

Sustainability of public debt, public capital formation, and endogenous growth in an overlapping generations setting

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Abstract

Under the golden rule of public finance for public investment with a constant budget deficit/GDP ratio, we show that for the sustainability of government budget deficits there is a threshold of the initial public debt for a given stock of public capital, and that this threshold level of public debt is increasing in the stock of public capital. If the initial public debt is greater than the threshold, the government can no longer sustain budget deficits, while if it is smaller, the government can conduct a permanent deficit policy, which eventually leads to a positive public debt/GDP ratio.

JEL classification: E62; H54; H62; O40

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1. Introduction

Since the outstanding work by Arrow and Kurz (1970), many authors have investigated the effects of public capital formation on the performance of the economy and the optimal fiscal policy in dynamic general equilibrium models.¹ In an endogenous growth setting, Futagami, Morita and Shibata (1993), among others, analyzed the growth-maximizing public investment size (the public investment/GDP ratio) assuming public capital formation rather than public flow expenditures of Barro (1990) type. In these theoretical contributions, however, they assumed that the government runs a balanced budget at any moment in time. Nonetheless, recently, an intensive debate has arisen regarding the long-term growth effects of public investment financed under various versions of the so-called golden rule of public finance (e.g. Greiner and Semmler, 2000; Ghosh and Mourmouras, 2004).² The golden rule of public finance is considered the fiscal rule according to which government expenditures for public consumption, transfer payments and interest payments must be smaller than the tax revenue. Under the rule, borrowing is allowed to finance only government investment.

Greiner and Semmler (2000) showed that the long-term growth effects of public capital depend on the exact budgetary regime adopted by the government, and that a less strict budgetary regime may not lead to a positive growth effect of a deficit-financed government investment. By comparing the welfare effects of allowing public borrowing under the standard dynamic government budget constraint and under the golden rule of public finance, Ghosh and Mourmouras (2004) showed that the golden rule of public finance can be an effective restriction on the composition of government

¹ Empirically, Aschauer (1989) and Iwamoto (1990) among others showed the substantially great growth-enhancing effects of public capital. However, Evans and Karras (1994) and Holtz-Eakin (1994) cast doubts on the empirical results.

² Among OECD countries, Japan has more than doubled its government debt/GDP ratio in the 1990s by raising the deficit finance ratio in the budget (0.9% in 1991, 6.6% in 1995 and 8.3% in 1999), and the growth rate declined drastically from the 1970s to the 1990s. The government debt/GDP ratio of Japan was 64.8% in 1991 and became 142.3% in 2001, the highest among OECD countries (OECD, 2004).

expenditure and that a less strict budgetary stance may lead to a lowering of welfare.

Under budget deficit policies, public debt accumulates and in turn affects the government budget. While Greiner and Semmler (2000) and Ghosh and Mourmouras (2004) did not focus their attention on accumulation of public debt, the sustainability of public debt has been examined, for example, by Hamilton and Flavin (1986) and Bohn (1998). Among others, pointing out that the transversality condition tests depend on sensitivity on the choice of discount rates and the cointegration tests generally do not adjust real levels of fiscal variables, Bohn (1998) proposed a new test that requires that primary surplus increases at least linearly with the ratio of debt to GDP at high debt-GDP ratios. In contrast, Chalk (2000) examined the sustainability of government budget deficits in an overlapping generations model of Diamond (1965) type, and showed that the present value budget balance may not be crucial to the sustainability of permanent deficits.

While it is well known that permanent budget deficits can be sustainable when the dynamic resource allocation is dynamically inefficient in an overlapping generations setting (e.g. Diamond, 1965; Tirole, 1985), Chalk (2000) also showed that, even when the growth rate is lower than the interest rate and hence the cost of debt finance is high, the government can run the primary deficits, and that the permanent deficits are sustainable only when the initial public debt is not too large. Bräuninger (2005) showed in an overlapping generations model with the AK production structure that under a fiscal rule in which the government purchase/GDP ratio and the budget deficit/GDP ratio are constant, the tax rate therefore being endogenously adjusted, there is a stable steady-growth path as long as the initial debt-capital ratio is lower than a certain level, and that an increase in the deficit rate reduces the growth rate. However, both Chalk (2000) and Bräuninger (2005), as well as most of the literature on public debt sustainability, assumed that government expenditures are public

consumption.³

Our purpose in this study is to analyze the sustainability of budget deficits, simultaneously taking into account the growth effects of a deficit-financed public investment, in an endogenous growth setting with the growth engine of public capital formation. For our purpose, we use the overlapping generations model pioneered by Diamond (1965), in which public debt can have real effects. We assume that the government not only controls the public investment/GDP ratio but also keeps the deficit finance ratio in public investment at less than one. Thus, the financing rule in this study is the mixture of the golden rule of public finance, as to the borrowing rule, and a deficit rule of keeping the budget deficits at a certain percentage of GDP, while the tax rate must be endogenously adjusted according to the government budget constraint.⁴ The public debt/GDP ratio is endogenously determined by the fiscal rule along the growth path.

We illustrate that there can be two long-term equilibria, one locally stable and one saddle-point stable, and that there is a threshold for the initial public debt in order for budget deficits to be sustainable at each level of public capital stock. The threshold of the initial public debt is represented by a point on the stable branch to the saddle-point equilibrium, and is increasing in the stock of public capital, i.e. the so-called public assets. If the initial public debt is greater than the threshold at a level of public capital, the government can no longer sustain the fiscal deficit policy. If the initial debt is smaller than the threshold, the economy converges to the stable equilibrium and the government can run the permanent fiscal deficit and public investment policy, which eventually leads to a positive public debt/GDP ratio. We also show that decreases in the public investment/GDP ratio and/or the deficit finance ratio will raise the threshold for a given level of public capital stock, and that the decreased deficit finance ratio leads

³ Greiner (2006) examined the sustainability and the growth effects of public investment in a dynamic model, incorporating the fiscal rule proposed by Bohn (1998) into a representative, infinitely-lived agent model of Futagami et al. (1993) type.

⁴ Ghosh and Mourmouras (2004) classified fiscal rules into four types; (i) balanced budget rules, (ii) deficit rules, (iii) borrowing rules, and (iv) debt/reserve rules.

to higher balanced growth, while an increase in the public investment ratio can have a growth-enhancing effect.

The remainder of the paper is organized as follows. We devote the next section to developing an overlapping generations model of Diamond (1965) type, which incorporates public capital formation. Section 3 analyzes dynamics of the economy and the long-term equilibrium. The effects of policy changes are analyzed in Section 4, while the last section presents concluding remarks.

2. Model

We consider a one-sector endogenous growth model, populated by two-period-lived generations and with accumulation of public capital as the growth engine. Assuming that the population of each generation in the economy is constant over time, we denote it as N .

2.1 Production

Production technology of a representative firm j is assumed to be

$$Y^j = \tilde{A}(K^j)^\alpha (GL^j)^{1-\alpha} \quad (0 < \alpha < 1; \tilde{A} > 0) \quad (1)$$

where Y^j , K^j and L^j stand for output, private capital stock and labor, respectively, employed by the firm. G is the stock of public capital which is available and common for all firms. We assume here that public capital stock enters the production function, as in Futagami et al. (1993), and that the use of capital is not subject to congestion. Denoting the interest rate and the wage rate as r and w , respectively, the profit maximizing conditions of the firm in competitive markets are given as

$$\partial Y^j / \partial K^j = \alpha(Y^j / K^j) = r, \quad (2a)$$

$$\partial Y^j / \partial L^j = (1 - \alpha)(Y^j / L^j) = w. \quad (2b)$$

2.2 Individuals

A representative individual works only when young, and the labor supply is inelastic and normalized to one. He consumes a part of wage income and saves the remainder for his retirement during the second period. The lifetime budget constraint of the individual can be written as

$$(1 - \tau)w = c^y + \frac{c_{+1}^o}{1 + (1 - \tau_{+1})r_{+1}} \quad (3)$$

where c^y and c_{+1}^o are consumption in the first and second period, respectively; τ stands for the income tax rate; and the variable with subscript $+1$ represents the value of the variable in the next period. We assume here perfect foresights of individuals for the future after-tax rate of return on savings.

