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ON THE EXCEPTIONAL SET IN GOLDBACH'S PROBLEM

By

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1. Introduction.

Let $E(x)$ denote the number of even integers not exceeding x that are not representable as a sum of two primes. The Goldbach conjecture asserts $E(x) \ll 1$. Unfortunately this is far from our reach. In 1923 G.H. Hardy and J.E. Littlewood [4] showed, on assuming the extended Riemann hypothesis, that

$$E(x) \ll x^{1/2+\varepsilon} \tag{1}$$

for any $\varepsilon>0$. After the fundamental work of I.M. Vinogradov [18], several authors have unconditionally given the non-trivial bounds for $E(x)$. The best one of these is due to H.L. Montgomery and R.C. Vaughan. In 1975 they [13] showed that there exists a positive constant δ such that

$$E(x) \ll x^{1-\delta}.$$

Chen J.-r. [2] gave an explicit value of δ, which is very small.

In 1973 K. Ramachandra [16] proved that, for any $A>0$,

$$E(x+x^{\theta})-E(x) \ll x^{\theta}(\log x)^{-A} \tag{2}$$

providing

$$\frac{7}{12}<\theta \leqq 1. \tag{3}$$

This bound 7/12 comes from a zero density estimate for the Dirichlet L-series. In 1981 Lou S.-t. and Yao Q. [9] attempted to sharpen the inequality (2). Later Yao [20] replaced, in the same range of θ as (3), the right hand side of (2) by $x^{\theta(1-\delta)}$ with some $\delta>0$.

It is of some interest, from the point of veiw of (1), to demonstrate the formula (2) for θ less than 1/2. We shall present such a result.

THEOREM. *Let $A>0$ and $7/48<\theta \leqq 1$ be given. Then we have*

$$E(x+x^{\theta})-E(x) \ll x^{\theta}(\log x)^{-A}$$

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where the implied constant depends only on A and θ.

This work is inspired by the article of D. Wolke [19]. On combining the argument of [16] with the result of [19], one may prove our assertion for $\theta>(7/12)\cdot(5/8)$, which is less than 1/2 surely. We appeal to the device of H. Iwaniec and M. Jutila [7]. See also [14]. Using the method in our previous papers [10, 11], we then conclude the bound $\theta>(7/12)\cdot(1/4)$.

Our notation is standard or self-explanatory. For a real number t, we write $\psi(x)=[x]-x+1/2$, $e(x)=e^{2\pi i x}$ and $\|x\|=\min_{n\in Z}|x-n|$. The convention $n\sim N$ means that $N<n\leqq N'\leqq 2N$. ρ in $\sum_{\rho(\chi)}$ runs through the set of non-trivial zeros of $L(s,\chi)$. c_i's denote certain positive absolute constants. For simplicity, we write $\mathcal{L}=\log x$. F in $\mathcal{L}^F$ stands for a positive numerical constant, which is not the same at each occurrence.

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2. Lemmas.

LEMMA 1. *Let $1\leqq a\leqq q$, $(a,q)=1$. The Hurwitz zeta-function $\zeta(s,a/q)$ is regular, except for a simple pole at $s=1$, of residue 1. Also, it satisfies the growth conditions*:

$$\zeta\left(s,\frac{a}{q}\right)\ll\begin{cases} q+\exp(c_1(\log\log x)^3)\,; & 1-c_2(\log x)^{-4/5}<\mathrm{Re}(s)\leqq 1+(\log x)^{-1} \\ & |\mathrm{Im}(s)|\leqq x \\ & |s-1|\geqq(\log x)^{-1} \\ q^{1/2}+x^{1/6}(\log x)\,; & \mathrm{Re}(s)=1/2 \\ & |\mathrm{Im}(s)|\leqq x \\ qx^{(1-\mathrm{Re}(s))/3}(\log x)\,; & 1/2\leqq\mathrm{Re}(s)\leqq 1+(\log x)^{-1} \\ & c_3\leqq|\mathrm{Im}(s)|\leqq x\,. \end{cases}$$

LEMMA 2. *Let $N(\sigma,T,\chi)$ denote the number of zeros of $L(s,\chi)$ in the rectangle*; $\mathrm{Re}(s)\geqq\sigma$, $|\mathrm{Im}(s)|\leqq T$. *Let $A>0$ be given. If $q\leqq(\log T)^A$ and $\sigma\geqq 1-c_4(\log T)^{-4/5}$ then*

$$\sum_{\chi(\mathrm{mod}\, q)}N(\sigma,T,\chi)=0\,.$$

For Lemmas 1 and 2, see [15, Kap. VIII Satz 5.3. Kap. IV Satz 5.3. Kap. IX Satz 4.2. Anhang Satz 9.1. Kap. VIII Satz 6.2.]. The following Lemma 3 is due to H. L. Montgomety [12, Theorem 12.1] and M. N. Huxley [5]. Lemma

4 follows from [12, Theorem 7.6] immediately.

LEMMA 3. *Let* $1/2 \leqq \sigma \leqq 1$. *For any* $\varepsilon > 0$,

$$\sum_{\chi(\mathrm{mod}\ q)} N(\sigma, T, \chi) \ll (qT)^{(12/5+\varepsilon)(1-\sigma)} (\log qT)^{14}$$

where the implied constant depends only on ε.

LEMMA 4. *For* $k=1, 2$,

$$\sum_{\chi(\mathrm{mod}\ q)} \sum_{\substack{\rho(\chi) \\ \mathrm{Re}(\rho) \geqq \sigma \\ |\mathrm{Im}(\rho)| \leqq T}} \Big| \sum_{n \sim N} \chi(n) \Lambda(n) n^{-\rho} \Big|^{2k} \ll (qT + N^k) N^{k(1-2\sigma)} (\log qN)^{12}.$$

Let Ω be a non-negative arithmetical function with a compact support, and $\mathcal{P}$ a set of primes. For $z > 2$, put

$$S(\Omega, \mathcal{P}, z) = \sum_{(a, P(z))=1} \Omega(a)$$

where

$$P(z) = \prod_{\substack{p < z \\ p \in \mathcal{P}}} p.$$

Suppose that, for a square-free d composed by primes $\in \mathcal{P}$,

$$\sum_{a \equiv 0 (\mathrm{mod}\ d)} \Omega(a) = \frac{\omega(d)}{d} X + r(\Omega, d)$$

where X is an approximation of $\sum \Omega(a)$ and $\omega(d)$ is a multiplicative function which satisfies some conditions of regularity. Write

$$V(z) = \prod_{\substack{p < z \\ p \in \mathcal{P}}} \Big(1 - \frac{\omega(p)}{p}\Big).$$

LEMMA 5. *Let* $z, D > 2$ *and* $s = \log D / \log z$. *For* $2 \leqq s \leqq 4$ *we have*

$$S(\Omega, \mathcal{P}, z) \geqq V(z) X \Big\{ \frac{2e^{\gamma} \log(s-1)}{s} + O((\log\log D)^{-1}) \Big\} + R,$$

where γ *is the Euler constant and the remainder term* R *has the form*

$$R = \sum_{d | P(z)} \lambda_d(D, z) r(\Omega, d).$$

Here, the weights $(\lambda_d) = (\lambda_d(D, z))$ *satisfy that*

$$\lambda_d = 0 \quad \textit{if} \quad d \geqq D,$$

$$|\lambda_d| \leqq \mu^2(d).$$

Moreover, on writing $z_1 = \exp((\log D)(\log\log D)^{-1/10})$, *if* $\lambda_d \neq 0$ *for* $d > z_1 z$ *then* λ_d *is decomposed into the shape*

(2.1)
$$\lambda_d=\sum_{h\leqq\log D}\sum_{\substack{m\\ d=mp}}\sum_{P_h\leqq p<P_h'}\alpha(h,m)$$
where
$$z_2\leqq P_h<D^{1/4},\quad P_h'=P_hz_9 \quad \text{and} \quad |\alpha(h,m)|\leqq 1.$$

This is the Rosser-Iwaniec linear sieve [6, 14]. We explain the decomposition (2.1). It follows from [6, 14] that if $\lambda_d\neq 0$ then d is the form
$$d=\nu \quad \text{or} \quad \nu p_1\cdots p_r.$$
Here,
$$\nu\leqq z_1 \qquad D_i\leqq p_i<D_iz_9$$
$$\nu|P(z_2) \qquad p_i|P(z)/P(z_2),$$
$(D_1,\cdots,D_r)$ is a subsequence of $(z_2z_9^n)_{n\geqq 0}$ such that
$$D_1\geqq\cdots\geqq D_r,$$
$$D_1\cdots D_{2l-1}D_{2l}^3<D \qquad \text{for all} \quad 1\leqq l\leqq r/2.$$
Moreover, the coefficient of d depends only on ν and $(D_1,\cdots,D_r)$. And, the number of $(D_1,\cdots,D_r)$'s is at most $\log D$.

Now, if $d>z_1z$ then $z_1z<d=\nu p_1\cdots p_r\leqq z_1z^r$, whence $r\geqq 2$. Thus, $D_2^4\leqq D_1D_2^3<D$. Since $D_i\geqq z_2$, we obtain (2.1).

Next lemma is the combinatrial identity of R.C. Vaughan [17].

LEMMA 6. *If $Y<l\leqq XY$ then*
$$\Lambda(l)=\sum_{\substack{l=mn\\ n>Y}}\Big(\sum_{\substack{d|m\\ d\leqq X}}\mu(d)\Big)\Lambda(n).$$

LEMMA 7. *Let $1<\Delta<x/2$. For arbitrary complex numbers a_n, put*
$$\mathcal{J}=\Big(\frac{\Delta}{2}\Big)^2\int_{-1/2\Delta}^{1/2\Delta}\Big|\sum_{x<n\leqq 2x}a_ne(\beta n)\Big|^2d\beta.$$
Then we have
$$\mathcal{J}\ll\int_x^{2x}\Big|\sum_{t<n\leqq t+\Delta}a_n\Big|^2dt+\Delta^3\Big(\sup_{x<n\leqq 2x}|a_n|\Big)^2.$$

LEMMA 8. *Under the assumption of Lemma 7, we have*
$$\mathcal{J}\leqq\Delta\sum_{x<n\leqq 2x}|a_n|^2+2\,\mathrm{Re}\sum_{0<r\leqq\Delta}(\Delta-r)\sum_{\substack{x<m,\,n\leqq 2x\\ m-n=r}}\sum a_m\bar{a}_n$$
$$\ll\Delta\sum_{x<n\leqq 2x}|a_n|^2+\Delta\sup_{t\leqq\Delta}\Big|\sum_{0<r\leqq t}\sum_{x<n\leqq 2x}\bar{a}_na_{n+r}\Big|+\Delta^3\Big(\sup_{x<n\leqq 2x}|a_n|\Big)^2.$$

Lemma 7 is [3, Lemma 1] and Lemma 8 follows from a familiar Fourier integral. We, however, reproduce the proof. It is well known that

$$\max(0, \Delta-|x|)=\int_{-\infty}^{\infty} K(y, \Delta)e(xy)dy$$

where

$$K(y, \Delta)=\left(\frac{\sin \pi\Delta y}{\pi y}\right)^2.$$

Put

$$\mathscr{D}=\sum_{x<m,\, n\leqq 2x}\sum a_m\bar{a}_n \max(0, \Delta-|m-n|),$$

$$\mathcal{S}=\int_x^{2x}\Big|\sum_{t<n\leqq t+\Delta} a_n\Big|^2 dt,$$

and

$$R=\Delta^3\Big(\sup_{x<n\leqq 2x}|a_n|\Big)^2.$$

We shall show $\mathcal{S}\geqq\mathscr{D}+O(R)$ and $\mathscr{D}\geqq\mathscr{F}$, from which Lemmas follow immediately. Now,

$$\mathcal{S}\geqq\int_x^{2x-\Delta}=\sum_{x<m,\, n\leqq 2x}\sum a_m\bar{a}_n \ \text{meas.}\left\{t:\begin{array}{c} x\leqq t\leqq 2x-\Delta \\ m-\Delta\leqq t<m \\ n-\Delta\leqq t<n\end{array}\right\}.$$

We note that meas. $\{t\}=0$ for $|m-n|\geqq\Delta$. If $x+\Delta<m$, $n\leqq 2x-\Delta$ then meas. $\{t\}=\Delta-|m-n|$. Otherwise, the number of (m, n) is $O(\Delta^2)$. Also, meas. $\{t\}\leqq\Delta$ trivially. We thus have

$$\begin{aligned}\mathcal{S}&\geqq\sum_{\substack{x+\Delta<m,\, n\leqq 2x-\Delta\\ |m-n|\leqq\Delta}}\sum a_m\bar{a}_n(\Delta-|m-n|)+O(R)\\ &=\sum_{x<m,\, n\leqq 2x}\sum a_m\bar{a}_n\max(0, \Delta-|m-n|)+O(R)\\ &=\mathscr{D}+O(R).\end{aligned}$$

Moreover,

$$\begin{aligned}\mathscr{D}&=\int_{-\infty}^{\infty}K(\beta, \Delta)\Big|\sum_{x<n\leqq 2x}a_n e(\beta n)\Big|^2 d\beta\\ &\geqq\int_{-1/2\Delta}^{1/2\Delta}\Big(\frac{\Delta}{2}\Big)^2\Big|\sum_{x<n\leqq 2x}a_n e(\beta n)\Big|^2 d\beta\\ &=\mathscr{F},\end{aligned}$$

since $K(y, \Delta)\geqq(\Delta/2)^2$ for $|y|\leqq 1/2\Delta$.

