + \theta w^{p-1}(t))$ is used and from (Ap. 8A), (Ap. 8B) with $\Delta = \varepsilon_E$ follows (7.23). As a consequence of (7.21)-(7.23), we obtain (7.17). Combining (7.8) and (7.17), we get

$$(7.29)\quad \begin{aligned} &|v^{p,p-1}|_{1,0,[0,T]} + |w^{p,p-1}|_{0,2,[0,T]} \\ &\leqq C_{10}T|v^{p-1,p-2}|_{1,0,[0,T]} + C_{11}(T+\varepsilon_E)|w^{p-1,p-2}|_{0,2,[0,T]}, \end{aligned}$$

where $C_l = C_l(K, B, \Lambda_H, \Lambda_E)$, $l=10, 11$. If we choose T and ε_E so that

$$(7.30)\quad C_{10}(K, B, \Lambda_H, \Lambda_E)T \leqq \frac{1}{2}, \qquad C_{11}(K, B, \Lambda_H, \Lambda_E)(T+\varepsilon_E) \leqq \frac{1}{2},$$

then we obtain (7.3).

Using (7.2), (7.3), one can prove the existence of a pair $(v(t), w(t)) \in Z$ satisfying (4.1), (4.5). In fact, from (7.3) it follows that the sequences $\{v^p\}$ and $\{w^p\}$ are Cauchy ones in $X_D^{1,0}([0, T], \Omega)$ and $X_D^{0,2}([0, T], \Omega)$, respectively. Applying the interpolation inequality, cf. [20], Lemma 7.1

$$(7.31)\quad \begin{aligned} &|D^M D_x^2(w^p - w^{p'})|_{0,0,[0,T]} \leqq C|D_x^2(w^p - w^{p'})|_{0,0,[0,T]}^{1-M/(K-3)} |D_x^2(w^p - w^{p'})|_{K-3,0,[0,T]}^{M/(K-3)} \\ &\leqq C|D_x^2(w^p - w^{p'})|_{0,0,[0,T]}^{1-M/(K-3)} (2\Lambda_E)^{M/(K-3)}, \qquad 0 \leqq M \leqq K-3, \end{aligned}$$

we can see that $\{w^p\}$ is a Cauchy sequence in $X_D^{K-3,2}([0, T], \Omega)$. In the same way one can prove that $\{v^p\}$ is a Cauchy sequence in $X_D^{K-2,0}([0, T], \Omega)$. In consequence, there exist $v(t) \in X_D^{K-2,0}([0, T], \Omega)$ and $w(t) \in X_D^{K-3,2}([0, T], \Omega)$ such that

(7.32) $$\lim_{p\to\infty}|v^p-v|_{K-2,0,[0,T]}=\lim_{p\to\infty}|w^p-w|_{K-3,2,[0,T]}=0.$$

Let us recall that the sequences $\{v^p\}$, $\{w^p\}$ are bounded in the spaces $Y_D^{K-1,0}([0, T], \Omega)$ and $Y_D^{K-2,2}([0, T], \Omega)$, respectively, i.e.

(7.33) $$|v^p|_{K-1,0,[0,T]}\leqq \Lambda_H,\ |w^p|_{K-2,2,[0,T]}\leqq \Lambda_E,\ |w^p|_{K-3,2,[0,T]}\leqq \varepsilon_E$$

for $p=1, 2, \cdots$. Using (7.32), (7.33) and repeating the standard argument, cf. [20], Lemma 7.2, one can prove that the obtained limits v, w satisfy the relations (4.21) and (4.23). Since (4.22) is valid for every v^p and w^p, from (7.32) it follows that the limit functions v, w also satisfy this condition. Since (4.24) follows from (4.22), (4.23) and (7.6), we have proved that the pair $(v(t), w(t))$ satisfies all conditions (4.21)-(4.24), i.e. $(v(t), w(t))\in Z$. Letting $p\to\infty$ in (4.28), (4.30) and using (7.32), (Ap. 5), (Ap. 6), (Ap. 7), (Ap. 8), we can check that the present $v(t)$, $w(t)$ satisfy (4.1) and (4.11). If we put $u(t)=u^0(t)+w(t)$, from the manner of deriving (4.11) from (4.5) we see that $v(t)$ and $u(t)$ satisfy (4.1), (4.5).