Individuals are assumed to derive their utility only from their own consumption and have no bequest motives. The utility function is $U = (1 - \delta)\ln c^y + \delta\ln c_{+1}^o$, where $0 < \delta < 1$. An individual chooses consumption allocation so as to maximize the lifetime utility subject to the lifetime budget constraint (3). The optimizing conditions give the savings of the individual $s \equiv (1 - \tau)w - c^y$ as

$$s = \delta(1 - \tau)w. \quad (4)$$

2.3 Government

The government budget consists of two components, current budget and capital budget. Abstracting from government consumption and transfer payments, and defining φ as the proportion of tax revenue to finance the current spending of interest payments on public debt, we have the current budget as $rD = \varphi T$ where $T = \tau(w + rs_{-1})N$ is income tax revenue and D stands for the outstanding stock of public debt in the economy, while the capital component of the government budget is $G_{+1} - G = D_{+1} - D + (1 - \varphi)T$, that is, public investment is financed partly by public bond issues and partly by income tax revenue. Thus, the integrated budget equation of the government is given as

$$(D_{+1} - D) + \tau(w + rs_{-1})N = (G_{+1} - G) + rD. \quad (5)$$

We assume here that the government invests a constant fraction of GDP, θ , in public capital and finances a proportion, λ , of the expenditure by issuing bonds, where $0 < \theta, \lambda < 1$, i.e.

$$G_{+1} - G = \theta Y, \quad (6)$$

$$D_{+1} - D = \lambda(G_{+1} - G) [= \lambda\theta Y] \quad (7)$$

where $Y = \sum_j Y^j$.⁵ As long as $0 < \lambda < 1$, on the one hand, we have from (5), (6) and

(7)

$$(D_{+1} - D) - (G_{+1} - G) = rD - T < 0 \quad (8)$$

The tax revenue is greater than interest payments on the outstanding debt, in other words, the portion of the tax revenue spent on current expenditure is less than one, $0 < \varphi < 1$. On the other hand, inserting (6) and (7) into (5), the budget equation becomes $\tau Y = \theta(1 - \lambda)Y + (1 - \tau)rD$. When θ and λ are kept constant, the government must adjust the tax rate τ in order to satisfy the budget equation (5).⁶ Thus, the fiscal policy, represented by (θ, λ) in (6) and (7), is the mixture of the regime (A) of the golden rule of public finance in Greiner and Semmler (2000: p 368) and the deficit rule in Bräuninger (2005).

2.4 Market equilibrium

Because of the linear homogeneity of the production function of each firm, the capital/(effective) labor ratio is the same for all firms, i.e. $(GL^j)/K^j = (GL)/K$ and hence $Y^j/K^j = Y/K$ for all j , where $K = \sum_j K^j$ and $L = \sum_j L^j [= N]$.

Therefore, from (2), we have

⁵ The Maastricht Treaty constrains the general government financial deficit not to exceed 3% of GDP. Buiter (2001) stated that by restricting the financing options for investment, the Treaty is likely to depress the volume of public capital formation. However, he did not take into account the growth-enhancing effect of public capital.

⁶ Where φ is endogenously given as $\varphi = rD / [\tau(w + rs_{-1})N]$.

$$r = \alpha A(G/K)^{1-\alpha} = \alpha(Y/K), \quad (9a)$$

$$w = (1-\alpha)A(G/K)^{1-\alpha} K = (1-\alpha)(Y/L). \quad (9b)$$

where $A = \tilde{A}L^{1-\alpha}$.

Since the assets which individuals can hold are only private capital and public bonds, and since only the working generations have the assets, the equilibrium condition in the capital market is

$$K_{+1} + D_{+1} = sN. \quad (10)$$

Making use of individuals' budget equations, distribution of outputs, government's budget equation and the equilibrium condition in the capital market, the resource constraint in the period is given as⁷

$$Y = c^y N + c^o N + (G_{+1} - G) + (K_{+1} - K). \quad (11)$$

Output is allocated among consumption of the two generations, private investment and public investment.

3. Dynamics and long-term equilibrium

Now we have a look at the dynamics of the economy. From (6), we have the evolution of public debt as

$$\frac{D_{+1}}{D} = 1 + \lambda\theta(Y/D). \quad (12)$$

Since the tax rate, τ , is set so as to satisfy the budget equation (7), and by inserting (5), (6) and (9) into (7), making use of the linearity of the aggregate production function and rearranging terms, we obtain

$$1 - \tau = \frac{1 - \theta(1 - \lambda)}{1 + \alpha(D/K)}. \quad (13)$$

⁷ The inter-period budget constraints of the two generations, $(1 - \tau)w = c^y + s$ and $[1 + (1 - \tau)r][(K + D)/N] = c^o$, are used.

Inserting s from (5), w from (9b), $1 - \tau$ from (13) and D from (7) into the condition (10), dividing both sides by K , and using the production function, we obtain the rate of change in private capital stock as

$$\frac{K_{+1}}{K} = \left[\frac{1 - \theta(1 - \lambda)}{1 + \alpha(D/K)} \delta(1 - \alpha) - \lambda\theta \right] A(G/K)^{1-\alpha} - (D/K). \quad (14)$$

For our purpose we assume $K_{+1}/K > 0$ in the following. The evolution of public capital is obtained, from (5), as

$$\frac{G_{+1}}{G} = 1 + \theta(Y/G). \quad (15)$$

The dynamics of the system can be represented by the three difference equations (12), (14) and (15), in private capital stock, K , public capital stock, G , and the stock of public debt, D , which are the state variables.

The balanced growth path is defined as a path on which the three state variables grow at the same rate, and the balanced growth rate is defined as

$$\frac{K_{+1}}{K} = \frac{G_{+1}}{G} = \frac{D_{+1}}{D} \equiv 1 + \gamma. \quad (16)$$

Defining new variables $g \equiv G/K$ and $x \equiv D/K$ as in the literature of endogenous growth, we can rewrite the dynamic system as the following two difference equations in terms of x and g : from (14) and (15),

$$\frac{g_{+1}}{g} = \frac{G_{+1}}{G} / \frac{K_{+1}}{K} = \frac{1 + \theta A g^{-\alpha}}{\left[\frac{1 - \theta(1 - \lambda)}{1 + \alpha x} \delta(1 - \alpha) - \lambda\theta \right] A g^{1-\alpha} - x} \quad (17)$$

and, from (12) and (14),

$$\frac{x_{+1}}{x} = \frac{D_{+1}}{D} / \frac{K_{+1}}{K} = \frac{1 + \lambda\theta A (g^{1-\alpha} / x)}{\left[\frac{1 - \theta(1 - \lambda)}{1 + \alpha x} \delta(1 - \alpha) - \lambda\theta \right] A g^{1-\alpha} - x}. \quad (18)$$

On the balanced growth path, therefore, we have

$$g_{+1}/g = 1, \quad (19a)$$

$$x_{+1}/x = 1 \quad (19b)$$

which give the steady state (g, x) as a solution to the dynamic system. From (17), (18) and (19), we can readily see that there exists such a steady state satisfying

$$\lambda g = x \quad (20)$$

as long as both conditions of (19) are satisfied.

3.1 Critical levels of public investment ratio and debt finance ratio

At this stage, we briefly examine the sustainability condition on government budget policy represented by the public investment ratio and the debt finance ratio, (θ, λ) . We assume that the initial outstanding debt is positive ($D > 0$), and that the debt finance ratio is changed to $\lambda = 0$, while a positive public investment is still undertaken at a positive ratio to output $\theta_0 (> 0)$. The income tax rate is determined so as to finance both public investment and interest payments on the outstanding debts for the historically given public capital/private capital ratio and the public debt/private capital ratio, (g, x) . In this case, we have $D_{+1}/D = 1$, while $K_{+1}/K > 1$ if the economy grows due to public investment. Therefore, the public debt/private capital ratio will be lower in the next period, i.e. $x_{+1}/x < 1$. Alternatively, if debt finance is allowed instead of the balanced budget, that is, if we have $\lambda > 0$ instead, the public debt/private capital in the next period becomes greater than that obtained when $\lambda = 0$, since we can show $d(x_{+1}/x)/d\lambda > 0$ from (18); and for the same public capital/private capital ratio and public debt/private capital ratio (g, x) , the greater the public debt/private capital in the next period, the greater the debt finance ratio. Thus, we have the minimum public debt ratio $\lambda_0 > 0$ such that $x_{+1}/x \geq 1$ holds for the public investment ratio, θ_0 . Since $d(x_{+1}/x)/d\theta > 0$, the debt finance ratio λ_1 which makes $x_{+1}/x \geq 1$ is smaller for a higher public investment ratio $\theta_1 (> \theta_0)$, i.e. $\lambda_1 < \lambda_0$. In other words, in order to satisfy $x_{+1}/x < 1$ for a given (g, x) , the higher the public investment ratio, the lower the debt finance ratio. In a similar way, from $d(g_{+1}/g)/d\lambda > 0$ and $d(g_{+1}/g)/d\theta > 0$ we can see that in order to satisfy

$g_{+1}/g < 1$ for a given (g, x) , the higher the public investment ratio, the lower the debt finance ratio. Thus, given the state variables (g, x) , we can find the minimum debt finance ratio such that either $g_{+1}/g \geq 1$ or $x_{+1}/x \geq 1$ holds for a given public investment ratio.

From the above consideration, we can say that a fiscal policy represented by a combination (θ, λ) , that makes either $g_{+1}/g \geq 1$ or $x_{+1}/x \geq 1$ for any (g, x) , is not sustainable in the sense that a balanced growth path of the economy may not exist. The government must decrease either the public investment ratio or the debt finance ratio, or both, in order to prevent the economy from diverging from the path of balanced growth. It should be noted that a decrease in the public investment ratio means a decrease in the public deficits/GDP ratio.