$\mathscr{F}$ reminds us of Circle method or Saddle point method. Also $\mathscr{D}$, Dispersion method or Kloostermania. $\mathcal{S}$, Large sieve or Dirichlet polynomial methods. This observation makes a feature of our argument below.

3. Proof of Theorem.

Let x be a sufficiently large parameter. We take $y=Y^{\theta_1}(1/4<\theta_1<1/2)$ and $Y=x^{\theta_2}(7/12<\theta_2<2/3)$, so that $y=x^{\theta}$, $7/48<\theta<1/3$. Put $\mathcal{K}=(x, x+y]$, $\mathcal{A}=(2Y, 4Y]$ and $\mathcal{B}=(x-3Y, x-2Y]$. Our aim is to give a lower bound of the sum

$$\mathcal{G}=\mathcal{G}(2k)=\sum_{\substack{2k=p+p'\\ p\in\mathcal{B}}}\log p,$$

for $2k\in\mathcal{K}$. Moreover, define

$$\mathcal{S}=\mathcal{S}(2k, z)=\sum_{\substack{2k=p+n\\ p\in\mathcal{B}\\ (n,\mathcal{P}(2k,z))=1}}\log p$$

where

$$\mathcal{P}(2k, z)=\prod_{\substack{p<z\\ p\nmid 2k}}p.$$

Suppose that

$$(4Y)^{1/3}<z<Y^{1/2}. \tag{3.1}$$

Since $2k\in\mathcal{K}$ and $p\in\mathcal{B}$ imply $n=2k-p\in\mathcal{A}$, n counted by $\mathcal{S}$ has at most two prime factors $\geqq z$. We thus have

$$\mathcal{S}-\mathcal{G}=\sum_{\substack{2k=p+n\\ p\in\mathcal{B}}}(\log p)\rho'(n)$$

where

$$\rho'(n)=\sum_{\substack{n=pp'\\ z\leqq p\leqq p'}}1.$$

For any $l\in\mathcal{A}$,

$$\begin{aligned}\rho'(l)&\leqq\sum_{\substack{l=mn\\ z\leqq m\leqq n}}\frac{\Lambda(m)\Lambda(n)}{\log m\ \log n}\\ &\leqq\sum_{\substack{l=mn\\ \sqrt{2Y}<n\leqq 4Y/z}}\frac{\Lambda(m)\Lambda(n)}{\log m\ \log n}\\ &\leqq\frac{1}{\log(Y/z)\ \log z}\sum_{\substack{l=mn\\ n\in\mathcal{I}}}\Lambda(m)\Lambda(n)\\ &=\Xi\rho(l),\quad\text{say}.\end{aligned}$$

Here, $\mathcal{I}=(\sqrt{2Y}, 4Y/z]$. We therefore have our fundamental inequality

$$\begin{aligned}\mathcal{S}-\mathcal{G}&\leqq\Xi\sum_{\substack{2k=p+n\\ p\in\mathcal{B}}}(\log p)\rho(n)\\ &=\Omega,\quad\text{say}.\end{aligned} \tag{3.2}$$

We proceed to evaluate Ω by means of Circle method. Define

$$Q=yQ_1^{-2}$$

$$Q_1=(\log Y)^D$$

$$\boldsymbol{M}=\bigcup_{q\leq Q_1}\bigcup_{a=1}^{q}{}^{*}I_{q,a} \qquad I_{q,a}=\left[\frac{a}{q}-\frac{1}{qQ},\ \frac{a}{q}+\frac{1}{qQ}\right]$$

$$\boldsymbol{m}=[1/Q,\ 1+1/Q]\backslash \boldsymbol{M},$$

where D is a constant specified later, and $*$ stands for the restriction $(a, q)=1$. Furtheremore,

$$T(\alpha)=\Xi\sum_{n\in\mathcal{A}}\rho(n)e(\alpha n),$$

$$W(\alpha)=\sum_{p\in\mathcal{B}}(\log p)e(\alpha p).$$

For $\alpha=a/q+\beta\in I_{q,a}$, we write

$$V(\alpha)=\Xi\sum_{n\in\mathcal{A}}\left(\rho(n)e\left(\frac{a}{q}n\right)-\Gamma\frac{\mu(q)}{\varphi(q)}\right)e(\beta n)$$

where

$$\Gamma=\frac{1}{2}\log\left(\frac{8Y}{z^2}\right).$$

Then,

$$\begin{aligned}\Omega&=\int_{1/Q}^{1+1/Q}T(\alpha)W(\alpha)e(-2k\alpha)d\alpha\\&=\int_{\boldsymbol{M}}(T(\alpha)-V(\alpha))W(\alpha)e(-2k\alpha)d\alpha\\&\quad+\int_{\boldsymbol{M}}V(\alpha)W(\alpha)e(-2k\alpha)d\alpha\\&\quad+\int_{\boldsymbol{m}}T(\alpha)W(\alpha)e(-2k\alpha)d\alpha\\&=\Omega_0-\mathfrak{R}_1-\mathfrak{R}_2,\quad\text{say}.\end{aligned}\tag{3.3}$$

We first consider Ω_0.

$$\begin{aligned}\Omega_0=&\sum_{q\leq Q_1}\sum_{a=1}^{q}{}^{*}\int_{-1/qQ}^{1/qQ}\Xi\Gamma\frac{\mu(q)}{\varphi(q)}\left(\sum_{n\in\mathcal{A}}e(\beta n)\right)\left(\sum_{p\in\mathcal{B}}(\log p)e\left(\left(\frac{a}{q}+\beta\right)p\right)\right)\\&\times e\left(-2k\left(\frac{a}{q}+\beta\right)\right)d\beta\\=&\,\Xi\Gamma\sum_{q\leq Q_1}\frac{\mu(q)}{\varphi(q)}\sum_{p\in\mathcal{B}}(\log p)c_q(p-2k)\cdot\sum_{n\in\mathcal{A}}\int_{-1/2}^{1/2}e((n+p-2k)\beta)d\beta\\&+O\left(\Xi\Gamma\sum_{q\leq Q_1}\frac{|\mu(q)|}{\varphi(q)}\sum_{a=1}^{q}{}^{*}\int_{1/qQ}^{1/2}\Big|\sum_{n\in\mathcal{A}}e(\beta n)\Big|\left|W\left(\frac{a}{q}+\beta\right)\right|d\beta\right).\end{aligned}$$

The above O-term is

$$\ll \mathcal{L}^{-1} \sum_{q \le Q_1} (qQ \cdot Y\mathcal{L})^{1/2}$$

$$\ll Y^{3/4}. \tag{3.4}$$

$2k \in \mathcal{K}$ and $p \in \mathcal{B}$ imply $2k - p \in \mathcal{A}$, namely

$$\sum_{n \in \mathcal{A}} \int_{-1/2}^{1/2} e((n+p-2k)\beta)d\beta \equiv 1.$$

Hence the main contribution of Ω_0 is equal to

$$\begin{aligned} &\Xi\Gamma \sum_{q \le Q_1} \frac{\mu(q)}{\varphi(q)} \sum_{\substack{d|q \\ (d,2k)=1}} d\mu\Big(\frac{q}{d}\Big) \sum_{\substack{p \in \mathcal{B} \\ p \equiv 2k(d)}} \log p + O\Big(\mathcal{L}^{-1} \sum_{q \le Q_1} \frac{|\mu(q)|}{\varphi(q)} \sum_{d|q} d\mathcal{L}^2\Big) \\ &= \Xi\Gamma \sum_{q \le Q_1} \frac{\mu(q)}{\varphi(q)} \sum_{\substack{d|q \\ (d,2k)=1}} d\mu\Big(\frac{q}{d}\Big) \frac{Y}{\varphi(d)} \\ &\quad + O\Big(\mathcal{L}^{-1} \sum_{q \le Q_1} \frac{|\mu(q)|}{\varphi(q)} \sum_{\substack{d|q \\ (d,2k)=1}} d \Big| \sum_{\substack{p \in \mathcal{B} \\ p \equiv 2k(d)}} \log p - \frac{Y}{\varphi(d)} \Big|\Big) + O(Q_1\mathcal{L}^2) \\ &= Y\Xi\Gamma \sum_{q \le Q_1} \frac{\mu^2(q)}{\varphi^2(q)} c_q(2k) \\ &\quad + O\Big(\sum_{\substack{d \le Q_1 \\ (d,2k)=1}} \Big| \sum_{\substack{x-3Y < p \le x-2Y \\ p \equiv 2k(d)}} \log p - \frac{Y}{\varphi(d)} \Big| + Y^{3/4}\Big). \end{aligned} \tag{3.5}$$

By Lemmas 2 and 3, the above O-term is

$$\ll Y\mathcal{L}^{-E}, \tag{3.6}$$

for any $E > 0$. Put

$$\mathfrak{R}_0 = \mathfrak{R}_0(2k) = Y\Xi\Gamma \sum_{q > Q_1} \frac{\mu^2(q)}{\varphi^2(q)} c_q(2k). \tag{3.7}$$

Then, by [15, Kap. VI p. 201], we see

$$\sum_{2k \in \mathcal{K}} \mathfrak{R}_0(2k)^2 \ll yY^2 Q_1^{-1}. \tag{3.8}$$

In conjuction with (3.4)-(3.7) we obtain

$$\Omega_0 = \mathfrak{S}(2k)Y\Xi\Gamma - \mathfrak{R}_0 + O(Y\mathcal{L}^{-3}), \tag{3.9}$$

where

$$\mathfrak{S}(2k) = 2\prod_{p>2}\Big(1 - \frac{1}{(p-1)^2}\Big)\prod_{\substack{p|k \\ p>2}}\Big(\frac{p-1}{p-2}\Big).$$

Next we consider $\mathfrak{R}_1$. Let $I_{q',a'}, I_{q,a} \subset M$, $a'/q' \ne a/q$. If $\alpha' \in I_{q',a'}$ and $\alpha \in I_{q,a}$ then we find

$$\begin{aligned}\|\alpha'-\alpha\| &\geqq \left\|\frac{a'}{q'}-\frac{a}{q}\right\|-\left\|\alpha'-\frac{a'}{q'}\right\|-\left\|\alpha-\frac{a}{q}\right\| \\ &\geqq \frac{1}{q'q}-\frac{1}{q'Q}-\frac{1}{qQ} \\ &\geqq (2Q_1)^{-2}.\end{aligned}$$

We therefore have

$$\begin{aligned}&\sum_{2k\in\mathcal{K}}\mathfrak{R}_1(2k)^2 \\ &\ll \int_M\int_M |V(\alpha')W(\alpha')|\,|V(\alpha)W(\alpha)|\min\left(y, \frac{1}{\|\alpha'-\alpha\|}\right)d\alpha' d\alpha \\ &\ll Q_1^2\left(\int_M |V(\alpha)W(\alpha)|\,d\alpha\right)^2 + y\sum_{q\leqq Q_1}\sum_{a=1}^{q}{}^*\left(\int_{I_{q,a}}|V(\alpha)W(\alpha)|\,d\alpha\right)^2 \\ &\ll Q_1^2\int_M |V(\alpha)|^2 d\alpha \int_M |W(\alpha)|^2 d\alpha \\ &\quad + y \sup_{\substack{(a,q)=1\\ q\leqq Q_1}} \int_{I_{q,a}} |V(\alpha)W(\alpha)|\,d\alpha\cdot\int_M |V(\alpha)W(\alpha)|\,d\alpha \\ &\ll Q_1^2 Y^2\mathcal{L} + yY^{3/2}\mathcal{L}\sup_{\substack{(a,q)=1\\ q\leqq Q_1}}\left(\int_{I_{q,a}}|V(\alpha)|^2 d\alpha\right)^{1/2},\end{aligned}$$

by Cauchy's inequality. Here we appeal to the following lemma. We shall prove Lemma 9 in the next section.