Now let us check that the present functions $u(t)$, $v(t)$ satisfy the relation $\partial_t u(t)=v(t)$ for $t\in[0, T]$. From the relation (4.21) we have $U(t)=(v(t), D_x^1 u(t))\in Y_D^{K-2,1}([0, T], \Omega)$. Applying Theorem Ap. 3 we see that (depending on $U(t)$) coefficients of the equation $(4.1)_1$ belong to the space $Y^{K-2,0}([0, T], \Omega)\subset X^{K-3,0}([0, T], \Omega)$ and the coefficients of $(4.1)_2$ belong to $Y^{K-2,1}([0, T], \Omega)\subset X^{K-3,1}([0, T], \Omega)$. The inequality $K-3\geqq[n/2]\geqq 1$ shows that we can differentiate $(4.5)_1$ with respect to t. Subtracting the obtained equation and equation $(4.1)_1$ side by side and putting $z(t)=\partial_t u(t)-v(t)$ we obtain

(7.34) $$\sum_{i,j=1}^{n} a_{ij}(t, U(t))\partial_i\partial_j z(t)+\sum_{l=1}^{n+1} a_l^*(t, U(t))\partial_l z(t)+\lambda z(t)=0 \quad \text{in } \Omega,$$

where we have posed

(7.35) $$\begin{aligned} a_l^*(t, U(t))\partial_l z(t)&=b_{\Omega l}(t, U(t))\partial_l z(t)+\sum{}'' a_{IJl}(t, U(t))\partial_l z(t) \\ &\times\partial_I^{sI}\partial_J^{sJ}u^{2-sI-sJ}(t)+\sum_{i,j=1}^{n} a_{ijl}(t, U(t))\partial_l z(t)\partial_i\partial_j u(t). \end{aligned}$$

Similar considerations on the boundary give

(7.36) $$\sum_{l=1}^{n+1}\Big(\sum_{i=1}^{n} n_i b_{il}(t, U(t))+b_{\Gamma l}(t, U(t))\Big)\partial_l z(t)=0,\ z_D(t)=0 \qquad \text{on } \Gamma.$$

Since $z(t)\in H_D^2(\Omega)$ for $t\in[0, T]$, then multiplying (7.34) by $z(t)$ and integrating by parts we obtain, that the left hand side of the inequality (5.19) with $b_{\Omega l}$ replaced by a_l^* is equel to zero. Thus $\|z(t)\|_1^2=0$ for $t\in[0, T]$, which implies $\partial_t u(t)=v(t)$ for $t\in[0, T]$. If we substitute the last relation into (4.5) we see that $u(t)$ satisfies (1.1)-(1.3).

In the final step of the proof we shall show that $u(t)\in X_D^{K,0}([0, T], \Omega)$. Let us observe that the function $v(t)$ may be regarded as a solution in $X_D^{2,0}([0, T], \Omega)$ to the linear problem (6.20)-(6.22). Applying Theorem 6.2 we see that since $(v(t), w(t))\in Z$, the solution $v(t)$ belongs to the space $X_D^{K-1,0}([0, T], \Omega)$. Since $\partial_t u(t)=v(t)$, to get $u(t)\in X_D^{K,0}([0, T], \Omega)$ it suffices to prove that $u(t)\in C^0([0, T], H_D^K(\Omega))$. In this purpose let t, s be two different elements of $[0, T]$. Let us put $U(t)=(v(t), D_x^1 u(t))$, $V(\theta)=\theta U(t)+(1-\theta)U(s)$ and apply some elementary calculations to (1.1) and Taylor formula to (1.2). We obtain

$$\sum_{i,j=1}^{n} q_{ij}^{\Omega}\partial_i\partial_j(u(t)-u(s))+\mu(u(t)-u(s))=h_\Omega \quad \text{in } \Omega \tag{7.37}$$

$$\sum_{j=1}^{n}\left[\sum_{i=1}^{n} q_{ij}n_i+q_j\right]\partial_j(u(t)-u(s))=h_\Gamma,\ (u(t)-u(s))_D=0 \quad \text{on } \Gamma \tag{7.38}$$

