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ON LOCKETT'S CONJECTURE FOR FINITE GROUPS

By

Aleksander GRYTCZUK and Nikolay T. VOROB'EV

All groups in this paper are finite. A class of groups F will be called the Fitting's class if there are satisfied the following two conditions:

1°. if $G\in F$ and $N\lhd G$ then $N\in F$

2°. if $G=N_1N_2$, $N_i\lhd G$ and $N_i\in F$; $i=1, 2$ then $G\in F$.

By the condition 2° it follows that if F is an arbitrary nonempty Fitting's class then the product G_F of all normal F-subgroups of G belongs to F. It has been remarked in the paper [3] that the description of construction of a Fitting class is very difficult problem in general case.

Many well-known results concerning soluble Fitting classes are strictly connected with Lockett's construction given in [5]. Namely, if F is any Fitting class then $F^*=\{G:(G\times G)_F$ is subdirect in $G\times G\}$ is also the Fitting class.

Moreover, we denote by F_* the intersection of all such Fitting classes X for which $X^*=F^*$.

In the paper [5] Lockett conjectured that every soluble Fitting class F is the same as $F^*\cap N(F)$, where $N(F)$ denotes the normal Fitting class generated by F.

Nonempty Fitting class X is called normal if a subgroup G_X of G is a maximal X-subgroup of G. In [2] Bryce and Cossey proved that the Lockett's condition is satisfied by soluble Fitting's class iff $F_*=F^*\cap\gamma_*$ where γ_* is a minimal normal Fitting's class and γ denotes the class of all soluble groups. They also proved that primitive saturated formations which are Fitting classes satisfy the Lockett conjecture. In [8] was proved that all soluble local Fitting classes F satisfy Lockett's condition whenever $F\supseteqq N_0$, where N_0 denotes the class of all nilpotent groups. Note that the classes XN_0 and $X\gamma_{\Pi}\gamma_{\Pi'}$, considered in [1] are particular cases of mentioned above. Now, we can consider nonempty class U of finite groups with respect to the operations S, Q, R, Ext_U described in [7].

The class U will be called universal class.

Then we see that fundamental problem concerning the construction of Fit-

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ting classes X and U is the following:

PROBLEM 1. For which Fitting's classes X and U we have

$$X_*=X^*\cap U_*. \tag{*}$$

In this paper we show that (*) is true when X is local and every group $G\in U$ satisfies $G/G_{x(p)}\in\gamma$ for all primes $p\in\Pi(X)$, where x denotes the function locally defining the class X.

Our result improves and generalizes the fundamental theorem of [8]. Namely if $U=\gamma$ then we obtain that any soluble and local Fitting class satisfies the Lockett's condition, so the condition $F\supseteqq N_0$ given in [8] can be omitted. The Fitting class F will be called local (see [9], [10]) or the Fitting class with radical function f, if $f(p)$ is the Fitting class for all primes p and

$$F=U_{\pi(F)}\cap(\bigcap_{p\in\pi(F)}f(p)U_pU_{p'}) .$$

By the product of two Fitting classes F and H we mean the class of all groups such that $G/G_F\in H$, (Cf. [7]).

LEMMA 1. *If F is local Fitting class then $F=F^*$.*

For the proof see [9].

LEMMA 2. *Let F and H be the Fitting classes. Then we have*
1° *If $F\subseteqq H$ then $F_*\subseteqq H_*$*
2° $(F^*)_*=F_*\subseteqq F\subseteqq F^*=(F_*)^*$.

For the proof see [5] and [8].

LEMMA 3. *Let F be a Fitting class and H denotes the Fitting class which is saturated homomorph.*
Then

$$(FH)^*=F^*H .$$

For the proof see [8].

If F is a Fitting class then the subgroup V of the group G will be called F-injector of G if $V\cap N$ is a maximal F-subgroup of N for any subnormal in G of the subgroup N.

LEMMA 4. *If $G\in F\gamma$ then there exist F-injectors in the group G and any two of them are conjugates.*

For the proof see [6].