While the conventional literature on sustainability of public deficits without public capital accumulation (e.g. Chalk, 2000; Bräuning, 2005) emphasized a critical level of government budget deficits, our analysis with public investment shows the existence of the critical budget rule (θ, λ) for the sustainability of budget deficits, although we can not express the policy explicitly in terms of budget deficits. Especially, as will be shown later, even if we have $x_{+1}/x \geq 1$ or $g_{+1}/g \geq 1$ in the transition converging to the long-term equilibrium, the fiscal policy (θ, λ) can be sustainable. Since we are rather concerned here with the sustainability of the budget deficits in relation to the (initial) public debt, we focus our attention on the cases in which both conditions of (19) hold.

3.2 Initial conditions for sustainability

Now we analyze the properties of the balanced growth paths. First, we derive the combinations (g, x) which satisfy (19a). (19a) is rewritten as

$$1 + \theta A g^{-\alpha} + x = \left[\frac{1 - \theta(1 - \lambda)}{1 + \alpha x} \delta(1 - \alpha) - \lambda \theta \right] A g^{1 - \alpha}. \quad (21)$$

Let the left and right hand side of (21) be $\beta(g, x)$ and $\varepsilon(g, x)$, respectively. Fig. 1

illustrates the two functions for a given value of the public capital/private capital ratio, g . There is a crossing of β and ε , and the value of x which satisfies (19a) or (21) for the given g is indicated by the vertical arrow.

[Please insert Fig. 1 about here]

An increase in g shifts the function β downward and ε rightward in the positive quadrant, respectively, and therefore raises x correspondingly. These shifts are depicted by the dotted lines in Fig. 1. Thus, the combination (g, x) satisfying (19a) can be depicted as an upward-sloping curve, GG , on the (g, x) plane in Fig. 3. Denoting the slope of the curve GG as $dg/dx|_{GG}$, we can see

$$\frac{dg}{dx}|_{GG} = \frac{Ag^{1-\alpha} \frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha)\alpha + 1}{(1-\alpha)Ag^{-\alpha} \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda\theta \right] + \alpha\theta Ag^{-\alpha-1}} > 0. \quad (22)$$

Since the right hand side of (21) must be positive, x has an upper limit, $\bar{x} = \{[1-\theta(1-\lambda)]\delta(1-\alpha) - \lambda\theta\}/(\alpha\lambda\theta)$, and since x is non-negative, g has a lower limit \underline{g} when $x = 0$, where \underline{g} satisfies $1 + \theta A \underline{g}^{-\alpha} = \{[1-\theta(1-\lambda)]\delta(1-\alpha) - \lambda\theta\} A \underline{g}^{1-\alpha}$.

Next, turning to (19b) and rewriting it, we have

$$1 + \lambda\theta A \frac{g^{1-\alpha}}{x} + x = \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda\theta \right] A g^{1-\alpha}. \quad (23)$$

Let the left hand side of (23) be $\eta(g, x)$, while the right hand side is the same as in (21), i.e. $\varepsilon(g, x)$. Fig. 2 illustrates the two functions, η and ε , for a given value of g . There are two intersections of the two functions, which give two values of x which satisfy (23) for the given g , calling the smaller one type (i) and the greater one type (ii). We can see that an increase in g shifts both $\varepsilon(g, x)$ and $\eta(g, x)$ upward, and that the upward shift of $\varepsilon(g, x)$ is greater than that of $\eta(g, x)$ at each value of x . These shifts are illustrated by the dotted lines in Fig. 2. Therefore, an increase in g

lowers x of type (i) and raises x of type (ii), respectively. Plotting the combinations of (g, x) satisfying (19b) and (23) on the (g, x) plane, we have a curve, XX , which is U-shaped as depicted in Fig. 3. Denoting the slope of the curve XX as $dg/dx|_{XX}$ and from (19b), we obtain the following:

$$\frac{dg}{dx}|_{XX} = \frac{\left[\frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha)\alpha - \frac{\lambda\theta}{x^2}\right] Ag^{1-\alpha} + 1}{(1-\alpha)Ag^{-\alpha} \left\{ \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda\theta\right] - \frac{\lambda\theta}{x} \right\}} \quad (24)$$

where the denominator of the right hand side is positive from (23) and $x > 0$. At x of type (i), we have $\partial\varepsilon/\partial x > \partial\eta/\partial x$, i.e.

$$\left[\frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha)\alpha - \frac{\lambda\theta}{x^2}\right] Ag^{1-\alpha} + 1 < 0 \quad (25)$$

and at type (ii), $\partial\varepsilon/\partial x < \partial\eta/\partial x$, i.e.

$$\left[\frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha)\alpha - \frac{\lambda\theta}{x^2}\right] Ag^{1-\alpha} + 1 > 0 \quad (26)$$

[Please insert Fig. 2 about here]

Therefore, we can see that $dg/dx|_{XX} < 0$ at type (i) and $dg/dx|_{XX} > 0$ at type (ii). As g approaches zero, $\eta(g, x)$ comes closer to the line $1+x$ and $\varepsilon(g, x)$ to the horizontal axis, respectively. In order for a positive x satisfying (23) to exist, there is a lower limit of g , $\underline{\underline{g}}$. Since the intercept of $\varepsilon(g, x)$ on the vertical line must be greater than 1, $\underline{\underline{g}}$ satisfies $1 < \{[1-\theta(1-\lambda)]\delta(1-\alpha) - \lambda\theta\} A(\underline{\underline{g}})^{1-\alpha}$.

Now we can analyze the balanced growth paths, represented by the steady states (g, x) satisfying (19a) and (19b) simultaneously. The steady state values of (g, x) are illustrated by the crossings of the curves, XX and GG , on the (g, x) plane in Fig. 3. Since (19) is not linear in g and x , we will generally have two long-term equilibria, which are illustrated by the intersections of the curves GG and XX in Fig. 3 as S

at low x and U at high x . The curve XX has a negative slope at equilibrium S and a positive slope at equilibrium U . When both conditions of (19) hold, we have (20), and therefore the equilibria are on the line (20).⁸ A phase diagram is drawn in Fig. 3 by using (17) and (18) (see Appendix A). The diagram shows that equilibrium S is locally stable and equilibrium U is saddle-point stable, respectively (for the proof, see Appendix B). The stable branches converging to the saddle point are illustrated by a dotted line in Fig. 3.

[Please insert Fig. 3 about here]

Since K , G and D are predetermined variables, the initial state of the economy is given by a point (g_0, x_0) on the (g, x) plane, where g_0 and x_0 are the initial values of the public capital/private capital ratio and the public debt/private capital ratio. We have three cases: (i) If the initial state (g_0, x_0) locates on the lower-right of the saddle-point stable branch, the economy will not have long-term equilibrium; (ii) if the initial point (g_0, x_0) is on the stable branch, the economy converges to the equilibrium U ; and (iii) if it is on the upper-left of the stable branch, the economy eventually converges to the stable equilibrium S .

It should be noted at this stage that we confine ourselves to cases in which

$$\left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda\theta \right] A g^{1-\alpha} - x > 0, \text{ or equivalently } K_{+1}/K > 0, \text{ so that}$$

$g_{+1}/g > 0$ and $x_{+1}/x > 0$. If this condition is not satisfied, both the public capital/private capital ratio and the public debt/private capital may go to zero in infinite time, as can be seen from (17) and (18). The combinations of (g, x) satisfying

⁸ We can not rule out the possibility that there is only one long-term equilibrium, at which the curve XX is tangent to the curve GG . In this case, there is a stable arm to the equilibrium only above the curve XX , while the long-term equilibrium is still on the line (20). It should be noted, however, that as long as conditions (19) are satisfied, we have at least one equilibrium.

$[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda\theta]Ag^{1-\alpha} - x = 0$ are plotted as the broken curve, BB , in Fig.

3.⁹ From (21) and (23) we can easily see that the curves XX and GG are above the curve BB .

The above result has important implications for the sustainability of budget deficit and public debt. When conditions (19) are satisfied, the economy converges to the stable balanced-growth equilibrium insofar as the initial public debt/private capital ratio is not too large relative to the public capital/private capital ratio. In other words, there is a threshold of initial public debt for a given stock of public capital stock. At the balanced-growth equilibrium, the stocks of public and private capital and public debt grow at the same rate, $\gamma = \theta Ag^{-\alpha}$, which is greater than, equal to, or smaller than, the interest rate $r = \alpha Ag^{1-\alpha}$, depending on whether g is smaller than, equal to, or greater than, θ/α . Interest on the public debt can be paid forever, although the public debt may not be paid off in the future, and the government can run the fiscal deficit and public investment policy permanently. In this sense the fiscal deficit is sustainable, and the fiscal policy (θ, λ) leads to the permanent, constant and positive, public debt/private capital ratio. It should be recalled here that the cost of debt finance is the after-tax interest rate.