LEMMA 9. *Let ε, B, $E>0$ be given. If $Y^{1/4+\varepsilon}\leqq\Delta\leqq Y^{1/2}$ and $4Y/z\leqq Y^{8/15}$ then*

$$\Delta^2\int_{-1/2\Delta}^{1/2\Delta}\left|V\left(\frac{a}{q}+\beta\right)\right|^2 d\beta \ll \Delta^2 Y\mathcal{L}^{-E}$$

uniformly for $(a, q)=1$ and $q\leqq(\log Y)^B$. Here the implied constant depends only on ε, B and E.

Now, we take $2\Delta=Q$ and

$$z=Y^{7/15+\delta} \tag{3.10}$$

with a sufficiently small $\delta>0$. Then this choice of z satisfies the assumption of Lemma 9, also (3.1). We thus obtain

$$\sum_{2k\in\mathcal{K}}\mathfrak{R}_1(2k)^2 \ll yY^2\mathcal{L}^{-E} \tag{3.11}$$

for any $E>0$.

We turn to $\mathfrak{R}_2$. For $2\Delta<y$, we have

$$\sum_{2k\in\mathcal{K}}\mathfrak{R}_2(2k)^2\ll\int_m\int_m|T(\alpha')W(\alpha')||T(\alpha)W(\alpha)|\min\Big(y,\ \frac{1}{\|\alpha'-\alpha\|}\Big)d\alpha'd\alpha$$

$$\ll\Delta\Big(\int_m|T(\alpha)W(\alpha)|d\alpha\Big)^2$$

$$+y\int_m|T(\alpha)W(\alpha)|\Big(\int_{\substack{m\\ \|\alpha'-\alpha\|\leq 1/2\Delta}}|T(\alpha')W(\alpha')|d\alpha'\Big)d\alpha$$

$$\ll\Delta\int_m|T(\alpha)|^2d\alpha\int_m|W(\alpha)|^2d\alpha$$

$$+y\Big(\sup_{\alpha\in m}\int_{\substack{m\\ \|\alpha'-\alpha\|\leq 1/2\Delta}}|T(\alpha')|^2d\alpha'\Big)^{1/2}\Big(\int_m|T(\alpha)|^2d\alpha\Big)^{1/2}\int_m|W(\alpha)|^2d\alpha$$

$$\ll\Delta Y^2\mathcal{L}^4+yY^{3/2}\mathcal{L}^3\sup_{\alpha\in m}\Big(\int_{-1/2\Delta}^{1/2\Delta}|T(\alpha+\beta)|^2d\beta\Big)^{1/2},\tag{3.12}$$

since $T(\alpha)$ has the period 1.

We now use the following lemma. We postpone the proof of Lemma 10 until the final section.

LEMMA 10. *Let* $1<\Delta<Y^{1/2}$. *Suppose that* $|\alpha-a/q|\leqq 1/q^2$ *with* $(a, q)=1$. *Let* $E, \varepsilon>0$ *be given. If* $4Y/z\leqq Y^{8/15-\varepsilon}$ *then*

$$\Delta^2\int_{-1/2\Delta}^{1/2\Delta}|T(\alpha+\beta)|^2d\beta\ll\Delta Y\mathcal{L}^F\{Y^{1/4}+\Delta q^{-1/2}+(q\Delta)^{1/2}\}+\Delta^2Y\mathcal{L}^{-E}$$

where the implied constant depends only on E *and* ε.

We take $\Delta=yQ_1^{-1}=QQ_1$. The assumption of Lemma 10 is satisfied, because of (3.10). For any $\alpha\in m$ there exist a and q such that $|\alpha-a/q|\leqq 1/q^2$, $(a, q)=1$ and $Q_1<q\leqq Q$. Thus, Lemma 10 yields

$$\sup_{\alpha\in m}\Delta^2\int_{-1/2\Delta}^{1/2\Delta}|T(\alpha+\beta)|^2d\beta\ll\sup_{Q_1<q\leq Q}\Delta Y\mathcal{L}^F\{Y^{1/4}+\Delta q^{-1/2}+(q\Delta)^{1/2}\}+\Delta^2Y\mathcal{L}^{-E}$$
$$\ll\Delta^2Y\mathcal{L}^FQ_1^{-1/2}.$$

Combining this with (3.12) we obtain

$$\sum_{2k\in\mathcal{K}}\mathfrak{R}_2(2k)^2\ll yY^2\mathcal{L}^FQ_1^{-1/4}.\tag{3.13}$$

For any given $E>0$, we take $D=4(E+F)$. Then (3.8), (3.11) and (3.13) become

$$\sum_{i=0}^{2}\sum_{2k\in\mathcal{K}}\mathfrak{R}_i(2k)^2\ll yY^2\mathcal{L}^{-E}.\tag{3.14}$$

Finally we calculate $\Xi\Gamma$. Because of (3.10),

$$(\log Y)\Xi\Gamma = \frac{(1/2)\log(8Y/z^2)\cdot(\log Y)}{\log(Y/z)\cdot\log z}$$

$$\leqq \frac{(1/2)\cdot(1/15)}{(8/15)\cdot(7/15)} + \delta_1$$

$$< \frac{1}{7} + \delta_1, \tag{3.15}$$

for sufficiently large Y and small δ_1. In conjunction with (3.3), (3.9), (3.14) and (3.15) we therefore have

PROPOSITION 1.

$$\Omega \leqq \left(\frac{1}{7} + \delta_1\right)\mathfrak{S}(2k)\frac{Y}{\log Y} - \sum_{i=0}^{2} \mathfrak{R}_i + O(Y(\log Y)^{-3}).$$

Here, $\mathfrak{R}_i$'s satisfy (3.14).

In the next stage of the proof we use Sieve method. We remember

$$\mathcal{S} = \sum_{\substack{2k=p+n \\ p\in\mathcal{B} \\ (n,\,\mathcal{P}(2k,\,z))=1}} \log p = \sum_{\substack{p\in\mathcal{B} \\ (2k-p,\,\mathcal{P}(2k,\,z))=1}} \log p.$$

For d with $(d, 2k)=1$, put

$$r(\mathcal{B}, d, 2k) = \sum_{\substack{p\in\mathcal{B} \\ p\equiv 2k(d)}} \log p - \frac{Y}{\varphi(d)}$$

$$= \sum_{\substack{2k=p+n \\ p\in\mathcal{B} \\ d\mid n}} \log p - \frac{Y}{\varphi(d)}.$$

Then Lemma 5 yields, on taking $D=Y^{1-\delta}>z^2$,

$$\mathcal{S} \geqq Y \prod_{\substack{p<z \\ p\nmid 2k}} \left(1-\frac{1}{\varphi(p)}\right)\left\{f\left(\frac{\log D}{\log z}\right) + O((\log \mathcal{L})^{-1})\right\}$$

$$+ \sum_{\substack{(d,\,2k)=1 \\ d<D}} \lambda_d r(\mathcal{B}, d, 2k)$$

$$= \Pi + \Sigma, \text{ say.} \tag{3.16}$$

We first consider Σ.

$$\Sigma = \sum_{\substack{(d,\,2k)=1 \\ d<D}} \lambda_d \sum_{\substack{p\in\mathcal{B} \\ p\equiv 2k(d)}} \log p - Y \sum_{\substack{(d,\,2k)=1 \\ d<D}} \frac{\lambda_d}{\varphi(d)}$$

$$= \sum_{d<D} \lambda_d \sum_{\substack{2k=p+n \\ p\in\mathcal{B} \\ d\mid n}} \log p - Y \sum_{d<D} \frac{\lambda_d}{d} \sum_{q\mid d} \frac{\mu(q)}{\varphi(q)} c_q(2k)$$

$$+ O\Bigg(\sum_{\substack{(d,\,2k)>1 \\ d<D}} |\lambda_d| \sum_{\substack{p\in\mathcal{B} \\ p\equiv 2k(d)}} \log p\Bigg)$$

$$=\sum_{\substack{2k=p+n\\ p\in\mathcal{B}}}(\log p)(\sum_{\substack{d\mid n\\ d<D}}\lambda_d)-Y\sum_{q\leq Q_1}\frac{\mu(q)}{\varphi(q)}\Big(\sum_{\substack{d<n\\ q\mid d}}\frac{\lambda_d}{d}\Big)c_q(2k)$$

$$+O\Big(D\mathcal{L}^2+Y\mathcal{L}\sum_{Q_1<q<D}\frac{|\mu(q)|}{\varphi(q)q}|c_q(2k)|\Big)$$

$$=Z-Z_1+\mathfrak{R}_3,\ \text{say.} \tag{3.17}$$

By [15, Kap. VI], as before, we see

$$\sum_{2k\in\mathcal{B}}\mathfrak{R}_3(2k)^2\ll yY^2\mathcal{L}^4Q_1^{-1}. \tag{3.18}$$

We next appeal to Circle method. Put

$$S(\alpha)=\sum_{n\in\mathcal{A}}(\sum_{\substack{d\mid n\\ d<D}}\lambda_d)e(\alpha n).$$

For $\alpha=a/q+\beta\in I_{q,a}$, we write

$$U(\alpha)=\sum_{n\in\mathcal{A}}\Big((\sum_{\substack{d\mid n\\ d<D}}\lambda_d)e\Big(\frac{a}{q}n\Big)-\Big(\sum_{\substack{d<D\\ q\mid d}}\frac{\lambda_d}{d}\Big)\Big)e(\beta n).$$

We then have

$$Z=\int_{1/Q}^{1+1/Q}S(\alpha)W(\alpha)e(-2k\alpha)d\alpha$$

$$=\int_M(S(\alpha)-U(\alpha))W(\alpha)e(-2k\alpha)d\alpha$$

$$+\int_M U(\alpha)W(\alpha)e(-2k\alpha)da$$

$$+\int_m S(\alpha)W(\alpha)e(-2k\alpha)d\alpha$$

$$=Z_0+\mathfrak{R}_4+\mathfrak{R}_5,\quad\text{say}. \tag{3.19}$$

By the argument similar to that for Ω, we infer that

$$Z_0=Z_1+O(Y\mathcal{L}^{-3}), \tag{3.20}$$

$$\sum_{2k\in\mathcal{K}}\mathfrak{R}_4(2k)^2\ll Q_1^2Y^2\mathcal{L}^4+yY^{3/2}\mathcal{L}^3\sup_{\substack{(a,q)=1\\ q\leq Q_1}}\Big(\int_{I_{q,a}}|U(\alpha)|^2d\alpha\Big)^{1/2},$$

$$\sum_{2k\in\mathcal{K}}\mathfrak{R}_5(2k)^2\ll\Delta Y^2\mathcal{L}^4+yY^{3/2}\mathcal{L}^3\sup_{\alpha\in m}\Big(\int_{-1/2\Delta}^{1/2\Delta}|S(\alpha+\beta)|^2d\beta\Big)^{1/2}.$$

We now use the following lemmas, which will be proven later.

LEMMA 11. *In Lemma* 9, *replace the sum* V *and the condition* $4Y/z\leqq Y^{8/15}$ *by* U *and* $D\leqq Y^{1-\varepsilon}$, *respectively. Then, the resulting assertion holds true.*

LEMMA 12. *In Lemma* 10, *replace the sum* T *and the condition* $4Y/z\leqq Y^{8/15-\varepsilon}$

by S and $D \leqq Y^{1-\varepsilon}$, respectively. Then, the resulting assertion holds true.