where

$$\begin{aligned}
&h_\Omega=f_\Omega(t)-f_\Omega(s)+\mu(u(t)-u(s))+I_1+I_2,\\
&I_1=-\Sigma'' a_{IJ}(s, U(s))\partial_I^{sI}\partial_J^{sJ}(u^{2-sI-sJ}(t)-u^{2-sI-sJ}(s)),\\
&I_2=-\sum_{I,J=0}^{n}[a_{IJ}(t, U(t))-a_{IJ}(s, U(s))]\partial_I\partial_J u(t)\\
&\qquad -(a_\Omega(t, U(t))-a_\Omega(s, U(s))),\\
&h_\Gamma=f_\Gamma(t)-f_\Gamma(s)-b_0(s, U(s))(v(t)-v(s))+I_3+I_4,\\
&I_3=-\int_0^1 d^2\Big(\sum_{i=1}^{n} n_i a_i+a_\Gamma\Big)(s, V(\theta))(D^1(u(t)-u(s)), D^1(u(t)-u(s)))d\theta,\\
&I_4=-\sum_{i=1}^{n} n_i(a_i(t, U(t))-a_i(s, U(t)))-(a_\Gamma(t, U(t))-a_\Gamma(s, U(t))),\\
&q_{ij}^V=q_{ij}^{V\infty}+q_{ij}^{Vs}, \qquad q_j^\Gamma=q_j^{\Gamma\infty}+q_j^{\Gamma s}, \qquad q_{ij}^{\Omega\infty}=a_{ij}(s, 0),\\
&q_{ij}^{\Gamma\infty}=b_{ij}(s, 0), \qquad q_j^{\Gamma\infty}=b_{\Gamma j}(s, 0), \qquad q_{ij}^{\Omega s}=(a_{ij})_1(s, U(s)),\\
&q_{ij}^{\Gamma s}=(b_{ij})_1(s, U(s)), \qquad q_j^{\Gamma s}=(b_{\Gamma j})_1(s, U(s)).
\end{aligned} \tag{7.39}$$

The problem (7.37), (7.38) is a special case of (5.1) (with $q_i^\Omega=0$). Applying Theorem Ap. 3 we can check that the coefficients given by (7.39) satisfy the condition

$$\begin{aligned}
&\sum_{V\in(\Omega,\Gamma)}\sum_{i,j=1}^{n}(\|q_{ij}^{V\infty}\|_{\infty,K-1}+\|q_{ij}^{Vs}\|_{K-1})+\sum_{j=1}^{n}(\|q_j^{\Gamma\infty}\|_{\infty,K-1}+\|q_j^{\Gamma}\|_{K-1})\\
&\leq C_{12}(K, B, \Lambda_H, \Lambda_E) \qquad \text{for } t, s\in[0, T].
\end{aligned} \tag{7.40}$$

Thus in the present case the constants γ_∞, γ_K in the inequalities (5.9), (5.13) are of the form

$$\gamma_K = C_{12}(K, B, \Lambda_H, \Lambda_E), \qquad \gamma_\infty = C_{13}(K, B, \Lambda_H, \Lambda_E), \tag{7.41}$$

and are independent of t, $s \in [0, T]$. Hence, there exists a μ depending only on K, B, Λ_H, Λ_E and independent of t, $s \in [0, T]$ such that the inequality (5.14) with $L = K$ and with w replaced by $u(t) - u(s)$ holds true:

$$\|u(t) - u(s)\|_K \leqq M\{\|h_\Omega\|_{K-2} + \langle\!\langle h_\Gamma \rangle\!\rangle_{K-3/2}\} \qquad \text{for} \quad t,\ s \in [0, T]. \tag{7.42}$$

Let us estimate the right hand side of (7.42). Using (Ap. 1), we see that $\|I_1\|_{K-2} \leqq M \sum_{I=0}^n \|\partial_I (v(t) - v(s))\|_{K-2}$. Applying (Ap. 1), the mean-value theorem and Theorem Ap. 3, we get $\|I_2\|_{K-2} \leqq M\{|t-s| + \|v(t) - v(s)\|_{K-2} + \|u(t) - u(s)\|_{K-1}\}$. Combining Theorem Ap. 1 and the estimate (Ap. 1) we can check that $\|I_3\|_{K-2} = \left\| \int_0^1 d^2 (\sum_{i=1}^n n_i a_i + a_\Gamma)(s, V(\theta))(D^1(u(t) - u(s)), D^1(u(t) - u(s))) d\theta \right\|_{K-2} \leqq M \|D^1(u(t) - u(s))\|_{K-2}^2 \leqq M\{\|u(t) - u(s)\|_{K-1}^2 + \|v(t) - v(s)\|_{K-2}^2\}$. Finally, from the mean-value theorem we obtain $\|I_4\|_{K-2} \leqq M|t-s|$. Substituting the obtained estimates into (7.42) we get