The transitional path to the stable equilibrium is as follows. When the initial public debt/private capital ratio, x , is great but less than the threshold, the tax rate will be relatively low. Therefore, private savings and hence private capital formation, relative to GDP, will be greater, lowering the public debt/private capital ratio along the transitional path. If in addition the initial public capital/private capital ratio is relatively high, the wage rate is higher, and so is the private savings for a given tax rate. High private savings reinforce private capital formation, lowering both the public capital/private capital ratio and the public debt/private capital ratio.

⁹ Increases in θ and/or λ shift the curve BB to the upper-left, although it passes through the origin.

In contrast, if the initial public debt is greater than the threshold, the budget rule, (θ, λ) , requires the government to revise the budget policy. The government borrowings hinder private capital formation, raising the marginal productivity of private capital, i.e. the interest rate. A greater amount of debt service requires the government to raise the income tax rate, thereby decelerating private capital formation. Since the size of public deficits is a constant fraction of GDP, the public debt/private capital ratio, and hence the tax rate, will be greater and greater. The growth rate of private capital becomes non-positive as the public debt/private capital ratio approaches the upper limit, \bar{x} . Therefore, the fiscal policy is not sustainable.¹⁰ The government must change the fiscal policy to reduce budget deficits. This policy change will be examined in the next section.

If the initial point is on the stable branch, the economy goes into the saddle-point equilibrium, U . The growth rate at equilibrium U may not necessarily be lower than the growth rate at equilibrium S , while the public debt/private capital ratio and the public capital/private capital ratio at U are higher than those at S . However, as will be shown in the next section, the government can lower the public debt/GDP ratio and raise the growth rate by increasing the deficit finance ratio, λ , at equilibrium U . Therefore, at equilibrium U , the government may have an incentive to increase the deficit finance ratio, and in this case the fiscal deficit policy could not be sustained by the government.¹¹

Thus, we have the following proposition:

Proposition 1

¹⁰ In our setting with the tax rate endogenously determined, the primary surplus/GDP ratio is given as $\tau - \theta = [\alpha(1 - \theta)x - \lambda\theta]/(1 + \alpha x)$. Since we can see that the primary surplus/GDP ratio increases with the debt/GDP ratio, i.e. $d(\tau - \theta)/dx > 0$, a sufficient condition for the sustainability, which is suggested by Bohn (1998), is satisfied even in this case.

¹¹ Once the deficit finance ratio or the public investment ratio is increased, the fiscal policy becomes unsustainable.

Under a public investment and fiscal deficit policy, parameterized by (θ, λ) , there is a threshold of public debt for each level of public capital in order for government to sustain the fiscal policy. The threshold of public debt is increasing in public capital stock.

It should be noted that the threshold of public debt, i.e. the constraint on the sustainability of budget deficits, is given in relation to the level of public, rather than private, capital, and that the threshold of public debt is greater as the stock of public capital becomes greater. This is in contrast to the sustainability literature without public capital accumulation, which shows the critical size of the initial public debt as a sustainability condition on the level of public debt in relation to a given initial private capital stock (e.g. Chalk, 2000; Bräuninger, 2005). If the economy has accumulated a greater stock of public capital in the past, it will be able to sustain a greater stock of public debt.¹² In our model with the growth engine of public capital accumulation, the government can run budget deficits even with a great stock of outstanding debt as long as sufficient public capital has been accumulated in the past. It is well known that the public investment/GDP ratio of Japan has been higher than those of other developed countries over the past several decades. Assuming that the higher investment is reflected in a greater stock of public capital, the budget deficits may be sustainable for an even relatively greater public debt.¹³ Fukuda and Teruyama (1994) among others

¹² Buiters (2001) pointed out that a prudent level of the debt-GDP ratio depends on many structural features of the economy, and noted that a one-size-fits-all figure, such as the 60% ceiling of the Maastricht Treaty, makes no sense at all. Our result confirms his statement in a growth context.

¹³ We here abstract from the efficiency problem of public investment. For example, in Japan the structure of Public Works has changed only slightly, i.e. most items changed no more than 2% points, over two decades (1980-1999), while the relative shares of primary, secondary and tertiary industries have changed from (3.6, 37.8, 58.7) to (1.7, 31.7, 66.7) respectively. Many authors in Japan (e.g. Doi and Nakazato, 2004) cast doubt on the productive efficiency. Yakita (2004) showed that when the elasticity of substitution in public investment is small, an increase in the income-tax-financed public investment ratio to GDP may result in a lower balanced growth rate even below the government size of the natural efficiency condition.

examined the sustainability of public debt in postwar Japan (for the period 1965-1992), based on the transversality condition test and the modified cointegration test, and concluded that it was sustainable. In contrast, Doi and Nakazato (2004) recently found that public debt is not sustainable, based on Bohn's (1998) test and using the data of the period 1955-2000. However, none of the above investigators considered the productivity effects of public expenditure explicitly. Our result shows that the sustainability of public debts must be considered by taking into account the relative magnitude of the stock of public capital.

4. Analysis of policy effects

We analyze the effects of changes in the public investment/GDP ratio and in the ratio of public bond financing.¹⁴

4.1 Changes in public investment/GDP ratio

While an increase in the public investment/GDP ratio, θ , shifts upward both curves GG and XX , we can see that the upward shift of the curve XX is greater than that of the curve GG (see Appendix C). Since changes in x and g satisfy equation (20), the stable equilibrium S shifts up-and-rightward and the saddle-point equilibrium U shifts left-and-downward. An increase in the public investment/GDP ratio raises both the (stable) long-term public capital/private capital ratio and public debt/private capital ratio. In contrast, the saddle-point equilibrium moves left-and-downward, and, at least in the vicinity of the equilibrium, the stable branch also shifts left-and-downward. Therefore, the threshold of public debt for a given value of public capital stock becomes lower, which implies that the range in which the deficit policy is not sustainable is widened by the increased public investment/GDP ratio. The economy lying initially near the saddle-point equilibrium becomes unable to sustain the public debt in the

¹⁴ The comparative statics make sense only for the stable long-term equilibrium.

sense that an increase in the public investment ratio enhances the instability of the economic system. Conversely, a decrease in the public investment ratio enhances the stability of the system in the sense that it may lead the economy to the stable long-term equilibrium when it was initially near, but on the lower-right of, the saddle-point equilibrium and with unsustainable (initial) public debt.

Since the public debt/GDP ratio can be written as $D/Y = (D/K)(K/Y) = xA^{-1}g^{\alpha-1}$, we have

$$\frac{d}{d\theta} \left(\frac{D}{Y} \right) = A^{-1} \left[g^{\alpha-1} \frac{dx}{d\theta} + (\alpha-1)g^{\alpha-2}x \frac{dg}{d\theta} \right]. \quad (27)$$

From (23) we can see $\text{sgn}[d(D/Y)/d\theta] = \text{sgn}[dg/d\theta]$, since at the stable equilibrium, $dx/d\theta = \lambda(dg/d\theta)$ from (20). Therefore, an increase (a decrease) in the public investment/GDP ratio raises (lowers) the public debt/GDP ratio of the long-term stable equilibrium. Then we have the following proposition:¹⁵

Proposition 2

A decrease (an increase) in the public investment/GDP ratio, keeping the debt finance ratio constant, will not only lower (raise) the public debt/GDP ratio in the long term, but also raise (lower) the threshold of the public debt/public capital ratio, so that the range of sustainable (initial) public debt will be enlarged (shrunk) by the policy change.

The tax rate is obtained from (13) as

$$\tau = \frac{\theta(1-\lambda) + \alpha x}{1 + \alpha x}. \quad (28)$$

The effect of an increase in the public investment/GDP ratio is two-fold: a short-term (direct) effect due to the increase in the ratio and a long-term (indirect) effect through changes in the public debt/private capital ratio. With an increase in the public capital/GDP ratio, both effects are obviously negative. Since the debt finance ratio is

¹⁵ The public debt/public capital, x/g , does not change in the long term.

kept constant, the increased public investment ratio not only requires an increase in the tax rate at the instant of the policy change, but also increases bond issues. The increased public debt in turn increases the interest payments to public debt, requiring a further increase in the tax rate. A decrease in the public investment ratio brings about a tax cut both in the short and long term.

Corollary 3

A decrease (an increase) in the public investment/GDP ratio leads to tax cuts (increases) not only directly in the short term but also indirectly through decreases (increases) in the interest payments to public debt in the long term.

4.2 Changes in public debt finance ratio

A rise in the public debt finance ratio of public investment shifts both the curves GG and XX upward. Thus, the effects of a rise in the public debt finance ratio on public capital/private capital ratio and the public debt/private capital ratio are qualitatively similar to those of increases in the public investment/GDP ratio (see Appendix C).