On taking the parameters as before, we find

$$\sum_{i=3}^{5} \sum_{2k \in \mathcal{K}} \mathfrak{R}_i(2k)^2 \ll y Y^2 \mathcal{L}^{-E}, \tag{3.21}$$

for any $E>0$. In conjunction with (3.17)-(3.20) we obtain

$$\Sigma = \sum_{i=3}^{5} \mathfrak{R}_i + O(Y\mathcal{L}^{-3}). \tag{3.22}$$

We turn now to Π. By Mertens' theorem,

$$\begin{aligned}
\prod_{\substack{p<z \\ p \nmid 2k}} \left(1-\frac{1}{\varphi(p)}\right) &= \prod_{2<p<z} \left(\frac{p-1}{p} \frac{p(p-2)}{(p-1)^2}\right) \cdot \prod_{\substack{2<p<z \\ p \mid k}} \left(\frac{p-1}{p-2}\right) \\
&= \prod_{p<z} \left(1-\frac{1}{p}\right) \cdot 2 \prod_{2<p<z} \left(1-\frac{1}{(p-1)^2}\right) \prod_{\substack{p \mid k \\ 2<p<z}} \left(\frac{p-1}{p-2}\right) \\
&= \prod_{p<z} \left(1-\frac{1}{p}\right) \cdot 2 \prod_{2<p<z} \left(1-\frac{1}{(p-1)^2}\right) \prod_{\substack{p \mid k \\ 2<p<z}} \left(\frac{p-1}{p-2}\right) \\
&= \frac{e^{-\gamma}}{\log z} \mathfrak{S}(2k)(1+O((\log z)^{-1})).
\end{aligned}$$

Thus,

$$\Pi = \mathfrak{S}(2k) Y \frac{e^{-\gamma}}{\log z} \frac{2e^{\gamma} \log(\log D/\log z - 1)}{\log D/\log z} (1+O((\log\log Y)^{-1}))$$

$$= \mathfrak{S}(2k) Y \cdot K, \quad \text{say}. \tag{3.23}$$

Since $\log x > 1-1/x$ for $x>1$, we see

$$\begin{aligned}
(\log Y)K &= \frac{2\log(\log D/\log z - 1)}{\log D/\log Y} (1+O((\log\log Y)^{-1})) \\
&\geqq 2\log\left(\frac{15}{7}-1\right) - \delta_2 \\
&> \frac{1}{4} - \delta_2,
\end{aligned}$$

for sufficiently large Y and small δ_2. Hence, (3.23) becomes

$$\Pi > \left(\frac{1}{4} - \delta_2\right) \frac{\mathfrak{S}(2k) Y}{\log Y}.$$

Combining this with (3.16) and (3.22) we therefore obtain

PROPOSITION 2.

$$\mathcal{S}>\left(\frac{1}{4}-\delta_2\right)\mathfrak{S}(2k)\frac{Y}{\log Y}+\sum_{i=3}^{5}\mathfrak{R}_i+O(Y(\log Y)^{-3}).$$

Here $\mathfrak{R}_i$*'s satisfy* (3.21).

In conjunction with (3.2), Propositions 1 and 2, we finally have

$$\sum_{\substack{2k=p+p'\\ p\in\mathcal{B}}}\log p>\mathcal{S}(2k)-\Omega(2k)$$

$$>\left(\frac{1}{4}-\frac{1}{7}-(\delta_1+\delta_2)\right)\mathfrak{S}(2k)\frac{Y}{\log Y}+\sum_{i=0}^{5}\mathfrak{R}_i(2k)+O(Y(\log Y)^{-3})$$

$$>\frac{\mathfrak{S}(2k)Y}{10\log Y}+R(2k), \tag{3.22}$$

for sufficiently large Y. Here, $R(2k)=\sum_{i=0}^{5}\mathfrak{R}_i(2k)$.

We shall derive Theorem from (3.22), (3.14) and (3.19). We first note $\mathfrak{S}(2k)>c_5$, see [15, Kap. VI]. Let $\mathcal{E}$ denote the exceptional set in Goldbach's problem. For all $2k\in\mathcal{E}$, the left hand side of (3.22) is zero. If $|R(2k)|\leqq\mathfrak{S}(2k)Y/11\log Y$ for some $2k\in\mathcal{K}=(x, x+y]$, then the right hand side of (3.22) is posititive, whence $2k\notin\mathcal{E}\cap\mathcal{K}$. Consecuently, it follows from (3.14) and (3.19) that, for any $A>0$,

$$\#\mathcal{E}\cap\mathcal{K}\left(\frac{Y}{\log Y}\right)^2\ll\sum_{2k\in\mathcal{E}\cap\mathcal{K}}|R(2k)|^2$$

$$\leqq\sum_{2k\in\mathcal{K}}|R(2k)|^2$$

$$\ll yY^2\mathcal{L}^{-(A+2)}$$

or

$$E(x+y)-E(x)=\#\mathcal{E}\cap\mathcal{K}\ll y\mathcal{L}^{-A},$$

as required.

This completes our proof of Theorem, apart from the verification of Lemmas 9, 10, 11 and 12.

4. Major arc.

Let ε', $B>0$ be given. Throughout this section we assume

$$x^{1/4+\varepsilon'}\leqq\Delta\leqq x^{1/2},\quad (a, q)=1,\quad q\leqq(\log x)^B. \tag{4.1}$$

We use the convention

$$|f|_i=\int_x^{2x}|f(t, \Delta, q, a)|^i dt,\qquad i=1, 2.$$

We call $f(t, \Delta, q, a)$ "admissible" if for any $E>0$

$$|f|_2 \ll \Delta^2 x \mathcal{L}^{-E}$$

where the implied constant depends only on ε', B and E. An admissible function is abbreviated to "A.R." in a formula. Under the assumption (4.1) we shall show Lemmas $9^\dagger$ and $11^\dagger$ below. By Lemma 7, Lemmas 9 and 11 follow from Lemmas $9^\dagger$ and $11^\dagger$, respectively.

LEMMA $9^\dagger$. *Define*

$$v=v(t, \Delta, q, a; M, N)=\sum_{\substack{t<mn\leqq t+\Delta\\ M<n\leqq N}} \Lambda(m)\Lambda(n)e\Big(\frac{a}{q}mn\Big).$$

If $x^{1/2}\leqq M<N\leqq x^{8/15}$ *then*

$$v=\frac{\mu(q)}{\varphi(q)}\Delta \log\Big(\frac{N}{M}\Big)+A.R..$$

PROOF. The terms with $(mn, q)>1$ contribute to v at most

$$\sum_{\substack{(mn,q)>1\\ t<mn\leqq t+\Delta\\ x^{\varepsilon}\ll m,\, n\ll x^{1-\varepsilon}}} \Lambda(m)\Lambda(n) \ll \sum_{\substack{x^{\varepsilon}\ll m\ll x^{1-\varepsilon}\\ (m,q)>1}} \Lambda(m)\Big(\frac{\Delta}{m}+1\Big)\mathcal{L}$$
$$\ll (\Delta x^{-\varepsilon}+1)\mathcal{L}^3,$$

which is admissible trivially. We then have

$$v=\frac{1}{\varphi(q)}\sum_{\chi(q)}\chi(a)\tau(\bar{\chi})\sum_{\substack{t<mn\leqq t+\Delta\\ n\in\mathcal{J}}}\chi(mn)\Lambda(m)\Lambda(n)+A.R..$$

Here $\mathcal{J}=(M, N]$. By the explicit formula [15, Kap. VII Satz 4.4.],

$$\sum_{t/n<m\leqq(t+\Delta)/n}\chi(m)\Lambda(m)=E_0\frac{\Delta}{n}$$
$$-\sum_{\substack{\rho(\chi)\\ |\mathrm{Im}(\rho)|\leqq x^2}}\frac{(t+\Delta)^{\rho}-t^{\rho}}{n^{\rho}\rho}$$
$$+O\Big(\frac{x/n}{x^2}(\log qx)^2+(\log qx)\sum_{j=0,1}\min\Big(1, \frac{x/n}{x^2\|(t+j\Delta)/n\|}\Big)\Big)$$

where $E_0=1$ when χ is the principal character and $E_0=0$ otherwise. Thus,

$$v=\frac{\mu(q)}{\varphi(q)}\Delta\sum_{\substack{n\in\mathcal{J}\\ (n,q)=1}}\frac{\Lambda(n)}{n}$$
$$-\frac{1}{\varphi(q)}\sum_{\chi(q)}\chi(a)\tau(\bar{\chi})\sum_{\substack{\rho(\chi)\\ |\mathrm{Im}(\rho)|\leqq x^2}}\frac{(t+\Delta)^{\rho}-t^{\rho}}{\rho}\sum_{n\in\mathcal{J}}\frac{\chi(n)\Lambda(n)}{n^{\rho}}$$
$$+O\Big(\sqrt{q}\Big(\mathcal{L}^3x^{-1}+\mathcal{L}^2\sum_{j=0,1}\sum_{n\in\mathcal{J}}\min\Big(1, \frac{1/x}{n\|(t+j\Delta)/n\|}\Big)\Big)\Big)+A.R.$$

$$=\frac{\mu(q)}{\varphi(q)}\Delta\log\left(\frac{N}{M}\right)$$

$$-\frac{1}{\varphi(q)}\sum_{\chi(q)}\chi(a)\tau(\bar{\chi})\Big(\sum_{\substack{\rho(\chi)\\ |\mathrm{Im}(\rho)|\le x}}+\sum_{\substack{\rho(\chi)\\ x<|\mathrm{Im}(\rho)|\le x^2}}\Big)\theta(\rho)N(\rho,\chi)$$

$$+O\Big(\sqrt{q}\mathcal{L}^2\sum_{j=0,1}\sum_{n\in\mathcal{I}}\min\Big(1,\frac{1}{nx\|(t+j\Delta)/n\|}\Big)\Big)+A.R.$$

(4.2) $$=v_0-(v_1+v_2)+O(v_3)+A.R.,\ \text{say,}$$

where

$$N(s,\chi)=\sum_{n\in\mathcal{I}}\chi(n)\Lambda(n)n^{-s},$$

$$\theta(s)=\theta(s,t,\Delta)=\frac{(t+\Delta)^s-t^s}{s}.$$

First we consider v_3.

$$|v_3|_1\ll\sqrt{q}\mathcal{L}^2\sum_{n\in\mathcal{I}}\int_x^{3x}\min\Big(1,\frac{1}{nx\|t/n\|}\Big)dt$$

$$\ll\sqrt{q}\mathcal{L}^2\sum_{n\in\mathcal{I}}x^{-1}\int_{x/n}^{3x/n}\min\Big(nx,\frac{1}{\|u\|}\Big)du$$

$$\ll\sqrt{q}\mathcal{L}^4.$$

Since $v_3\ll\sqrt{q}\mathcal{L}^2N$ trivially, we have

(4.3) $$|v_3|_2\ll qN\mathcal{L}^6.$$

Hence v_3 is admissible.

We proceed to v_2. We note that the number of ρ with $T<|\mathrm{Im}(\rho)|\leqq T+1$ is $O(\log T)$. By partial summation and Lemma 4, we then have

$$|v_2|_2\ll\mathcal{L}\sum_{\chi(q)}\int_x^{2x}\Big|\sum_{\substack{\rho(\chi)\\ x<|\mathrm{Im}(\rho)|\le x^2}}\frac{t^\rho}{\rho}N(\rho,\chi)\Big|^2dt$$

$$\ll\mathcal{L}\sum_{\chi(q)}\sum_{\substack{\rho(\chi)\\ x<|\mathrm{Im}(\rho)|\le x^2}}\frac{x^{1+2\mathrm{Re}(\rho)}}{|\rho|^2}|N(\rho,\chi)|^2\sum_{\substack{\rho'(\chi)\\ |\mathrm{Im}(\rho')|\le x^2}}\frac{1}{1+|\rho'+\bar{\rho}|}$$

$$\ll\mathcal{L}^5\sup_{\substack{x<U\le x^2\\ 0\le\sigma\le1}}\frac{x^{1+2\sigma}}{U^2}(qU+N)M^{1-2\sigma}\mathcal{L}^{12}$$

$$\ll\mathcal{L}^{17}\sup_\sigma\Big(qM\Big(\frac{x}{N}\Big)^{2\sigma}+N\frac{x}{M}\Big)^{2\sigma-1}\Big)$$

(4.4) $$\ll qx^{3/2}\mathcal{L}^{17}.$$

Because of (4.1), v_2 is also admissible.