$$\begin{aligned} \|u(t) - u(s)\|_K \leqq M\{ & \|f_\Omega(t) - f_\Omega(s)\|_{K-2} + \langle\!\langle f_\Gamma(t) - f_\Gamma(s) \rangle\!\rangle_{K-3/2} \\ & + |t-s| + \sum_{I=0}^n \|\partial_I (v(t) - v(s))\|_{K-2} + \|v(t) - v(s)\|_{K-2} + \|u(t) - u(s)\|_{K-1} \\ & + \|v(t) - v(s)\|_{K-2}^2 + \|u(t) - u(s)\|_{K-1}^2\}, \qquad \text{for} \quad s,\ t \in [0, T], \end{aligned} \tag{7.43}$$

with a constant M independent of s and t. Recall that we have checked that $v(t) \in X^{K-1}([0, T], \Omega)$ and $u(t) \in Y_D^{K-2,2}([0, T], \Omega) \subset C^0([0, T], H_D^{K-1}(\Omega))$. Using the hypotheses (1.14) we can see that from (7.43) it follows that $u(t) \in C^0([0, T], H_D^K(\Omega))$. The proof of Theorem 1.1 is complete.

APPENDIX. ESTIMATES OF SOME NONLINEAR TERMS.

In this appendix we present some facts which follow from Sobolev imbedding theorem (cf. for example [1], p. 97) and are frequently used in the text. We omit the proofs since they are similar to those given in sections 7.2, 7.3 of the monograph [14] and in the Appendix of the paper [20], (the only exception is the proof of Theorem Ap. 5b).

Let Ω be a n-dimensional domain with a smooth boundary and $K \geqq [n/2] + 3$.

THEOREM AP. 1A. *If α, β are real numbers and γ an integer such that α, $\beta \geqq \gamma \geqq 0$ and $\alpha + \beta - \gamma > (n/2)$ then the relations $u_1 \in H^\alpha(\Omega)$, $u_2 \in H^\beta(\Omega)$ imply $u_1 u_2 \in H^\gamma(\Omega)$ and $\|u_1 u_2\|_\gamma \leqq C(n, \gamma) \|u_1\|_\alpha \|u_2\|_\beta$.*

THEOREM AP. 1B. *It $r_1, \cdots, r_k$, $k \geqq 2$ and S be nonnegative real numbers* *and* L *a nonnegative integer such that* $S>n/2$, $S\geqq r_1+\cdots+r_k+L$ *and* $u_j\in H^{S-r_j}(\Omega)$, $j=1, \cdots, k$, *then the product* $u_1\cdots u_k$ *belongs to* $H^L(\Omega)$ *and* $\|u_1\cdots u_k\|_L \leqq C(k, L)\|u_1\|_{S-r_1}\cdots\|u_k\|_{S-r_k}$.

THEOREM AP.2. *Let* J *be an interval of* R *and* L, M *integers such that* $L, M\geqq 0$ *and* $L+M>n/2$. *If* $u_j\in Z^{L,M}(J, \Omega)$, $j=1, \cdots, k$ *and* $Z=X$ *or* $Z=Y$ *then their product* $u_1\cdots u_k$ *belongs to* $Z^{L,M}(J, \Omega)$. *Furthermore if* $Z=X$ *then* $\|D^L(u_1\cdots u_k)\|_M\leqq C(k, L, M)\|D^Lu_1\|_M\cdots\|D^Lu_k\|_M$ *for* $t\in J$.

THEOREM AP. 3. *Let* L, M *be as in Theorem Ap.* 2. *Let* $F(t, x, u)\in B^\infty(J\times\bar{\Omega}\times\{|u|\leqq u_0\})$, $F(t, x, 0)=0$ *for* $(t, x)\in J\times\bar{\Omega}$ *and* $u\in Z^{L,M}(J, \Omega)$, $Z=X$ *or* $Z=Y$, $\|u(t)\|_{\infty,0}\leqq u_0$ *for* $t\in J$. *Then* $F(t, x, u(t, x))\in Z^{L,M}(J, \Omega)$. *Furthermore, when* $Z=X$, $\|D^LF(t, \cdot, u(t, \cdot))\|_M\leqq C(L, M, F)\{1+\|D^Lu(t)\|_M\}^{L+M-1}\|D^Lu(t)\|_M$.