Proposition 4

When the public investment/GDP ratio is kept constant, a decrease (an increase) in the public debt finance ratio of the public investment not only reduces (raises) the long-term public debt/private capital ratio, but also raises (lowers) the threshold of the initial public debt/public capital ratio so that the range of sustainable (initial) public debt will be enlarged (shrunk) by the policy change.

However, in contrast to the effect of an increase in the public investment/GDP ratio, the short-term effect of the increased public debt finance ratio on the tax rate is negative, since it reduces the ratio of tax finance at the moment of the policy change. The indirect effect through increases in the public debt/private capital ratio will, at least

partly, offset the negative effect, although the long-term tax rate may be lower, constant or higher than the rate before the policy change.¹⁶

Corollary 5

A decrease in the debt finance ratio brings about a tax increase (a cut) in the short term, although the indirect effect through changes in the public debt/private capital ratio at least partly offsets the tax increase (the tax cut). The short-term effect may not be necessarily entirely offset by the long-term effect.

4.3 Effects on the balanced growth rate

From (14) the balanced growth rate is rewritten as

$$1 + \gamma = \left[\frac{1 - \theta(1 - \lambda)}{1 + \alpha x} \delta(1 - \alpha) - \lambda \theta \right] A g^{1 - \alpha} - x. \quad (29)$$

Making use of the fact that $\lambda g = x$ holds in the long-term equilibrium, and from (29), we have

$$\begin{aligned} \frac{d\gamma}{d\theta} = H^{-1} & \left(A g^{-\alpha} \left\{ \lambda \left[\frac{1 - \theta(1 - \lambda)}{(1 + \alpha x)^2} \delta(1 - \alpha) \alpha A g^{1 - \alpha} + 1 \right] \right. \right. \\ & \left. \left. - (1 - \alpha) A g^{-\alpha} \left[\frac{1 - \theta(1 - \lambda)}{1 + \alpha x} \delta(1 - \alpha) - \lambda \theta \right] \right\} \right. \\ & \left. + \alpha \theta A^2 g^{-2\alpha} \left[\frac{1 - \theta(1 - \lambda)}{1 + \alpha x} \delta(1 - \alpha) + \lambda \right] \right) \end{aligned} \quad (30)$$

and

$$\frac{d\gamma}{d\lambda} = H^{-1} \alpha \theta A g^{-\alpha} \left\{ A g^{-\alpha} \left(1 - \frac{\delta(1 - \alpha)}{1 + \alpha x} \right) + \left[\frac{1 - \theta(1 - \lambda)}{(1 + \alpha x)^2} \delta(1 - \alpha) \alpha A g^{1 - \alpha} + 1 \right] \right\} \quad (31)$$

where

¹⁶ An increase in λ raises the balanced-growth public debt/public capital ratio and moves the line (20).

$$\begin{aligned}
H = & \lambda \left[\frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha) A g^{1-\alpha} + 1 \right] \\
& - A g^{-\alpha-1} \left\{ (1-\alpha) g \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda \theta \right] + \alpha \theta \right\} \quad (32)
\end{aligned}$$

(see Appendix D). While the numerator of the right hand side of (31) is positive, we can see that the numerator of the right hand side of (30) is positive if H is positive. From (22) we can see that $H > 0$ when the curve GG is steeper than the line $g = x/\lambda$, and that $H < 0$ when the line is steeper than the curve GG . Thus, at a saddle-point equilibrium such as U in Fig. 3, we obtain $d\gamma/d\theta > 0$ and $d\gamma/d\lambda > 0$, that is, an increase in the public investment ratio *per se* boosts economic growth, making the interest rate higher due to debt finance, while an increase in the debt finance ratio reduces the tax finance ratio, stimulating private capital formation and thereby economic growth, but increasing interest payments at the same time. However, the comparative statics at the saddle-point-stable equilibrium may not make sense. In contrast, since the line $g = x/\lambda$ is steeper than the curve GG at a stable equilibrium such as S in Fig. 3, H is negative. Therefore, we can see that $d\gamma/d\lambda < 0$, that is, a cut in the public debt finance ratio, leads to higher growth since it stimulates private capital formation. This result is consistent with that in Bräuninger (2005), who assumed away public capital accumulation. However, the effect of an increase in the public investment/GDP ratio, θ , on the growth rate may be positive, zero, or negative. A rise in the public investment/GDP ratio affects economic growth negatively through corresponding increases in the tax rate *per se* and in the public debt/private capital ratio, whereas it has a positive effect on growth by raising the marginal productivity of private capital. If the latter positive effect is sufficiently great, we can not exclude the possibility that the increased public investment ratio will raise the balanced growth rate.

5. Concluding remarks

In an endogenous growth model, populated by two-period-lived generations and with an engine of public capital formation, we have analyzed the sustainability of public debt policy, assuming that the public capital/GDP ratio and the public debt finance ratio of public investment are kept constant, and that the tax rate is adjusted so as to satisfy the government budget equation. With Cobb-Douglas production function and the log-linear utility function, we have shown that there is a threshold for the initial stock of public debt at each level of public debt in order for the public investment and deficit policy to be sustainable, and that the threshold is increasing in the stock of public capital. This contrasts with the condition of initial indebtedness of the economy given in the conventional literature without public investment in which the critical initial level of public debt is given in relation to private capital. When public debt is greater than the threshold, the economy can no longer sustain the budget deficit and hence the balanced growth. This implies that an economy which has accumulated only small public capital in the past may seriously diminish its set of feasible policy alternatives.

In our study with public capital formation, the sustainable fiscal policy is also conditioned by the public investment/GDP ratio and the debt finance ratio of public investment, rather than the size of government budget deficit itself. Although the critical combination of the ratios can not be explicitly obtained, the result is also in contrast to the literature without public capital accumulation.

So far we have analyzed the sustainability of government deficit policy under the golden rule of public finance, assuming that the public investment ratio and the debt finance ratio are kept constant and that the tax rate is endogenously determined to satisfy the government budget. The problems analyzed here are rather positive models in the sense that the fiscal policy may not necessarily be optimally arranged. In a representative, infinitely-lived agent model, Ghosh and Mourmouras (2004) showed that the optimal fiscal policy depends on the budgetary regime taken by government. While many authors (e.g. Pestieau, 1974) examined the optimal public

capital formation and taxation policy in decentralized economies populated by overlapping generations in which public debt can have real effects, most of them did not take into account the restriction of the golden rule of public finance. Analyzing the optimal policy under various fiscal regimes in such a setting is an interesting issue for future research.

Appendix A:

From (18) we obtain the marginal effect of g on x_{+1}/x , evaluated on the curve XX , as

$$\frac{d}{dg}\left(\frac{x_{+1}}{x}\right)|_{XX} = \frac{-(1-\alpha)Ag^{-\alpha}\left\{\left[\frac{1-\theta(1-\lambda)}{1+\alpha x}\delta(1-\alpha)-\lambda\theta\right]+\lambda\theta\right\}}{\left\{\left[\frac{1-\theta(1-\lambda)}{1+\alpha x}\delta(1-\alpha)-\lambda\theta\right]Ag^{1-\alpha}-x\right\}^2} < 0. \quad (A1)$$

Therefore, we have $x_{+1}/x < 1$ above the curve XX and $x_{+1}/x > 1$ below the curve.

From (17) we have the marginal effect of g on g_{+1}/g , evaluated on the curve GG , as

$$\frac{d}{dx}\left(\frac{g_{+1}}{g}\right)|_{GG} = \frac{(1+\theta Ag^{-\alpha})\left[\frac{1-\theta(1-\lambda)}{(1+\alpha x)^2}\delta(1-\alpha)Ag^{1-\alpha}+1\right]}{\left\{\left[\frac{1-\theta(1-\lambda)}{1+\alpha x}\delta(1-\alpha)-\lambda\theta\right]Ag^{1-\alpha}-x\right\}^2} > 0. \quad (A2)$$

Therefore, $g_{+1}/g < 1$ on the left of the curve GG , and $g_{+1}/g > 1$ on the right of the curve GG .