We turn now to v_1. We write $\theta(\rho)=\int_1^{1+\Delta/t} u^{\rho-1}du\cdot t^\rho$ for $|\mathrm{Im}(\rho)|\leqq x/\Delta$. On splitting the remaining range of $|\mathrm{Im}(\rho)|$ into intervals $((x/\Delta)2^j, (x/\Delta)2^{j+1}]$, we use Cauchy's inequality. Thus,

$$|\sum_\rho \theta(\rho)N(\rho,\chi)|_2 \ll \left(\frac{\Delta}{x}\right)^2 \sup_{1\leqq u\leqq 1+\Delta/x}\int_x^{2x}|\sum_{\substack{\rho(\chi)\\ |\mathrm{Im}(\rho)|\leqq x/\Delta}} u^{\rho-1}t^\rho N(\rho,\chi)|^2 dt$$

$$+\mathcal{L}^2 \sup_{x/\Delta\leqq U\leqq x/2}\int_x^{3x}\left|\sum_{\substack{\rho(\chi)\\ U<|\mathrm{Im}(\rho)|\leqq 2U}} \frac{t^\rho}{\rho}N(\rho,\chi)\right|^2 dt$$

$$\ll \mathcal{L}^4 \sup_{x/\Delta\leqq U\leqq x}\sum_{\substack{\rho(\chi)\\ |\mathrm{Im}(\rho)|\leqq U}} \frac{x^{1+2\mathrm{Re}(\rho)}}{U^2}|N(\rho,\chi)|^2 .$$

Hence, by partial summation,

$$|v_1|_2 \leqq \frac{q}{\varphi(q)}\sum_{\chi(q)}|\sum_{\rho(\chi)}\theta N|_2$$

$$\ll \mathcal{L}^6 \sup_{\substack{x/\Delta\leqq U\leqq x\\ 0\leqq\sigma\leqq 1}} \frac{x^{1+2\sigma}}{U^2}\sum_{\chi(q)}\sum_{\substack{\rho(\chi)\\ \mathrm{Re}(\rho)\geqq\sigma\\ |\mathrm{Im}(\rho)|\leqq U}} |N(\rho,\chi)|^2$$

$$=\mathcal{L}^6 \sup_{\substack{x/\Delta\leqq U\leqq x\\ 0\leqq\sigma\leqq 1}} x^{1+2\sigma}U^{-2}K_\sigma(U), \quad \text{say}. \tag{4.5}$$

In case of $0\leqq\sigma\leqq 1/2$ Lemma 4 yields

$$K_\sigma(U)\ll \mathcal{L}^F(qU+N)N^{1-2\sigma},$$

since $1-2\sigma$, $2-2\sigma\geqq 0$. Moreover the supremum over U is attained at $U=x/\Delta$. This contributes to $|v_1|_2$

$$\ll \mathcal{L}^F \sup_{0\leqq\sigma\leqq 1/2}\Delta^2 x^{2\sigma-1}\left(q\frac{x}{\Delta}+N\right)N^{1-2\sigma}$$

$$\ll \mathcal{L}^F\Delta^2 x \sup_{0\leqq\sigma\leqq 1/2}\left(\frac{q}{\Delta}\left(\frac{x}{N}\right)^{2\sigma-1}+\left(\frac{x}{N}\right)^{2\sigma-2}\right)$$

$$\ll \Delta^2 x\mathcal{L}^F\left(\frac{q}{\Delta}+\frac{N}{x}\right)$$

$$\ll \Delta^2 x^{3/4}. \tag{4.6}$$

In another case $1/2\leqq\sigma\leqq 1$, Lemma 2 yields that the supremum may be taken over $1/2\leqq\sigma\leqq 1-\eta(x)$ only, where $\eta(x)=c_4(\log x)^{-4/5}$. Furtheremore, by Lemmas 3 and 4, we have

$$K_\sigma(U)\leqq\Big(\sum_{\chi(q)}\sum_{\substack{\rho(\chi)\\ \mathrm{Re}(\rho)\geqq\sigma\\ |\mathrm{Im}(\rho)|\leqq U}} 1\Big)^{1/2}\Big(\sum_{\chi(q)}\sum_{\substack{\rho(\chi)\\ \mathrm{Re}(\rho)\geqq\sigma\\ |\mathrm{Im}(\rho)|\leqq U}} |N(\rho,\chi)|^4\Big)^{1/2}$$

$$\ll \mathcal{L}^F((qU)^{k(1-\sigma)})^{1/2}(\sup_{M\leqq V\leqq N}(qU+V^2)V^{2(1-2\sigma)})^{1/2}.$$

Here $k>12/5$. We see the supremum over U is attained at $U=x/\Delta$ again. Then, since $x/\Delta\leqq M^2$ and $1-\sigma\geqq 0$, we have that the contribution to $|v_1|_2$ is at most

$$\Delta^2 xq\mathcal{L}^F \sup_{1/2\leqq\sigma\leqq 1-\eta(x)}\left(\left(\frac{x}{\Delta}\right)^{k/4}\frac{N}{x}\right)^{2(1-\sigma)}. \tag{4.7}$$

We now note that

$$\left(1-\frac{1}{4}\right)\frac{12}{5}\frac{1}{4}+\left(\frac{8}{15}-1\right)=-\frac{1}{60}.$$

Hence, there exists a constant $\xi>0$ such that

$$\left(\frac{x}{\Delta}\right)^{k/4}\frac{N}{x}\leqq x^{-\xi}.$$

Thus (4.7) becomes

$$\begin{aligned}
&\ll\Delta^2 xq\mathcal{L}^F \sup_{1/2\leqq\sigma\leqq 1-\eta(x)} x^{-2\xi(1-\sigma)}\\
&\ll\Delta^2 xq\mathcal{L}^F \exp(-c_6(\log x)^{-1/5})\\
&\ll\Delta^2 x\mathcal{L}^{-E}, \qquad (4.8)
\end{aligned}$$

for any $E>0$.

In conjunction with (4.5), (4.6) and (4.8), v_1 is admissible. Combining this with (4.2), (4.3) and (4.4), we conclude that $v-v_0$ is also admissible, as required.

LEMMA 11†. *Define, for arbitrary sequence* (λ_d) *with* $|\lambda_d|\leqq 1$,

$$u=u(t,\Delta,q,a;D)=\sum_{t<n\leqq t+\Delta}\Big(\sum_{\substack{d|n\\ d<D}}\lambda_d\Big)e\left(\frac{a}{q}n\right).$$

For any $\varepsilon>0$, *if* $D\leqq x^{1-\varepsilon}$ *then*

$$u=\Big(\sum_{\substack{d<D\\ q|d}}\frac{\lambda_d}{d}\Big)\Delta+A.R..$$

PROOF. Put

$$f(s)=\sum_{n=1}^{\infty}\Big(\sum_{\substack{d|n\\ d<D}}\lambda_d\Big)e\left(\frac{a}{q}n\right)n^{-s},$$

which is absolutely convergent for $\mathrm{Re}(s)>1$, since $|f(s)|\leqq\zeta^2(\mathrm{Re}(s))$. On writing $n=dm$, $m=(d,q)l$ and $q=(d,q)r$, we eliminate n and m. Then the conditions on l and r are

$$r|q, \qquad (l,r)=1.$$

Thus, we have

$$f(s)=\sum_{d<D}\sum_{r\mid q}\sum_{b=1}^{r}{}^{*}\sum_{\substack{l>0\\ l\equiv b(r)}}\lambda_d e\Big(\frac{a}{q}\frac{dql}{r}\Big)\Big(\frac{dql}{r}\Big)^{-s}$$

$$=\frac{1}{q^s}\sum_{r\mid q}\sum_{b=1}^{r}{}^{*}\Big(\sum_{d<D}\lambda_d e\Big(\frac{abd}{r}\Big)d^{-s}\Big)r^s\Big(\sum_{\substack{l>0\\ l\equiv b(r)}}l^{-s}\Big)$$

$$=\frac{1}{q^s}\sum_{r\mid q}\sum_{b=1}^{r}{}^{*}D\Big(s,\frac{b}{r}\Big)\zeta\Big(s,\frac{b}{r}\Big),\quad\text{say.}\tag{4.9}$$

Hence $f(s)$ is regular, except for a simple pole at $s=1$. By Perron's formula [15, Anhang Satz 3.1.],

$$u=\frac{1}{2\pi i}\int_{c-ix}^{c+ix}f(s)\theta(s)ds+O((\log x)^3),$$

where $c=1+(\log x)^{-1}$. Now,

$$D\Big(s,\frac{b}{r}\Big)=\sum_{d\leq x/\Delta}+\sum_{x/\Delta<d<D}=D_1+D_2,\quad\text{say}.$$

Let the corresponding expressions be f_1 and f_2, respectively. Put $\eta=\eta(x)=c_2(\log x)^{-4/5}$. On moving the line of integration, we have

$$u=\operatorname*{Res}_{s=1}f(s)\theta(s)$$

$$+\frac{1}{2\pi i}\int_{1/2-ix}^{1/2+ix}f_1(s)\theta(s)ds+O\Big(\int_{1/2\pm ix}^{c\pm ix}|f_1(s)\theta(s)||ds|\Big)$$

$$+\frac{1}{2\pi i}\int_{1-\eta-ix}^{1-\eta+ix}f_2(s)\theta(s)ds+O\Big(\int_{1-\eta\pm ix}^{c\pm ix}|f_2(s)\theta(s)||ds|\Big)+A.R.$$

$$=u_0+u_1+u_2+O(u_3)+A.R.,\quad\text{say.}\tag{4.10}$$

First,

$$u_0=\operatorname*{Res}_{s=1}=\frac{\Delta}{q}\sum_{r\mid q}\sum_{b=1}^{r}{}^{*}D\Big(1,\frac{b}{r}\Big)=\Delta\Big(\sum_{\substack{d<D\\ q\mid d}}\frac{\lambda_d}{d}\Big).\tag{4.11}$$

Since $\theta(s)\ll x^{\mathrm{Re}(s)-1}$ for $|\mathrm{Im}(s)|=x$, we have

$$u_3\ll\sup_{\substack{1/2\leq\sigma\leq c\\ j=1,2}}|f_j(\sigma\pm ix)|x^{\sigma-1}$$

$$\ll\sup_{1/2\leq\sigma\leq c}\sum_{r\mid q}\sum_{b=1}^{r}{}^{*}D^{1-\sigma}\mathcal{L}\Big|\zeta\Big(\sigma\pm ix,\frac{b}{r}\Big)\Big|x^{\sigma-1}$$

$$\ll\sup_{1/2\leq\sigma\leq c}q^2\mathcal{L}^2(Dx^{-2/3})^{1-\sigma}$$

$$\ll q^2x^{1/6},$$

by Lemma 1. Hence, because of (4.1), u_3 is admissible.

We proceed to u_1. Lemma 1 yields that

$$|u_1|_2 \ll \mathcal{L}^2 \sup_{x/\Delta \leq U \leq x} \left(\frac{x}{U}\right)^2 \sum_{r|q} \sum_{b=1}^{r}{}^* \int_{1/2-iU}^{1/2+iU} \left| D_1\left(s, \frac{b}{r}\right) \zeta\left(s, \frac{b}{r}\right)\right|^2 |ds|$$

$$\ll \mathcal{L}^2 \sup_{x/\Delta \leq U \leq x} \left(\frac{x}{U}\right)^2 \sum_{r|q} \sum_{b=1}^{r}{}^* \Big(\sup_{\substack{\mathrm{Re}(s)=1/2 \\ |\mathrm{Im}(s)| \leq U}} \left|\zeta\left(s, \frac{b}{r}\right)\right|^2 \Big) \int_{1/2-iU}^{1/2+iU} \left| D_1\left(s, \frac{b}{r}\right)\right|^2 |ds|$$

$$\ll \mathcal{L}^5 \sup_{x/\Delta \leq U \leq x} \left(\frac{x}{U}\right)^2 q U^{1/3}\left(U + \frac{x}{\Delta}\right)$$

$$\ll \mathcal{L}^5 \Delta^2 q \left(\frac{x}{\Delta}\right)^{4/3}. \tag{4.13}$$

Since $\Delta \geqq x^{1/4+\varepsilon'}$, u_1 is admissible.