REMARK AP. 1. When u_j, u, F do not depend on t, Theorems Ap. 2, Ap. 3 are valid if we put $L=0$ and $Z^{L,M}(J, \Omega)=H^M(\Omega)$.

In the following estimates we always assume that $J=[0, T]$, $G(t, x, u)\in B^\infty(J\times\bar{\Omega}\times\{|u|\leqq u_0\})$, $H(x, u)\in B^\infty(\bar{\Omega}\times\{|u|\leqq u_0\})$.

(AP. 1) Let K, N be nonnegative integers such that $K-2\leqq N+M\leqq K-1$. If $u(t)\in Z^{N,M}(J, \Omega)$, $v(t)\in Z^{N,M}(J, \Omega)$, $Z=X$ or $Z=Y$ and $\|u(t)\|_{\infty,0}\leqq u_0$ for $t\in J$ then $G(t, u(t))v(t)\in Z^{N,M}(J, \Omega)$. Furthermore, when $Z=X$, $\|D^N(G(t, u(t))v(t))\|_M \leqq C(M, N)\{\|D^NG(t, 0)\|_{\infty,M}+\|D^N(G(t, u(t))-G(t, 0))\|_M\}\|D^Nv(t)\|_M$.

(AP. 2) Let $u(t)\in X^{K-2,1}(J, \Omega)$ be such that $\|u(t)\|_{\infty,0}\leqq u_0$ for $t\in J$ and $v(t)\in X^{K-2,N}(J, \Omega)$, $N=0, 1$. Put $I(t)=\{G(t, \cdot, u(t))-G(0, \cdot, u(0))\}v(t)$. Then $I(t)\in X^{K-2,N}(J, \Omega)$ and

$$|I|_{K-2,N,J}\leqq C(K, |u|_{K-2,1,J})\{T|v|_{K-2,N,J}+|v|_{K-3,1,J}\}.$$

(AP. 3A) If $u(t)$, $I(t)$ are the same as in (Ap. 2) and $v(t)\in X^{K-3,0}(J, \Omega)$ then $I\in X^{K-3,0}(J, \Omega)$ and

$$|I|_{K-3,0,J}\leqq C(K, B, |u|_{K-2,1,J})T|v|_{K-3,0,J}.$$

(AP. 3B) If $u(t)$, $I(t)$ are the same as in (Ap. 2) and $v(t)\in X^{K-2,0}(J, \Omega)$ then $I(t)\in X^{K-2,0}(J, \Omega)$ and

$$|I|_{K-3,1,J}\leqq C(K, |u|_{K-2,1,J})T|v|_{K-2,0,J}+C(K, \|D^{K-2}u(0)\|_1)|v|_{K-3,0,J}.$$

(AP. 4) Let $u(t)$ and $v(t)$ be the same as in (Ap. 2). Put $I(t)=G(t, \cdot, u(t))\times v(t)v(t)$. Then $I(t)\in X^{K-2,N}(J, \Omega)$ and

$$|I|_{K-2,N,J}\leqq C(K,\ |u|_{K-2,1,J})|v|_{K-2,N,J}|v|_{K-3,1,J}.$$

(AP. 5) Let $N=0$ or 1, $H(x, 0)=0$. If $u_j\in H^{K-2}(\Omega)$, $\|u_j\|_{\infty,0}\leqq u_0$, $j=1, 2$ then $\|H(\cdot, u_1)-H(\cdot, u_2)\|_N\leqq C(K,\ \|u_1\|_{K-2},\ \|u_2\|_{K-2})\|u_1-u_2\|_N$.

(AP. 6) Let $N=0$ or 1. If u_j, $v_j\in H^{K-2}(\Omega)$ and $\|u_j\|_{\infty,0}\leqq u_0$, $j=1, 2$ then $\|H(\cdot, u_1)v_1-H(\cdot, u_2)v_2\|_N\leqq C(K,\ \|u_1\|_{K-2},\ \|u_2\|_{K-2})\{\|v_1-v_2\|_N+\|v_2\|_{K-2}\|u_1-u_2\|_N\}$.