Appendix B:

Approximating (17) and (18) linearly in the neighborhood of the steady state, (\bar{g}, \bar{x}) , we obtain

$$\begin{pmatrix} g_{+1} - \bar{g} \\ x_{+1} - \bar{x} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_{+1}}{\partial g} & \frac{\partial g_{+1}}{\partial x} \\ \frac{\partial x_{+1}}{\partial g} & \frac{\partial x_{+1}}{\partial x} \end{pmatrix} \begin{pmatrix} g - \bar{g} \\ x - \bar{x} \end{pmatrix} \quad (A3)$$

where $\frac{\partial g_{+1}}{\partial g} = g \cdot \partial(g_{+1}/g)/\partial g + 1$, $\frac{\partial g_{+1}}{\partial x} = g \cdot \partial(g_{+1}/g)/\partial x$, $\frac{\partial x_{+1}}{\partial x} = x \cdot \partial(x_{+1}/x)/\partial x + 1$ and $\frac{\partial x_{+1}}{\partial g} = x \cdot \partial(x_{+1}/x)/\partial g$, which are evaluated at the steady equilibrium. The characteristic equation is

$$\phi(\mu) \equiv \mu^2 - \left(\frac{\partial g_{+1}}{\partial g} + \frac{\partial x_{+1}}{\partial x}\right)\mu + \left(\frac{\partial g_{+1}}{\partial g} \frac{\partial x_{+1}}{\partial x} - \frac{\partial g_{+1}}{\partial x} \frac{\partial x_{+1}}{\partial g}\right) = 0. \quad (A4)$$

We can see

$$\begin{aligned}
\phi(0) &= \frac{\partial g_{+1}}{\partial g} \frac{\partial x_{+1}}{\partial x} - \frac{\partial g_{+1}}{\partial x} \frac{\partial x_{+1}}{\partial g} \\
&= \left\{ \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda\theta \right] A g^{1-\alpha} - x \right\}^{-2} \\
&\quad \cdot \left[\alpha(1+x) + x \frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha) \alpha A g^{1-\alpha} \right] > 0
\end{aligned} \tag{A5}$$

and

$$\begin{aligned}
\phi(1) &= 1 - \left(\frac{\partial g_{+1}}{\partial g} + \frac{\partial x_{+1}}{\partial x} \right) + \left(\frac{\partial g_{+1}}{\partial g} \frac{\partial x_{+1}}{\partial x} - \frac{\partial g_{+1}}{\partial x} \frac{\partial x_{+1}}{\partial g} \right) \\
&= x g \left(\frac{\partial(g_{+1}/g)}{\partial g} \frac{\partial(x_{+1}/x)}{\partial x} - \frac{\partial(g_{+1}/g)}{\partial x} \frac{\partial(x_{+1}/x)}{\partial g} \right).
\end{aligned} \tag{A6}$$

From (17) and (18) we can see

$$\frac{\partial(g_{+1}/g)}{\partial g} = - \frac{\alpha \theta A g^{-\alpha-1} + (1-\alpha) A g^{-\alpha} \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda\theta \right]}{1 + \theta A g^{-\alpha}} < 0, \tag{A7}$$

$$\frac{\partial(g_{+1}/g)}{\partial x} = \frac{1 + \frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha) \alpha A g^{1-\alpha}}{1 + \theta A g^{-\alpha}} > 0, \tag{A8}$$

$$\frac{\partial(x_{+1}/x)}{\partial g} = - \frac{(1-\alpha) A g^{-\alpha} \left\{ \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda\theta \right] - \frac{\lambda\theta}{x} \right\}}{1 + \theta A g^{-\alpha}} < 0, \tag{A9}$$

$$\begin{aligned}
\frac{\partial(x_{+1}/x)}{\partial x} &= \frac{\left[\frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha) \alpha - \frac{\lambda\theta}{x^2} \right] A g^{1-\alpha} + 1}{1 + \theta A g^{-\alpha}} > 0 \\
&\text{as } \left[\frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha) \alpha - \frac{\lambda\theta}{x^2} \right] A g^{1-\alpha} + 1 > 0.
\end{aligned} \tag{A10}$$

The discriminant of (A4) is

$$\begin{aligned}
& \left(\frac{\partial g_{+1}}{\partial g} + \frac{\partial x_{+1}}{\partial x}\right)^2 - 4\left(\frac{\partial g_{+1}}{\partial g} \frac{\partial x_{+1}}{\partial x} - \frac{\partial g_{+1}}{\partial x} \frac{\partial x_{+1}}{\partial g}\right) \\
& = \left(\frac{\partial g_{+1}}{\partial g} - \frac{\partial x_{+1}}{\partial x}\right)^2 + 4\frac{\partial g_{+1}}{\partial x} \frac{\partial x_{+1}}{\partial g} > 0.
\end{aligned} \tag{A11}$$

We can see that when $\partial(x_{+1}/x)/\partial x > 0$, the equation has real value solutions, while when $\partial(x_{+1}/x)/\partial x < 0$, the equation may have imaginary value solutions, as shown below.

Since

$$\left.\frac{dg}{dx}\right|_{XX} = -\frac{\frac{\partial(x_{+1}/x)}{\partial x}}{\frac{\partial(x_{+1}/x)}{\partial g}} > \left.\frac{dg}{dx}\right|_{GG} = -\frac{\frac{\partial(g_{+1}/g)}{\partial x}}{\frac{\partial(g_{+1}/g)}{\partial g}} (> 0) \tag{A12}$$

at equilibrium U , where $[\frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha)\alpha - \frac{\lambda\theta}{x^2}]Ag^{1-\alpha} + 1 > 0$, we can see from

(A6) that the characteristic equation has real value solutions and $\phi(1) < 0$. Therefore, we have $0 < \mu_1 < 1 < \mu_2$ where μ_i 's are the eigenvalues. This implies the equilibrium is a saddle point.

On the other hand, at equilibrium S , where

$[\frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha)\alpha - \frac{\lambda\theta}{x^2}]Ag^{1-\alpha} + 1 < 0$, we may have the imaginary value solutions.

At equilibrium S we have

$$\left.\frac{dg}{dx}\right|_{XX} = -\frac{\frac{\partial(x_{+1}/x)}{\partial x}}{\frac{\partial(x_{+1}/x)}{\partial g}} < 0 < \left.\frac{dg}{dx}\right|_{GG} = -\frac{\frac{\partial(g_{+1}/g)}{\partial x}}{\frac{\partial(g_{+1}/g)}{\partial g}}. \tag{A13}$$

Therefore, we obtain $\phi(1) > 0$ from (A6) and $\phi'(1) = 2 - (\frac{\partial g_{+1}}{\partial g} + \frac{\partial x_{+1}}{\partial x})$

$= -(g \frac{\partial(g_{+1}/g)}{\partial g} + x \frac{\partial(x_{+1}/x)}{\partial x}) > 0$ from (A7) and (A10). We also have

$$\phi'(0) = -\left(\frac{\partial g_{+1}}{\partial g} + \frac{\partial x_{+1}}{\partial x}\right) < 0 \quad \text{since} \quad \left(\frac{\partial g_{+1}}{\partial g} + \frac{\partial x_{+1}}{\partial x}\right)$$

$$= \frac{1+x \frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha) \alpha A g^{1-\alpha} + \alpha(1+x)}{\left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda \theta\right] A g^{1-\alpha} - x} > 0 \quad \text{from (17) and (18).} \quad \text{Therefore,}$$

when the solutions are real values, we have $0 < \mu_1, \mu_2 < 1$. The equilibrium S is locally stable and the system monotonically converges to equilibrium in its vicinity. On the other hand, if we have imaginary value solutions, i.e. if the discriminant (A11) is negative, the system is oscillatory and converges to the steady equilibrium S as long as (A5) is smaller than one. Since the slope of the curve GG is smaller than the line (20) at equilibrium S , we have

$$\frac{1}{\lambda} > \frac{\frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha) \alpha A g^{1-\alpha} + 1}{(1-\alpha) \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda \theta \right] A g^{-\alpha} + \alpha \theta A g^{-\alpha-1}}.$$

Making use of (18), (19) and (20), this condition becomes

$$1 + \lambda \theta A \frac{g^{1-\alpha}}{x} > \alpha(1+x) + x \frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha) \alpha A g^{1-\alpha}. \quad (\text{A14})$$

Thus, together with (18) and (19), we can see that (A5) is smaller than one.

Therefore, whether the eigenvalues of the equation (A4) are real or imaginary, the equilibrium S is locally stable.