We turn now to u_2. As before, by Lemma 1, we have

$$|u_2|_2 \ll \mathcal{L}^2 \sup_{x/\Delta \leq U \leq x} x^{1+2(1-\eta)} U^{-2} \sum_{r|q} \sum_{b=1}^{r}{}^* \int_{1-\eta-iU}^{1-\eta+iU} \left| D_2\left(s, \frac{b}{r}\right) \zeta\left(s, \frac{b}{r}\right)\right|^2 |ds|$$

$$\ll q \mathcal{L}^4 \exp(2c_1 (\log\log x)^3) x^{3-2\eta} \sup_{\substack{x/\Delta \leq U \leq x \\ x/\Delta \leq N \leq D}} U^{-2}(U+N) N^{1-2(1-\eta)}$$

$$\ll q \mathcal{L}^4 \exp(2c_1 (\log\log x)^3) \Delta^2 x \left(\frac{D}{x}\right)^{2\eta}.$$

Since $\eta = c_2 (\log x)^{-4/5}$, $D \leqq x^{1-\varepsilon}$ and $q \leqq (\log x)^B$, we finally obtain

$$|u_2|_2 \ll \Delta^2 x \exp(-\varepsilon c_2 (\log x)^{-1/6}). \tag{4.14}$$

Thus, u_2 is also admissible.

In conjunction with (4.10)-(4.14), we have

$$u = u_0 + A.R.,$$

as required.

5. Auxiliary results.

In this section we provide for the proof of Lemmas 16 and 17 in the next section. For real numbers a, b and c, $a \leqq b$, we set

$$\begin{aligned} \Phi(a, b; c) &= \psi(b-c) - \psi(a-c) \\ &= [b-c] - [a-c] - (b-a). \end{aligned}$$

Let α, β, γ and δ be arbitrary sequences with modulus $\leqq 1$. Moreover, let 1 denote the arithmetical function $1(n) \equiv 1$ or $\equiv \log n$. We consider the following linear forms involving Φ.

$$\begin{aligned} F_1 &= F_1(K, D, M, N; x, s, t) \\ &= \sum_{0<k\leq K} \sum_{d \sim D} \sum_{\substack{m \sim M \\ (d, mn)=1 \\ mn \leq t}} \sum_{n \sim N} \alpha_d \beta_m 1(n) \delta_k \Phi\left(\frac{2x}{dmn}, \frac{\max(x, ds, mns)}{dmn}; k\frac{\overline{mn}}{d}\right). \end{aligned}$$

$$F_2=F_2(K, D, M, N;\ x, s, t)$$
$$=\sum_{0<k\leq K}\sum_{d\sim D}\sum_{\substack{m\sim M\\ (d,mn)=1\\ mn\leq t}}\sum_{n\sim N}\alpha_d\beta_m\gamma_n\delta_k\Phi\Big(\frac{2x}{dmn},\ \frac{\max(x, ds, mns)}{dmn};\ k\frac{\bar{d}}{mn}\Big).$$

Here, $1\ll K, D, M, N\ll x$, $MN\leqq t$ and $s\leqq 2x/D, 2x/MN$.

Our aim is to prove $F\ll Kx^{1-\varepsilon}$, $\varepsilon>0$. Since $F\ll KDMN$ trivially, we assume

$$DMN>x^{1-2\varepsilon}. \tag{5.1}$$

LEMMA 13. *For any $\varepsilon>0$, if $MN, DM\leqq x^{2/3-4\varepsilon}$ then*

$$F_1\ll Kx^{1-\varepsilon}.$$

PROOF. We appeal to the Erdös-Turán inequality [8, Lemme 2], which runs as follows. For arbitrary real numbers x_n and $H>2$,

$$|\sum_{n\sim N}\psi(x_n)|\ll\frac{N}{H}+\sum_{0<h\leq H}\frac{1}{h}|\sum_{n\sim N}e(hx_n)|.$$

Now, by partial summation, we have

$$\sum_{\substack{n\sim N\\ (n,d)=1\\ mn\leq t}}1(n)\Phi\ll\mathcal{L}\sup_{\substack{y\leq 2x/DM\\ \mathcal{I},\mathcal{J}\subset[N,2N]}}\Big(\Big|\sum_{\substack{n\in\mathcal{I}\\ (n,d)=1}}\psi\Big(\frac{y}{n}-k\frac{\overline{mn}}{d}\Big)\Big|+\Big|\sum_{\substack{n\in\mathcal{J}\\ (n,d)=1}}\psi\Big(\frac{s}{d}-k\frac{\overline{mn}}{d}\Big)\Big|\Big)$$

$$\ll\mathcal{L}\sup_{\substack{y\leq 2x/DM\\ \mathcal{I}\subset[N,2N]}}\Big(\frac{N}{H}+\sum_{0<h\leq H}\frac{1}{h}\Big|\sum_{\substack{n\in\mathcal{I}\\ (n,d)=1}}e\Big(\frac{hy}{n}\Big)e\Big(-hk\frac{\overline{mn}}{d}\Big)\Big|\Big).$$

It follows from the argument of [11, p. 37] that

$$F_1\ll\mathcal{L}\frac{KDMN}{H}+\mathcal{L}^2\sum_{0<h\leq H}\frac{1}{h}x^{\varepsilon}\Big(1+\frac{Hx}{DMN}\Big)\tau(h)KM(D^{3/2}+D^{1/2}N)$$

$$\ll\mathcal{L}\frac{KDMN}{H}+\mathcal{L}^4x^{\varepsilon}\Big(1+\frac{Hx}{DMN}\Big)Kx^{1-6\varepsilon}.$$

On taking $H=DMN/x^{1-3\varepsilon}>2$, because of (5.1), we obtain Lemma 13.

LEMMA 14. *For any $\varepsilon>0$, if $D, MN\leqq x^{8/15-4\varepsilon}$ and $N\geqq x^{1/15}$ then*

$$F_2\ll Kx^{1-\varepsilon}.$$

PROOF. We follow the argument of [11] with a minor modification. We use a well known Fourier expansion. For $H>2$,

$$\Phi(a, b;\ c)=\sum_{0<|h|\leq H}\frac{e(hb)-e(ha)}{2\pi ih}e(-hc) \tag{5.2}$$
$$+O(\min(1, 1/H\|b-c\|)+\min(1, 1/H\|a-c\|)).$$

Also,

(5.3) $$\min(1, 1/H\|x\|)=\sum_{h\in Z} A_h e(hx)$$

where

$$|A_h|\ll\min\left(\frac{\log H}{H}, \frac{H}{h^2}\right).$$

Now, Φ in F_2 is expressed as a sum of the main part of Fourier series, say Φ', and the tail part, say Φ''. On putting the resulting linear forms to F_2' and F_2'', we may write

$$F_2=F_2'+F_2''.$$

First we consider F_2''. (5.3) yields that

$$\begin{aligned}F_2''&\ll\sum_h|A_h|\Bigg(\Bigg|\sum_{0<k\le K}\sum_{d\sim D}\sum_{\substack{m\sim M\\(d,mn)=1}}\sum_{n\sim N}e\left(\frac{2hx}{dmn}\right)e\left(-hk\frac{\bar d}{mn}\right)\Bigg|\\&\quad+\Bigg|\sum_{0<k\le K}\sum_{d\sim D}\sum_{\substack{m\sim M\\(d,mn=1)}}\sum_{n\sim N}e\left(\frac{h\max(x, ds, mns)}{dmn}\right)e\left(-hk\frac{\bar d}{mn}\right)\Bigg|\Bigg)\\&=\sum_h|A_h|S(h),\text{ say.}\end{aligned}$$

Summing by parts, we see

$$S(h)\ll\sum_{0<k\le K}\sum_{m\sim M}\sum_{n\sim N}\left(1+\frac{hx}{DMN}\right)\sup_{\mathcal{I}\subset[D,2D]}\Bigg|\sum_{\substack{d\in\mathcal{I}\\(d,mn)=1}}e\left(hk\frac{\bar d}{mn}\right)\Bigg|.$$

By the same argument as that in [11, (17)(18)] with $q=1$, we then obtain

$$F_2''\ll Kx^{1-2\varepsilon}+\mathcal{L}^3x^{4\varepsilon}K\{(MN)^{3/2}+D(MN)^{1/2}\}.$$

Here we have taken $H=DMNx^{3\varepsilon-1}>2$, because of (5.1). Since D, $MN<x^{2/3-4\varepsilon}$, we have

$$F_2''\ll Kx^{1-\varepsilon}.$$

We turn now to F_2'. By (5.2),

$$\begin{aligned}F_2'=&\sum_{0<k\le K}\sum_{d\sim D}\sum_{\substack{m\sim N\\(d,mn)=1\\mn\le t}}\sum_{n\sim N}\alpha_d\beta_m\gamma_n\delta_k\\&\times\sum_{0<|h|\le H}\int_{\max(x,ds,mns)g/dmn}^{2xg/dmn}e\left(\frac{h}{g}y\right)\frac{dy}{g}\cdot e\left(-hk\frac{\bar d}{mn}\right),\end{aligned}$$

where g is arbitrary. After some elementary rearrengement, we reach the inequality

(5.4) $$F_2'\ll\frac{x}{DMN}\sup_B\sum_{d\sim D}\sum_{\substack{m\sim M\\(d,m)=1}}\Bigg|\sum_{0<h\le H}\sum_{0<k\le K}\sum_{\substack{n\sim N\\(d,n)=1\\mn\le t}}B(h, k, n)e\left(hk\frac{\bar d}{mn}\right)\Bigg|,$$

where the supremum is over all arithmetical functions B with $|B|\leqq 1$. To deal with the condition $mn\leqq t$, we use a lemma in Fourier analysis [1, Lemma 2.2.], which runs as follows. Let a_n be arbitrary finite sequence. If $X\leqq Y$ then

$$\left|\sum_{n\leqq X} a_n\right| \leqq \int_{-\infty}^{\infty} f(\theta, Y)\left|\sum_n a_n e(\theta n)\right| d\theta .$$

Here, the kernel $f\geqq 0$ satisfies

$$\int_{-\infty}^{\infty} f(\theta, Y)d\theta \ll \log Y .$$

Now, applying this lemma with $Y=t/M$, we may remove the condition $mn\leqq t$ in (5.4) with the cost of $(\log x)$ time. Thus, the method in [11, p. 38, 39] is applicable. We therefore have

$$F_2' \ll \mathcal{L} x^{3\varepsilon}\{(KDM)^{1/2}x^{1/2-\varepsilon}+Kx^{5\varepsilon/2}(D^{1/2}M^{5/4}N^{3/2}+DM^{3/4}N^{1/2})\}.$$

Since $DM<x^{8/15-4\varepsilon}\cdot x^{7/15-4\varepsilon}=x^{1-8\varepsilon}$, the above first term is

$$\begin{aligned}&\ll \mathcal{L} x^{3\varepsilon}K^{1/2}x^{1-5\varepsilon}\\ &\ll Kx^{1-\varepsilon}.\end{aligned}$$

We may assume $M\geqq N$, because of the symmetry of F_2, so that $M\geqq (MN)^{1/2}$. Hence, the second term is

$$\begin{aligned}&\ll Kx^{6\varepsilon}\{D^{1/2}(MN)^{3/2-1/8}+D(MN)^{1/2}M^{1/4}\}\\ &\ll Kx^{6\varepsilon}\{(x^{8/15-4\varepsilon})^{15/8}+(x^{8/15-4\varepsilon})^{3/2}(x^{7/15-4\varepsilon})^{1/4}\}\\ &\ll Kx^{1-\varepsilon}.\end{aligned}$$

We thus obtain Lemma 14.