(AP. 7A) Let $N=0$ or 1 and $u_j(t)\in X^{K-2,1}(J, \Omega)$, $v_j(t)\in X^{K-2,N}(J, \Omega)$, $j=1, 2$. Assume that $\|u_j\|_{\infty,0}\leqq u_0$ for $t\in J$, $j=1, 2$ and $u_1(0)=u_2(0)$. Put $I(t)=I_1(t)v_1(t)-I_2(t)v_2(t)$ where $I_j(t)=G(t, \cdot, u_j(t))-G(0, \cdot, u_j(0))$. Then $|I|_{0,N,J}\leqq C(K,\ |u_1|_{K-2,1,J},\ |u_2|_{K-2,1,J})\{T|v_1-v_2|_{0,N,J}+|v_2|_{0,K-3+N,J}|u_1-u_2|_{0,1,J}\}$.

(AP. 7B) Let $u_j(t)$, $I_j(t)$ be as in (Ap. 7A) and $v_j(t)$, $w_j(t)\in X^{K-2,0}(J, \Omega)$ for $j=1, 2$. Put $I(t)=I_1(t)v_1(t)w_1(t)-I_2(t)v_2(t)w_2(t)$. Then $|I|_{0,0,J}\leqq C(K,\ |u_1|_{K-2,1,J},\ |u_2|_{K-2,1,J},\ |w_1|_{K-2,0,J},\ |w_2|_{K-2,0,J})\{|v_1|_{0,K-2,J}|w_1-w_2|_{0,0,J}+T|v_1-v_2|_{0,1,J}+|v_2|_{0,K-2,J}|u_1-u_2|_{0,1,J}\}$.

(AP. 8A) If u_j, $v_j\in H^{K-2}(\Omega)$, $\|u_j\|_{\infty,0}\leqq u_0$ and $\|v_j\|_{K-2}\leqq\Delta<1$ for $j=1, 2$, then $\|H(\cdot, u_1)v_1v_1-H(\cdot, u_2)v_2v_2\|_N\leqq C(K,\ \|u_1\|_{K-2},\ \|u_2\|_{K-2})\times\Delta\{\|u_1-u_2\|_N+\|v_1-v_2\|_N\}$.

(AP. 8B) If additionally $w_j\in H^{K-2}(\Omega)$, then

$$\|H(\cdot, u_1)v_1v_1w_1-H(\cdot, u_2)v_2v_2w_2\|_N\leqq C(K,\ \|u_1\|_{K-2},\ \|u_2\|_{K-2},\ \|v_2\|_{K-2})\times\Delta$$
$$\times\{\|u_1-u_2\|_N+\|v_1-v_2\|_N+\|w_1-w_2\|_N\}.$$

(AP. 9) Let $u(t)\in X^{K-2,1}(J, \Omega)$, $\|u(t)\|_{\infty,0}\leqq u_0$ for $t\in J$. Then $\|G(t, \cdot, u(t))\|_{\infty,0}\leqq C_1+C_2T|u|_{K-2,1,J}$ for $t\in J$, where $C_1=\sup\{|G(t, x, u)|: (t, x)\in J\times\bar{\Omega},\ |u|\leqq u_0\}$, $C_2=\sup\{|\partial_tG(t, x, u)|+|dG(t, x, u)|: (t, x)\in J\times\bar{\Omega},\ |u|\leqq u_0\}$.

(AP. 10) If u_j, $v_j\in X^{K-2,0}(J, \Omega)$, $j=1, 2$ then $\|u_1(t)v_1(t)-u_2(t)v_2(t)\|_0\leqq$

$$C(K)\{|u_1-u_2|_{0,1,J}T|v_1|_{K-2,0,J}+|u_2|_{K-3,1,J}|v_1-v_2|_{0,0,J}\}.$$

THEOREM AP. 4A. *There exists a constant $C=C(\Gamma)>0$ such that*

$$\langle\!\langle u\rangle\!\rangle_{1/2}\leqq C\|u\|_1 \qquad \textit{for all} \quad u\in H^1(\Omega).$$