Appendix C:

From (17) and (19a) we obtain

$$\begin{aligned} & -A g^{-\alpha-1} \left\{ (1-\alpha) g \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda \theta \right] + \alpha \theta \right\} dg \\ & + \left[\frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha) \alpha A g^{1-\alpha} + 1 \right] dx \end{aligned}$$

$$= -Ag^{-\alpha} \left\{ \left[\frac{1-\lambda}{1+\alpha x} \delta(1-\alpha) - \lambda \right] g + 1 \right\} d\theta + \theta Ag^{1-\alpha} \left[\frac{1}{1+\alpha x} \delta(1-\alpha) - 1 \right] d\lambda. \quad (\text{A15})$$

Therefore, the shifts of the curve GG due to the policy changes are given as

$$\frac{dg}{d\theta} \Big|_{GG} = \frac{Ag^{-\alpha} \left\{ \left[\frac{1-\lambda}{1+\alpha x} \delta(1-\alpha) + \lambda \right] g + 1 \right\}}{Ag^{-\alpha-1} \left\{ (1-\alpha)g \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\lambda) - \lambda\theta \right] + \alpha\theta \right\}} (> 0), \quad (\text{A16})$$

$$\frac{dg}{d\lambda} \Big|_{GG} = \frac{-\theta Ag^{1-\alpha} \left[\frac{1}{1+\alpha x} \delta(1-\alpha) - 1 \right]}{Ag^{-\alpha-1} \left\{ (1-\alpha)g \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\lambda) - \lambda\theta \right] + \alpha\theta \right\}} (> 0). \quad (\text{A17})$$

Therefore, both changes in the public investment/GDP ratio and the public debt finance ratio shift the curve GG upward. Similarly, from (18) and (19b), we obtain

$$\begin{aligned} & -(1-\alpha)Ag^{-\alpha} \left\{ \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda\theta \right] - \frac{\lambda\theta}{x} \right\} dg \\ & + \left[1 - Ag^{1-\alpha} \frac{\lambda\theta}{x^2} + \frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha)\alpha Ag^{1-\alpha} \right] dx \\ & = \theta Ag^{1-\alpha} \left[\frac{1}{1+\alpha x} \delta(1-\alpha) - 1 - \frac{1}{x} \right] d\lambda - Ag^{1-\alpha} \left[\frac{1-\lambda}{1+\alpha x} \delta(1-\alpha) + \lambda \left(1 + \frac{1}{x} \right) \right] d\theta \end{aligned} \quad (\text{A18})$$

from which the shift of the curve XX is given as

$$\frac{dg}{d\theta} \Big|_{XX} = \frac{Ag^{1-\alpha} \left[\frac{1-\lambda}{1+\alpha x} \delta(1-\alpha) + \lambda \left(1 + \frac{1}{x} \right) \right]}{(1-\alpha)Ag^{-\alpha} \left\{ \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\lambda) - \lambda\theta \right] - \frac{\lambda\theta}{x} \right\}} (> 0), \quad (\text{A19})$$

$$\frac{dg}{d\lambda} \Big|_{XX} = \frac{\theta Ag^{1-\alpha} \left[1 + \frac{1}{x} - \frac{1}{1+\alpha x} \delta(1-\alpha) \right]}{(1-\alpha)Ag^{-\alpha} \left\{ \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\lambda) - \lambda\theta \right] - \frac{\lambda\theta}{x} \right\}} (> 0) \quad (\text{A20})$$

where the denominators on the right hand side of (A19) and (A20) are positive from (19b). Thus, increases in the public investment/GDP ratio and the public debt finance

ratio shift the curve XX upward.

Since the numerator of the right hand side of (A16) is the same as that of (A19), we have

$$\begin{aligned}
& \frac{dg}{d\theta} \Big|_{XX} - \frac{dg}{d\theta} \Big|_{GG} \\
&= \frac{Ag^{-\alpha} \left\{ \left[\frac{1-\lambda}{1+\alpha x} \delta(1-\alpha) + \lambda \right] g + 1 \right\}}{\left[Ag^{-\alpha-1} \left\{ (1-\alpha) g \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\lambda) - \lambda\theta \right] + \alpha\theta \right\} \right] \left[(1-\alpha) Ag^{-\alpha} \left\{ \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\lambda) - \lambda\theta \right] - \frac{\lambda\theta}{x} \right\} \right]} \\
&\times \left(Ag^{-\alpha-1} \left\{ (1-\alpha) g \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\lambda) - \lambda\theta \right] + \alpha\theta \right\} - (1-\alpha) Ag^{-\alpha} \left\{ \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\lambda) - \lambda\theta \right] - \frac{\lambda\theta}{x} \right\} \right) \\
&= \frac{\theta \left\{ \left[\frac{1-\lambda}{1+\alpha x} \delta(1-\alpha) + \lambda \right] g + 1 \right\}}{\left\{ (1-\alpha) g \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\lambda) - \lambda\theta \right] + \alpha\theta \right\} (1-\alpha) \left\{ \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\lambda) - \lambda\theta \right] - \frac{\lambda\theta}{x} \right\}} > 0. \quad (\text{A21})
\end{aligned}$$

Similarly, from (A17) and (A20), we have

$$\begin{aligned}
& \frac{dg}{d\lambda} \Big|_{XX} - \frac{dg}{d\lambda} \Big|_{GG} \\
&= \frac{\theta \left[1 - \frac{\delta(1-\alpha)}{1+\alpha x} \right] + \frac{1}{x} \left\{ \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda\theta \right] (1-\alpha) g + \alpha\theta \right\}}{\left\{ (1-\alpha) g \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\lambda) - \lambda\theta \right] + \alpha\theta \right\} (1-\alpha) \left\{ \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\lambda) - \lambda\theta \right] - \frac{\lambda\theta}{x} \right\}} > 0. \quad (\text{A22})
\end{aligned}$$

Therefore, both increases in the public investment/GDP ratio and the public debt finance ratio shift the curve XX upward more than the curve GG .

Appendix D:

From (20) we have

$$gd\lambda + \lambda dg = dx. \quad (\text{A23})$$

Making use of (A15) and (A23), we obtain

$$\frac{dg}{d\theta} = H^{-1} \left(-Ag^{-\alpha} \left\{ 1 + \left[\frac{1-\lambda}{1+\alpha x} \delta(1-\alpha) + \lambda \right] g \right\} \right), \quad (\text{A24})$$

$$\frac{dx}{d\theta} = \lambda \frac{dg}{d\theta} = H^{-1} \left(-\lambda Ag^{-\alpha} \left\{ 1 + \left[\frac{1-\lambda}{1+\alpha x} \delta(1-\alpha) + \lambda \right] g \right\} \right). \quad (\text{A25})$$

As noted in the text, since the curve GG crosses the line (20) from the left-above to the right-below at the stable equilibrium S , we have $H < 0$ from (22). However, since the curve GG is steeper than the line (20) at the saddle-point-stable equilibrium U , we have $H > 0$. Thus, we have $dg/d\theta > 0$ and $dx/d\theta > 0$ at the equilibrium S , and $dg/d\theta < 0$ and $dx/d\theta < 0$ at the equilibrium U . Similarly, since

$$\frac{dg}{d\lambda} = H^{-1} \left(\theta Ag^{1-\alpha} \left(\frac{\delta(1-\lambda)}{1+\alpha x} - 1 \right) - \left[\frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha) \alpha Ag^{1-\alpha} + 1 \right] g \right), \quad (\text{A26})$$

$$\begin{aligned} \frac{dx}{d\lambda} = H^{-1} \left(\lambda \theta Ag^{1-\alpha} \left(\frac{\delta(1-\lambda)}{1+\alpha x} - 1 \right) \right. \\ \left. - Ag^{-\alpha} \left\{ (1-\alpha) g \left[\frac{1-\theta(1-\alpha)}{1+\alpha x} \delta(1-\alpha) - \lambda \theta \right] + \alpha \theta \right\} \right), \quad (\text{A27}) \end{aligned}$$

we have $dg/d\lambda > 0$ and $dx/d\lambda > 0$ at the equilibrium S , and $dg/d\lambda < 0$ and $dx/d\lambda < 0$ at the equilibrium U .

From (29) we obtain

$$\begin{aligned} \frac{d\gamma}{d\theta} = (1-\alpha) Ag^{-\alpha} \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda \theta \right] \frac{dg}{d\theta} \\ - \left[\frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha) \alpha Ag^{1-\alpha} + 1 \right] \frac{dx}{d\theta} - \left[\frac{1-\lambda}{1+\alpha x} \delta(1-\alpha) + \lambda \right] Ag^{1-\alpha}. \quad (\text{A28}) \end{aligned}$$

Inserting (A24) and (A25) into (A28), and rearranging terms, we have

$$\frac{d\gamma}{d\theta} = H^{-1} \left(Ag^{-\alpha} \left\{ -(1-\alpha) Ag^{-\alpha} \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda \theta \right] \right. \right.$$

$$+ \lambda \left[\frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha) \alpha A g^{1-\alpha} + 1 \right] + \alpha \theta A^2 g^{-2\alpha} \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) + \lambda \right] \Bigg). \quad (\text{A29})$$

When the curve GG is steeper than the line (20) as at the equilibrium U , we have

$$\begin{aligned} & \lambda \left[\frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha) \alpha A g^{1-\alpha} + 1 \right] \\ & > (1-\alpha) A g^{-\alpha} \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda \theta \right] + \alpha \theta A g^{-\alpha-1} \\ & > (1-\alpha) A g^{-\alpha} \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda \theta \right]. \end{aligned} \quad (\text{A30})$$

In this case, $H > 0$, and the numerator of the right hand side of (A29) is positive. Therefore, we have $d\gamma/d\theta > 0$. However, even when $H < 0$, the numerator of the right hand side of (A29) can be negative.