Let $Y(F, H)$ denotes the class of all groups such that F-injectors exist and are H-subgroups. We have

LEMMA 5. *Let $F=FU_p$ and H be the Fitting classes. If every group $G\in U$ satisfies $G/G_F\in\gamma$ then $Y(F, HU_{p'})$ is also Fitting class and moreover if H is local then*

$$Y(F, (HU_{p'})^*)=(Y(F, HU_{p'}))^* .$$

The proof of Lemma 5 follows by Lemma 4, Theorem 3.3 of [4] Lemma 1 of [1].

LEMMA 6. *Every local Fitting class F is defined by a radical function f such that $f(p)U_p=f(p)\subseteqq F$ for all primes p.*

The proof of Lemma 6 follows by slighty modification of the proof of Lemma 1 from [8].

Now, we can prove the main result:

THEOREM. *If X is the Fitting class with radical function x and every group $G\in U$, where U is the universal class, satisfies $G/G_{X(p)}\in\gamma$ for all primes $p\in \Pi(X)$ then*

$$X_*=X^*\cap U_* .$$

PROOF. By Lemma 1 we have $X^*=X$. Thus we can prove that $X_*=X\cap U_*$, From Lemma 2 we have $X_*\subseteqq U_*$ and $X_*\subseteqq X$ and therefore we obtain $X_*\subseteqq X\cap U_*$. It suffices to prove that $X\cap U_*\subseteqq X_*$. By Lemma 6 we can assume without loss of generality that for all primes p, $x(p)U_p=x(p)\subseteqq X$. Let $G\in U$. Then by Lemma 4 we have that in G there exist $x(p)$-injectors and any two of them are conjugates in G. From Lemma 3 and Lemma 2 we obtain

$$(XU_{p'})^*=X^*U_{p'}=(X_*)^*U_{p'}=(X_*U_{p'})^* .$$

By last equality it follows that $x(p)\subseteqq(XU_{p'})^*$.

Consequently $Y(x(p), (X_*U_{p'})^*)=U$.

Thus by Lemma 5 we obtain $(Y(x(p), (X_*U_{p'}))^*=U$. But by the second part of the assertion of Lemma 2 we have $U_*=(Y(x(p), X_*U_{p'}))_*$.

From Lemma 2 and Lemma 5 we obtain that U_* is contained in the Fitting class $Y(x(p), X_*U_{p'})$.

Thus we have

$$x(p)\cap U_*\subseteqq x(p)\cap Y(x(p),\ X_*U_{p'})=x(p)\cap X_*U_{p'}\subseteqq X_*U_{p'}$$

and consequently

$$(\bigcap_{p\in\pi(X)} x(p)U_{p'})\cap U_*(\bigcap_{p\in\pi(X)} U_{p'})\subseteqq X_*(\bigcap_{p\in\pi(X)} U_{p'}).$$

Hence

$$X\cap U_*U_{\pi'(x)}\subseteqq X_*U_{\pi'(x)}\cap U_{\pi(x)}$$

and therefore

$$X\cap U_*\subseteqq X_*$$

and the proof our Theorem is complete.

COROLLARY. *Every soluble and local Fitting class F satisfies*

$$F_*=F^*\cap\gamma_*.$$

REMARK. There exist non-local soluble Fitting classes for which Lockett's condition is true.

Indeed, let F be a normal class of Fitting such that $F^*=\gamma$.

Then by Lemma 1 we have that F is non-local Fitting class. But from Lemma 2 we have $F_*=\gamma_*$ and consequently $F_*=F^*\cap\gamma_*$.

So it would be of some interest to solve the following:

PROBLEM 2. Characterize non-normal and non-local soluble Fitting classes satisfying Lockett's condition.

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A. Grytczuk
Institute of Mathematics
Pedagogical University
Zielona Gora, Poland

N. T. Vorob'ev
Department of Algebra
Pedagogical Institute
Vitebsk, Belyorussia