THEOREM AP. 4B. *For any $\varepsilon>0$ there exists a constant $C(\varepsilon, \Gamma)$ such that*

$$\langle\!\langle u\rangle\!\rangle_0^2\leqq\varepsilon\|u\|_1^2+C(\varepsilon, \Gamma)\|u\|_0^2 \qquad \textit{for} \quad u\in H^1(\Omega).$$

THEOREM AP. 5A. *If $u_M\in H^{K-M}(\Omega)$ for $0\leqq M\leqq K$, then there exists a $v(t)\in X^{K,0}(R, \Omega)$ such that $(\partial_t^Mv)(0)=u_M$ in Ω for $0\leqq M\leqq K$ and*

$$\|D^K v(t)\| \leqq C(K) \sum_{M=0}^{K} \|u_M\|_{K-M} \qquad for \quad t \in R.$$

THEOREM AP. 5B. *If* $u_M \in H_D^{K-M}(\Omega)$ *for* $0 \leqq M \leqq K$, *then there exists a* $u(t) \in X_D^{K-2,2}(R, \Omega)$ *such that* $(\partial_t^M u)(0) = u_M$ *in* Ω *for* $0 \leqq M \leqq K-2$ *and*

$$|u|_{K-2,2,R} \leqq C(K) \sum_{M=0}^{K} \|u_M\|_{K-M} \qquad for \quad t \in R.$$

PROOF. If $a \notin M_D$, then we define $u^a(t)$ as $v^a(t)$ where $v(t)$ is the same function as in from Theorem Ap. 5A. It remains to define the function $u_D(t)$ (cf. (1.6)). In this purpose let us consider the elliptic boundary value problem

$$(*) \qquad \Delta u_D(t) = \Delta v_D(t) \text{ in } \Omega, \ u_D(t) = 0 \text{ on } \Gamma, \qquad \text{for} \quad t \in R.$$

In the same way as Theorem 5.2 we can prove the existence of an unique $u_D(t) \in X^{K-2,2}(R, \Omega)$ satisfying (*). Let us note that the functions u_{MD}, $0 \leqq M \leqq K-2$ satisfy the conditions

$$(**) \qquad \Delta \partial_t^M u_D(0) = \Delta \partial_t^M v_D(0) \text{ in } \Omega, \qquad \partial_t^M u_D(0) = 0 \text{ on } \Gamma,$$

obtained from (*) by differentiation with respect to t and putting $t=0$. From the uniquenes of solutions to the problem (**) we have $\partial_t^M u_D(0) = u_{MD}$ in Ω for $0 \leqq M \leqq K-2$. Using known estimates for Dirichlet problem we can prove that $|u_D(t)|_{K-2,2,R} \leqq C |\Delta v_D(t)|_{K-2,0,R} \leqq C |v_D(t)|_{K-2,2,R} \leqq C(K) \sum_{M=0}^{K} \|u_M\|_{K-M}$. The proof is complete.

THEOREM AP. 6. *Let* $T>0$ *and let* L *and* M *be nonnegative integers. If* $u(t) \in Y^{L,M}([0, T], \Omega)$ *then there exist* $v(t) \in Y^{L,M}(R, \Omega)$ *such that* $v(t) = u(t)$ *for* $t \in [0, T]$ *and*

$$|v|_{L,M,R} \leqq C(M, L) \left\{ |u|_{L,M,[0,T]} + \sum_{N=0}^{L-1} \|\partial_t^N u(0)\|_{L+M-N} \right\}.$$

THEOREM AP. 7. *Let* $F(t, x, U) \in B^\infty([0, T] \times \bar{\Omega} \times \{|U| < U_0\})$ *and let* $u(t) \in Y^{K-2,1}([0, T], \Omega)$ *be such that* $\|u(t)\|_{\infty,0} < U_0$ *for* $t \in [0, T]$. *Then* $\|F(t, \cdot, u(t)) - F(0, \cdot, u(0))\|_{\infty,1} \leqq C(K, |u|_{K-2,1,[0,T]}) \{T + C(\varepsilon) T^\varepsilon\}$ *for* $t \in [0, T]$, *where* ε *is a constant in* $(0, [n/2]+1-n/2)$. *In the special case* $F=U$, $\|U(t) - U(0)\|_{\infty,1} \leqq C(K) t^\varepsilon |U|_{K-2,1,[0,T]}$.

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