6. Minor arc, preliminaries.

In this section we provide for the final section. We re-present the method of [10] with a few simplifications. Let $a(n)$, $b(n)$ and $c(n)$ be arbitrary arithmetical functions which are bounded by

$$\tau_{c_7}(n)(\log n)^{c_8}.$$

Also, $1(n)\equiv 1$ or $\equiv \log n$. Furthermore, we put

$$j\,\mathrm{I}(h;\ s)=\sum_{\substack{h=dl\\ l\geqq s}} a(d),$$

$$j\,\mathrm{II}(h;\ D, V)=\sum_{\substack{h=lmn\\ mn\leqq D\\ m\leqq V}} 1(l)a(m)1(n),$$

$$j\mathrm{III}(h;\ D, U)=\sum_{\substack{h=lmn\\ mn\leqq D\\ m,n\geqq U}} 1(l)a(m)b(n),$$

$$j\mathrm{IV}(h;\ U, V, \mathcal{J})=\sum_{\substack{h=mnk\\ V<m\leqq U\\ mk\in\mathcal{J}}} a(m)b(n)c(k).$$

Here, s, D, U, V's are parameters and $\mathcal{J}$ is an interval. We are devoted to estimating the mean value of trigonometrical sums

$$J\boldsymbol{i}(*)=J\boldsymbol{i}(\alpha, x, \Delta;\ *)=\Delta^2\int_{-1/2\Delta}^{1/2\Delta}\Big|\sum_{x<h\leqq 2x} j\boldsymbol{i}(h;\ *)e((\alpha+\beta)h)\Big|^2 d\beta$$

$$(\boldsymbol{i}=\mathrm{I},\ \mathrm{II},\ \mathrm{III},\ \mathrm{IV})$$

under the assumption

(61). $$\left|\alpha-\frac{a}{q}\right|\leqq\frac{1}{q^2}\quad\text{with}\quad (a, q)=1\quad\text{and}\quad 1<q<\Delta<x/2.$$

LEMMA 15. *For any $\varepsilon>0$,*

$$J\,\mathrm{I}\,(s)\ll\Delta x\mathcal{L}^F\{\Delta q^{-1/2}+(q\Delta)^{1/2}\}+\Delta^2\left(\frac{x}{s}\right)^2\mathcal{L}^F+\Delta^3x^{\varepsilon}.$$

PROOF. On writing $w_n=j\,\mathrm{I}\,(n;\ s)$, Lemma 8 yields that

$$J\,\mathrm{I}\ll\Delta\sum_{n\sim x}|w_n|^2+\Delta\sup_{t\leqq\Delta}\Big|\sum_{0<r\leqq t}e(\alpha r)\sum_{n\sim x}\overline{w}_n w_{n+r}\Big|+\Delta^3(\sup|w_n|)^2$$

(6.2) $$\ll\Delta x\mathcal{L}^F+\Delta\sup|\textstyle\sum|+\Delta^3x^{\varepsilon},$$

where

$$\sum=\sum_{0<r\leqq t}e(\alpha r)\sum_{d_1}\sum_{d_2}a(d_1)a(d_2)\#\left\{n:\ \begin{array}{c}\max(x, d_1s, d_2s)<n\leqq 2x\\ n\equiv 0\ (\mathrm{mod}\ d_1)\\ n\equiv r\ (\mathrm{mod}\ d_2)\end{array}\right\}+O(\Delta^2x^{\varepsilon}).$$

The above simultaneous congruences are soluble if and only if $(d_1, d_2)|r$. We write $d_i^*=d_i/(d_1, d_2)$ and $r^*=r/(d_1, d_2)$. We then have, with the notation in section 5, that

$$\#\{n\}=\#\left\{l:\ \frac{\max(x, d_1s, d_2s)}{[d_1, d_2]}<r^*\frac{\overline{d_1^*}}{d_2^*}+l\leqq\frac{2x}{[d_1, d_2]}\right\}$$

$$=\frac{2x-\max(x, d_1s, d_2s)}{[d_1, d_2]}+\Phi\left(\frac{2x}{[d_1, d_2]}, \frac{\max(x, d_1s, d_2s)}{[d_1, d_2]};\ r^*\frac{\overline{d_1^*}}{d_2^*}\right)$$

$$=\Phi_0+\Phi,\quad\text{say.}$$

If d_1 or $d_2>2x/s$ then $\#\{n\}=0$. Hence

$$\sum=\sum_{0<r\leqq t}e(\alpha r)\sum_{\substack{d_1, d_2\leqq 2x/s\\ (d_1, d_2)|r}}a(d_1)a(d_2)(\Phi_0+\Phi)+O(\Delta^2x^{\varepsilon})$$

(6.3) $$=\textstyle\sum_0+\sum_1+O(\Delta^2x^{\varepsilon}),\quad\text{say.}$$

It follows from [15, Kap. IV Lemma 6.3.] that

$$\Sigma_0 \ll x \sum_{\substack{d_1, d_2 \leqq 2x/s \\ (d_1, d_2) \leqq t}} \frac{|a(d_1)a(d_2)|}{[d_1, d_2]} \Big| \sum_{\substack{0<r\leqq t \\ (d_1, d_2) \mid r}} e(\alpha r) \Big|$$

$$\ll x \sum_{0<\delta\leqq\Delta} \Big(\delta \sum_{\substack{d \leqq 2x/s \\ \delta \mid d}} \frac{|a(d)|}{d}\Big)^2 \delta^{-1} \min\Big(\frac{\Delta}{\delta}, \frac{1}{\|\alpha\delta\|}\Big)$$

$$\ll x \Big(\sum_{\delta\leqq\Delta} \Big(\delta \sum_{\substack{d\leqq 2x \\ \delta \mid d}} \frac{|a(d)|}{d}\Big)^4 \delta^{-2}\Big)^{1/2} \Big(\sum_{0<\delta\leqq\Delta} \frac{\Delta}{\delta} \min\Big(\frac{\Delta}{\delta}, \frac{1}{\|\alpha\delta\|}\Big)\Big)^{1/2}$$

$$\ll x \mathcal{L}^F \Delta^{1/2} \Big(\frac{\Delta}{q} + q\Big)^{1/2}. \tag{6.4}$$

Also, since $\Phi \ll 1$,

$$\Sigma_1 \ll \Delta\Big(\sum_{d \leqq 2x/s} |a(d)|\Big)^2$$

$$\ll \Delta \Big(\frac{x}{s}\Big)^2 \mathcal{L}^F. \tag{6.5}$$

In conjunction with (6.2)-(6.5) we obtain Lemma 15.

LEMMA 16. *For any* $\varepsilon>0$, *if* $DV \leqq x^{2/3-8\varepsilon}$ *then*

$$J\,\mathrm{II}(D, V) \ll \Delta x \mathcal{L}^F \{\Delta q^{-1/2} + (q\Delta)^{1/2}\} + \Delta^2 x^{1-\varepsilon} + \Delta^3 x^{\varepsilon}.$$

PROOF. We have, by the same argument as before, that

$$J\,\mathrm{II} \ll \Delta x \mathcal{L}^F \{\Delta q^{-1/2} + (q\Delta)^{1/2}\} + \Delta^3 x^{\varepsilon} + \Delta^2 \mathcal{L}^F x^{1-2\varepsilon} + \sup_{\substack{t\leqq\Delta \\ s \leqq x^{1/2+\varepsilon}}} |\Sigma_{s,t}|, \tag{6.6}$$

where

$$\Sigma = \sum_{0<r\leqq t} e(\alpha r) \sum_{\substack{d_1, d_2 \leqq \min(2x/s, D) \\ (d_1, d_2) \mid r}} w(d_1) w(d_2)$$

$$\times \Phi\Big(\frac{2x}{[d_1, d_2]}, \frac{\max(x, d_1 s, d_2 s)}{[d_1, d_2]}; r^* \frac{\overline{d_1^*}}{d_2^*}\Big)$$

with

$$w(d) = \sum_{\substack{d=m'n' \\ m' \leqq V}} a(m') 1(n').$$

Write

$$(d_1, d_2) = \delta,$$

$$d_1 = \delta m n, \quad \delta = ef, \quad m' = em, \quad n' = fn,$$

$$d_2 = \delta d,$$

$$r^* = k.$$

We then find

$$\Sigma=\sum_{0<\delta k\leq t}\sum e(\alpha\delta k)\sum_{\substack{ef=\delta\\ e\leq V}}\sum_{\substack{mn,\, d\leq\min(D/\delta,\, 2x/\delta s)\\ m\leq V/e\\ (mn,\, d)=1}}\sum\sum a(em)1(fn)w(\delta d)$$

$$\times\Phi\left(\frac{2x/\delta}{mnd},\ \frac{\max(x/\delta,\, mns,\, ds)}{mnd};\ k\frac{\overline{mn}}{d}\right).$$

On breaking up the ranges for m, n and d into $O((\log x)^3)$ intervals of the shape $[2^i, 2^{i+1}]$, we conclude that

$$\Sigma\ll x^{\varepsilon}\sum_{\delta\leq\Delta}\sup_{\substack{|\alpha|,|\beta|,|\gamma|\leq 1\\ u,MN,D'\leq D/\delta\\ M\leq V}}\left|\sum_{0<r\leq\Delta/\delta}\sum_{d\sim D'}\sum_{\substack{m\sim M\\ (d,mn)=1\\ mn\leq u}}\sum_{n\sim N}\alpha_d\beta_m 1(n)\gamma_k\Phi\right|$$

$$=x^{\varepsilon}\sum_{\delta\leq\Delta}\sup_{\substack{D',MN\leq D/\delta\\ M\leq V}}\left|F_1\left(\frac{\Delta}{\delta},\, D',\, M,\, N;\ \frac{x}{\delta},\, s,\, u\right)\right|,$$

where F_1 is defined in section 5. Since

$$\frac{DV}{\delta},\ \frac{D}{\delta}\leqq\left(\frac{x}{\delta}\right)^{2/3-8\varepsilon},$$

Lemma 13 yields

$$\Sigma\ll x^{\varepsilon}\sum_{\delta\leq\Delta}\left(\frac{\Delta}{\delta}\right)\left(\frac{x}{\delta}\right)^{1-2\varepsilon}$$

$$\ll\Delta x^{1-\varepsilon}.$$

Combining this with (6.6) we obtain Lemma 16.

LEMMA 17. *For any $\varepsilon>0$, if $D\leqq x^{8/15-12\varepsilon}$ and $U\geqq x^{1/5}$ then*

$$J\text{III}(D, U)\ll\Delta x\mathcal{L}^F\{\Delta q^{-1/2}+(q\Delta)^{1/2}\}+\Delta^2x^{1-\varepsilon}+\Delta^3x^{\varepsilon}.$$

PROOF. It suffices to show

$$\sum_{\delta\leq\Delta}\sup_{\substack{u,D',MN\leq D/\delta\\ M,N>\max(U/\delta,1)\\ s\leq x^{1/2+\varepsilon}}}\left|F_2\left(\frac{\Delta}{\delta},\, D',\, M,\, N;\ \frac{x}{\delta},\, s,\, u\right)\right|\ll\Delta x^{1-2\varepsilon}.$$

When $\delta>D^2x^{2\varepsilon-1}$, we use the trivial bound

$$F_2\ll\frac{\Delta}{\delta}\left(\frac{D}{\delta}\right)^2.$$

Otherwise, we make use of Lemma 14. Since

$$\frac{D}{\delta}\leqq\left(\frac{x}{\delta}\right)^{8/15-12\varepsilon}\quad\text{and}\quad\frac{U}{\delta}\geqq\left(\frac{x}{\delta}\right)^{1/15},$$

Lemma 14 yields

$$F_2\ll\left(\frac{\Delta}{\delta}\right)x^{1-3\varepsilon}.$$

We therefore have

$$\sum_{\delta\leqq\Delta}\sup|F_2|\ll\sum_{\delta\leqq D^2x^{2\varepsilon-1}}\Big(\frac{\Delta}{\delta}\Big)x^{1-3\varepsilon}+\sum_{D^2x^{2\varepsilon-1}<\delta\leqq\Delta}\Big(\frac{\Delta}{\delta}\Big)\Big(\frac{D}{\delta}\Big)^2$$
$$\ll\Delta x^{1-2\varepsilon},$$

as required.