From (29), we have

$$\begin{aligned} \frac{d\gamma}{d\lambda} &= (1-\alpha) A g^{-\alpha} \left[\frac{1-\theta(1-\lambda)}{1+\alpha x} \delta(1-\alpha) - \lambda \theta \right] \frac{dg}{d\lambda} \\ &- \left[\frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha) \alpha A g^{1-\alpha} + 1 \right] \frac{dx}{d\lambda} + \left[\frac{1}{1+\alpha x} \delta(1-\alpha) - 1 \right] \theta A g^{1-\alpha}. \end{aligned} \quad (\text{A31})$$

Inserting (A26) and (A27) into (A31), and rearranging terms, we have

$$\frac{d\gamma}{d\lambda} = H^{-1} \alpha \theta A g^{-\alpha} \left\{ A g^{-\alpha} \left(1 - \frac{\delta(1-\alpha)}{1+\alpha x} \right) + \left[\frac{1-\theta(1-\lambda)}{(1+\alpha x)^2} \delta(1-\alpha) \alpha A g^{1-\alpha} + 1 \right] \right\}. \quad (\text{A32})$$

Since the numerator on the right hand side of (A32) is positive, we have $\text{sgn}(d\gamma/d\lambda) = \text{sgn}(H)$. Therefore, we have $d\gamma/d\lambda < 0$ at the stable equilibrium S where $H < 0$, and $d\gamma/d\lambda > 0$ at the saddle-point-stable equilibrium U where $H > 0$.

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References

- Arrow, K.J., Kurz, M., 1970. Public Investment, the Rate of Return, and Optimal Fiscal Policy. The John Hopkins Press, Baltimore.
- Aschauer, D., 1989. Is public expenditure productive? *Journal of Monetary Economics* 23, 177-200.
- Barro, R.J., 1990. Government spending in a simple model of endogenous growth. *Journal of Political Economy* 98, S103-S125.
- Bohn, H., 1998. The behavior of U.S. public debt and deficits. *Quarterly Journal of Economics* 113, 949-963.
- Bräuning, M., 2005. The budget deficit, public debt, and endogenous growth. *Journal of Public Economic Theory* 7, 827-840.
- Buiter, W.H., 2001. Notes on 'A code for fiscal stability.' *Oxford Economic Papers* 53, 1-19.
- Chalk, N.A., 2000. The sustainability of bond-financed deficits: An overlapping generations approach. *Journal of Monetary Economics* 45, 293-328.
- Diamond, P.A., 1965. National debt in a neoclassical growth model. *American Economic Review* 55, 1126-1150.
- Doi, T., Nakazato, T., 2004. Sustainability of public debt. In: Ihori, T. (Ed.), *Fiscal deficits in Japan*, Iwanami Shoten, Tokyo, pp. 27-51. (in Japanese)
- Evans, P., Karras, G., 1994. Is government capital productive? Evidence from a panel of

- seven countries. *Journal of Macroeconomics* 16, 271-279.
- Fukuda, S., Teruyama, H., 1994. The sustainability of budget deficits in Japan. *Hitotsubashi Journal of Economics* 35, 109-119.
- Futagami, K., Morita, Y., Shibata, A., 1993. Dynamic analysis of an endogenous growth model with public capital. *Scandinavian Journal of Economics* 95, 607-625.
- Ghosh, S., Mourmouras, I.A., 2004. Endogenous growth, welfare and budgetary regimes. *Journal of Macroeconomics* 26, 623-635.
- Greiner, A., Semmler, W., 2000. Endogenous growth, government debt and budgetary regimes. *Journal of Macroeconomics* 22, 363-384.
- Greiner, A., 2006. An endogenous growth model with public capital and sustainable government debt. *Japanese Economic Review* (forthcoming).
- Hamilton, J.D., Flavin, M.A., 1986. On the limitations of government borrowing: A framework for empirical testing. *American Economic Review* 76, 808-816.
- Holtz-Eakin, D., 1994. Public sector and the productivity puzzle. *Review of Economics and Statistics* 76, 12-21.
- Iwamoto, Y., 1990. An evaluation of public investment policy in postwar Japan. *Economic Review* 41, 250-261 (in Japanese).
- OECD, 2004. *Economic Outlook*.
- Pestieau, P., 1974. Optimal taxation and discount rate for public investment in a growth setting. *Journal of Public Economics* 3, 217-235.
- Tirole, J., 1985. Asset bubbles and overlapping generations. *Econometrica* 53, 1499-1528.
- Yakita, A., 2004. Elasticity of substitution in public capital formation and economic growth. *Journal of Macroeconomics* 26, 391-408.

Fig. 1

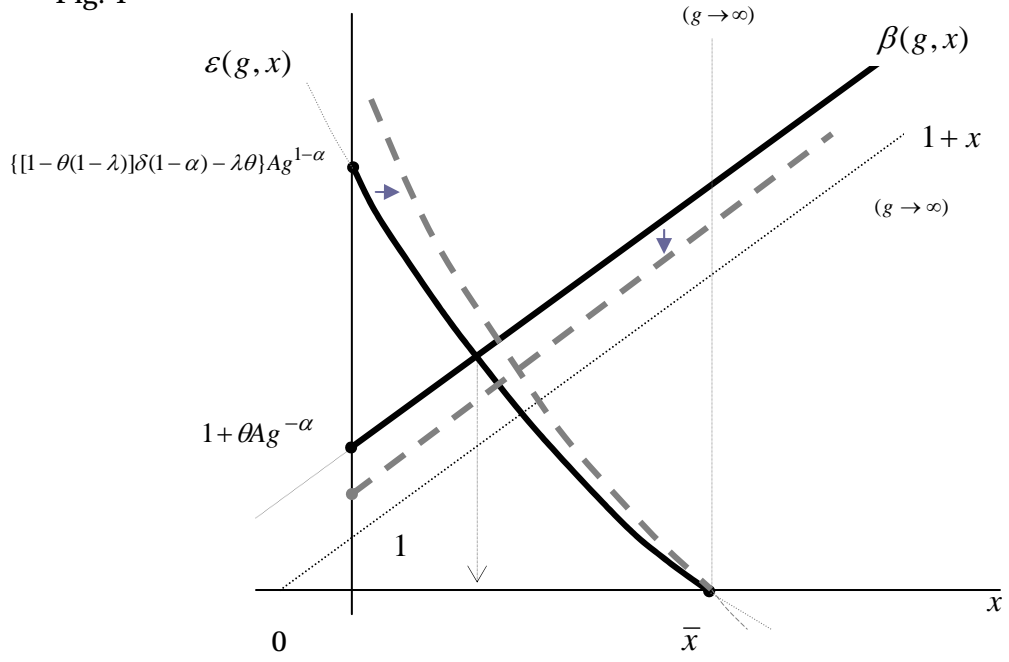


Fig. 2

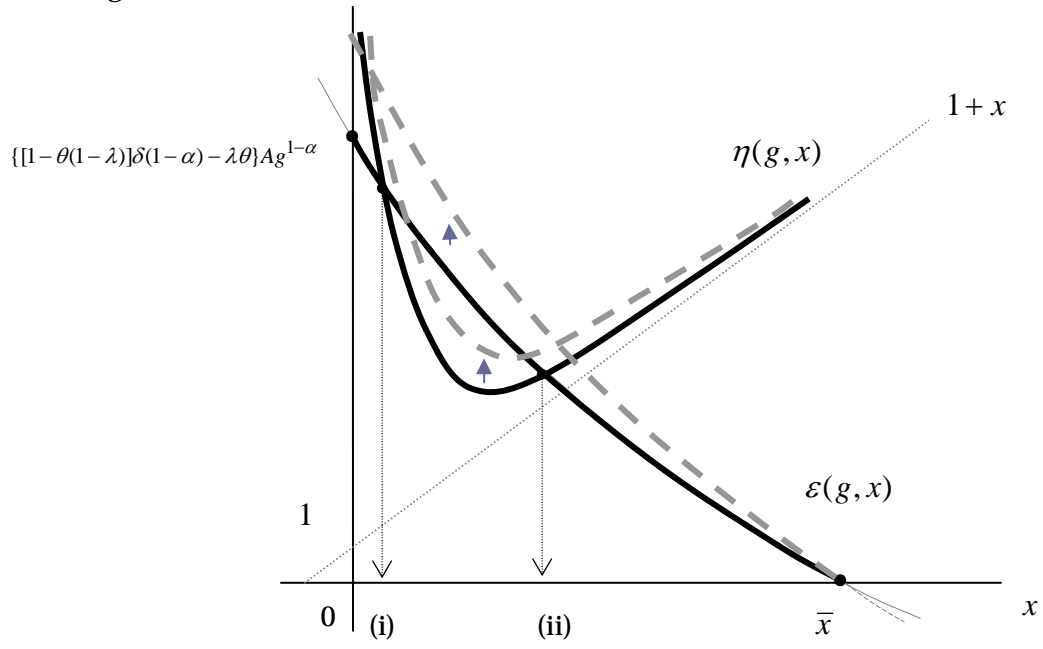


Fig. 3