LEMMA 18.

$$J\mathrm{IV}(U,V,\mathcal{I})\ll\Delta x\mathcal{L}^F\Big(U+\frac{\Delta}{q}+\frac{\Delta}{V}+q\Big).$$

PROOF. We may assume $U\leqq\Delta$, for otherwise Lemma is trivial. We split the ranges for m, n and k into $O((\log x)^3)$ intervals of the form $(M, 2M]$, $(N, 2N]$ and $(K, 2K]$, so that

$$x\ll MNK\ll x \quad\text{and}\quad V\leqq M\leqq U.$$

Using Cauchy's inequality twice, we have

$$J\mathrm{IV}\ll\mathcal{L}^F\sup_{\substack{V\leqq M\leqq U\\ x\ll MNK\ll x}}M\cdot\sum_{m\sim M}\Delta^2\int_{-1/2\Delta}^{1/2\Delta}\Big|\sum_{\substack{mnk\sim x\\ m\sim N}}\sum_{\substack{k\sim K\\ mk\in\mathcal{I}}}b(n)c(k)e((\alpha+\beta)mnk)\Big|^2d\beta \tag{6.7}$$
$$\ll\mathcal{L}^F\sup M\cdot J,\quad\text{say.}$$

By Lemma 8,

$$J\ll\sum_{m\sim M}\Delta\sum_{NK<l\leqq 4NK}\Big(\sum_{l=nk}|b(n)c(k)|\Big)^2$$
$$+\sum_{0<r\leqq\Delta}\sum_{\substack{r=nk-n'k'\\ n,n'\sim N\\ k,k'\sim K}}\sum|b(n)c(k)b(n')c(k')|\cdot\Big|\sum_{\substack{m\sim M\\ km,k'm\in\mathcal{I}}}\max(0,\Delta-mr)e(\alpha mr)\Big|$$
$$\ll\Delta\mathcal{L}^FMNK$$
$$+NK\mathcal{L}^F\sum_{0<r\leqq\Delta}\sup_{\mathcal{I}}\Big|\sum_{\substack{m\in\mathcal{I}\\ m\geqq M}}\max(0,\Delta-mr)e(\alpha mr)\Big|$$
$$\ll\Delta x\mathcal{L}^F+NK\mathcal{L}^F\sum_{0<r\leqq\Delta/M}\sup_{\mathcal{I}}\Big|\sum_{\substack{m\in\mathcal{I}\\ m\geqq M\\ 0<mr\leqq\Delta}}(\Delta-mr)e(\alpha mr)\Big|$$
$$=\Delta x\mathcal{L}^F+NK\mathcal{L}^F\cdot\mathcal{T},\quad\text{say.}\tag{6.8}$$

Summing by parts, we have

$$\mathcal{T}\ll\sum_{0<r\leqq\Delta/M}\Delta\min\Big(\frac{\Delta}{r},\frac{1}{\|\alpha r\|}\Big)$$
$$\ll\Delta\Big(\frac{\Delta}{q}+\frac{\Delta}{M}+q\Big),$$

by [15, Kap. VI Lemma 6.3.]. Combining this with (6.7) and (6.8), we finally obtain

$$\begin{aligned} J\mathrm{IV} &\ll \mathcal{L}^F \sup_{\substack{V\leqq M\leqq U\\ x\ll MNK\ll x}} M\left\{\Delta x+NK\cdot\Delta\left(\frac{\Delta}{q}+\frac{\Delta}{M}+q\right)\right\} \\ &\ll \Delta x\mathcal{L}^F\left(U+\frac{\Delta}{q}+\frac{\Delta}{V}+q\right). \end{aligned}$$

7. Minor arc.

In this section we prove Lemmas $10^{\dagger}$ and $12^{\dagger}$ below, which cover Lemmas 10 and 12, respectively. Throughout this section we assume that

$$\left|\alpha-\frac{a}{q}\right|\leqq\frac{1}{q^2} \quad \text{with} \quad (a, q)=1, \quad \text{and} \quad 1<\Delta<x^{2/3}.$$

LEMMA $10^{\dagger}$. *Put*

$$\mathcal{T}=\mathcal{T}(\alpha, x, \Delta, N)=\Delta^2\int_{-1/2\Delta}^{1/2\Delta}\Big|\sum_{\substack{x<mn\leqq 2x\\ \sqrt{x}<n\leqq N}}\Lambda(m)\Lambda(n)e((\alpha+\beta)mn)\Big|^2 d\beta.$$

For any E, $\varepsilon>0$, if $N\leqq x^{8/15-\varepsilon}$ then

$$\mathcal{T}\ll\Delta x\mathcal{L}^F\{x^{1/4}+\Delta q^{-1/2}+(q\Delta)^{1/2}\}+\Delta^2 x\mathcal{L}^{-E},$$

where the implied constant depends only on E and ε.

PROOF. We may assume $q\leqq\Delta$, for otherwise Lemma is trivial. For $x<h\leqq 2x$, put

$$\rho(h)=\sum_{\substack{h=mn\\ n\in\mathcal{J}}}\Lambda(m)\Lambda(n), \qquad \mathcal{J}=(\sqrt{x}, N].$$

We shall decompose ρ into $O(1)$ sums of the ji-type weight, which is defined in section 6. In order to apply Lemma 6, we introduce the prameters u, v and w such that

$$\begin{aligned} u&=2x^{1/4}\\ v&=Nx^{-1/4}\\ w&=x^{\xi}, \qquad \varepsilon=20\xi. \end{aligned}$$

Then, for $x<mn\leqq 2x$ and $n\in\mathcal{J}$,

$$u<\frac{x}{N}<m\leqq 2\sqrt{x}<u^2 \quad \text{and} \quad v<\sqrt{x}<n\leqq N<uv.$$

Lemma 6 yields that, for $x<h\leqq 2x$,

$$\rho(h)=\sum_{\substack{h=mm'nn'\\ nn'\in\mathcal{G}\\ m'>u\\ n'>v}}\Big(\sum_{\substack{d\mid m\\ d\leq u}}\mu(d)\Big)\Lambda(m')\Big(\sum_{\substack{e\mid n\\ e\leq u}}\mu(e)\Big)\Lambda(n').$$

We now divide the range of variables in ρ as follows.

$$d\in(0,\ u]=H_2\cup H_3,$$

$$e\in(0,\ u]=H_2\cup H_3,$$

$$m'\in(u,\ \infty)=H_1\backslash H_3,$$

$$n'\in(v,\ \infty)=H_1\backslash(H_3\cup H_4).$$

Here, $H_1=(w,\ \infty)$, $H_2=(0,\ w]$, $H_3=(w,\ u]$ and $H_4=(u,\ v]$. Thus, we may write

$$\rho(h)=\sum_{\substack{\delta=2\ 3\\ \varepsilon=2,3\\ \mu=1,3\\ \nu=1,3,4}}(\pm 1)\quad\sum_{\substack{h=mm'nn'\\ nn'\in\mathcal{G}}}\quad\sum_{(d,e,m',n')\in H_\delta\times H_\varepsilon\times H_\mu\times H_\nu}\Big(\sum_{d\mid m}\mu(d)\Big)\Lambda(m')\Big(\sum_{e\mid n}\mu(e)\Big)\Lambda(n').$$

Let ρ_2 and ρ_3 be the sums corresponding to

$$\begin{matrix}\delta=2\\ \varepsilon=2\\ \mu=1\\ \nu=1\end{matrix}\quad\text{and}\quad\begin{matrix}\delta=2\\ \varepsilon=2,\ 3\\ \mu=1\\ \nu=4\end{matrix}\ ,$$

respectively. Moreover, let ρ_4 denote the remaining sums, so that

$$\rho(h)=\rho_2(h)-\rho_3(h)+\rho_4(h).$$

We first consider ρ_4. We easily see that at least one of δ, ε, μ or ν must be 3. Namely, at least one of d, e, m' or n' lies in the interval $(w,\ u]$. We therefore have that ρ_4 is splitted into 21 sums of the $j\mathrm{IV}(h\,;\ u,\ w,\ \mathcal{G})$-type sum, with $\mathcal{G}=\mathcal{G}$ or $(0,\ 2x]$.

We proceed to ρ_2. On writing $mm'=df$ and $nn'=eg$, we have

$$\begin{aligned}\rho_2(h)&=\sum_{\substack{h=mm'nn'\\ nn'\in\mathcal{G}\\ m,m'>w}}\sum\Big(\sum_{\substack{d\mid m\\ d\leq w}}\mu(d)\Big)\Lambda(m')\Big(\sum_{\substack{e\mid n\\ e\leq w}}\mu(e)\Big)\Lambda(n')\\ &=\sum_{\substack{h=d\,feg\\ eg\in\mathcal{G}}}\sum\sum_{d\leq w}\mu(d)\Big(\sum_{\substack{m'\mid f\\ m'>w}}\Lambda(m')\Big)\sum_{e\leq w}\mu(e)\Big(\sum_{\substack{n'\mid g\\ n'>w}}\Lambda(n')\Big)\\ &=\sum_{\substack{h=dfeg\\ d,e\leq w\\ eg\in\mathcal{G}}}\sum\sum\sum\mu(d)\Big\{(\log f)-\sum_{\substack{m'\mid f\\ m'\leq w}}\Lambda(m')\Big\}\mu(e)\Big\{(\log g)-\sum_{\substack{n'\mid g\\ n'\leq w}}\Lambda(n')\Big\}\end{aligned}$$

Thus, $\rho_2(h)$ is divided into 4 sums of the $j\mathrm{II}(h\,;\ w^2N,\ w^4)$-type sum.

We turn now to ρ_3.

$$\rho_3(h)=\sum_{\substack{h=mm'\\ nn'\in\mathcal{J}\\ m'>w\\ u<n'\leq v}}\sum_{nn'}\Big(\sum_{\substack{d\mid m\\ d\leq w}}\mu(d)\Big)\Lambda(m')\Big(\sum_{\substack{e\mid n\\ e\leq u}}\mu(e)\Big)\Lambda(n')$$

$$=\sum_{\substack{h=dfnn'\\ d\leq w\\ nn'\in\mathcal{J}\\ u<n\\ u<n'\leq v}}\sum\sum\sum\mu(d)\{(\log f)-\sum_{\substack{m'\mid f\\ m'\leq w}}\Lambda(m')\}\Big(\sum_{\substack{e\mid n\\ e\leq u}}\mu(e)\Big)\Lambda(n'),$$

since $nn'\in\mathcal{J}$ and $u<n'\leqq v$ imply $n>1$. Hence, $\rho_3(h)$ is a sum of two $j\mathrm{III}(h;w^2N, u)$-type sums.

We therefore conclude that

$$\mathcal{T}\ll J\,\mathrm{II}(w^2N, w^4)+J\,\mathrm{III}(w^2N, u)+J\mathrm{IV}(u, w, \mathcal{J}).$$

We note that

$$w^6N\leqq x^{8/15-\varepsilon+12\xi}\leqq x^{2/3-8\xi},$$

$$w^2N\leqq x^{8/15-\varepsilon+4\xi}\leqq x^{8/15-12\xi},$$

$$u=2x^{1/4}\geqq x^{1/5}.$$

Thus, Lemmas 16, 17 and 18 yield

$$\mathcal{T}\ll\Delta x\mathcal{L}^F\{\Delta q^{-1/2}+(q\Delta)^{1/2}\}+\Delta^2x^{1-\xi}+\Delta x\mathcal{L}^F\Big(u+\frac{\Delta}{q}+\frac{\Delta}{w}+q\Big)$$

$$\ll\Delta x\mathcal{L}^F\{x^{1/4}+\Delta q^{-1/2}+(q\Delta)^{1/2}\}+\Delta^2x^{1-\xi},$$

as required.

LEMMA 12†. *Let* $(\lambda_d)=(\lambda_d(D, z))$ *be the weights introduced in Lemma* 5. *Put*

$$\mathcal{S}=\mathcal{S}(\alpha, x, \Delta, D, z)=\Delta^2\int_{-1/2\Delta}^{1/2\Delta}\Big|\sum_{x<n\leq 2x}\Big(\sum_{d\mid n}\lambda_d\Big)e((\alpha+\beta)n)\Big|^2d\beta.$$

For any E, $\varepsilon>0$, *if* $4<z^2\leqq D\leqq x^{1-\varepsilon}$ *then*

$$\mathcal{S}\ll\Delta x\mathcal{L}^F\{x^{1/4}+\Delta q^{-1/2}+(q\Delta)^{1/2}\}+\Delta^2x\mathcal{L}^{-E},$$

where the implied constant depends only on E *and* ε.

PROOF. We may assume $q\leqq\Delta$. By Lemma 5, we have

$$\mathcal{S}\ll J\,\mathrm{I}\,(x/z_1z)+\mathcal{L}^2J\mathrm{IV}(D^{1/4}z_9, z_2, (0, 2x]).$$

Here,

$$z_i=\exp((\log D)(\log\log D)^{-i/10}).$$

Lemmas 15 and 18 yield that

$$\mathcal{S}\ll\Delta x\mathcal{L}^F\{\Delta q^{-1/2}+(q\Delta)^{1/2}\}+\Delta(z_1z)^2\mathcal{L}^F$$

$$+\Delta x\mathcal{L}^F\Big(D^{1/4}z_9+\frac{\Delta}{q}+\frac{\Delta}{z_2}+q\Big)$$

$$\ll \Delta x \mathcal{L}^F \{D^{1/4} z_9 + \Delta q^{-1/2} + (q\Delta)^{1/2}\} + \Delta^2 \mathcal{L}^F z_1^2 D + \Delta^2 x \mathcal{L}^F z_2^{-1}$$

$$\ll \Delta x \mathcal{L}^F \{x^{1/4} + \Delta q^{-1/2} + (q\Delta)^{1/2}\} + \Delta^2 x \mathcal{L}^{-E},$$

for any $E>0$, as required.

This completes our proof of Theorem.

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