HAUSDORFF HYPERSPACES OF $\mathbb{R}^m$ AND THEIR DENSE SUBSPACES

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Abstract. Let $\mathrm{Bd}_H(\mathbb{R}^m)$ be the hyperspace of nonempty bounded closed subsets of Euclidean space $\mathbb{R}^m$ endowed with the Hausdorff metric. It is well known that $\mathrm{Bd}_H(\mathbb{R}^m)$ is homeomorphic to the Hilbert cube minus a point. We prove that natural dense subspaces of $\mathrm{Bd}_H(\mathbb{R}^m)$ of all nowhere dense closed sets, of all perfect sets, of all Cantor sets and of all Lebesgue measure zero sets are homeomorphic to the Hilbert space ℓ_2. For each $0 \leqslant 1 < m$, let

$$\nu_k^m = \{x = (x_i)_{i=1}^m \in \mathbb{R}^m : x_i \in \mathbb{R} \setminus \mathbb{Q} \text{ except for at most } k \text{ many } i\},$$

where ν_k^{2k+1} is the k-dimensional Nöbeling space and $\nu_0^m = (\mathbb{R} \setminus \mathbb{Q})^m$. It is also proved that the spaces $\mathrm{Bd}_H(\nu_0^1)$ and $\mathrm{Bd}_H(\nu_k^m)$, $0 \leqslant k < m-1$, are homeomorphic to ℓ_2. Moreover, we investigate the hyperspace $\mathrm{Cld}_H(\mathbb{R})$ of all nonempty closed subsets of the real line $\mathbb{R}$ with the Hausdorff (infinite-valued) metric. It is shown that a nonseparable component $\mathcal{H}$ of $\mathrm{Cld}_H(\mathbb{R})$ is homeomorphic to the Hilbert space $\ell_2(2^{\aleph_0})$ of weight $2^{\aleph_0}$ in case where $\mathcal{H} \not\ni \mathbb{R}, [0,\infty), (-\infty, 0]$.

Introduction

In this paper, we consider metric spaces and their hyperspaces endowed with the Hausdorff metric. Specifically, given a metric space $X = \langle X, d \rangle$, we shall denote by $\mathrm{Cld}(X)$ and $\mathrm{Bd}(X)$ the hyperspaces consisting of all nonempty closed sets and of all nonempty bounded closed sets in X respectively and by d_H the Hausdorff metric, which is infinite-valued on $\mathrm{Cld}(X)$ if X is unbounded. We shall sometimes write $\mathrm{Cld}_H(X)$ or $\mathrm{Bd}_H(X)$ to emphasize the fact that we consider this space with the Hausdorff metric topology.

A theorem of Antosiewicz and Cellina [2] states that, given a convex set X in a normed linear space, every continuous multivalued map $\varphi\colon Y \to \mathrm{Bd}_H(X)$ from a closed subset Y of a metric space Z, can be extended to a continuous map $\overline{f}\colon Z \to \mathrm{Bd}_H(X)$. Using the language of topology, this theorem says that, under the above assumptions, $\mathrm{Bd}_H(X)$ is an absolute extensor or an absolute retract (in the class of metric spaces). In [8], it is proved that the above result is still valid when X is replaced by a dense

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subset of a convex set in a normed linear space. More generally, $\mathrm{Bd}_H(X)$ is an absolute retract, whenever the metric on X is *almost convex* (see §3 for the definition). This condition was further weakened in [17], which has turned out to be actually a necessary and sufficient one by Banakh and Voytsitskyy [4]. In the last paper, several equivalent conditions are given, which are too technical to mention them here. We refer to [4] for the details.

It is a natural question whether $\mathrm{Bd}_H(X)$ and some of its natural subspaces are homeomorphic to some standard spaces, like the Hilbert cube/space, etc. Since the Hausdorff metric topology coincides with the Vietoris topology on the hyperspace $\exp(X)$ of nonempty compact sets, the above question was already answered, applying known results, in case where bounded closed sets in X are compact. Among the known results, let us mention the theorem of Curtis and Schori [10] (cf. [19, Chapter 8]), saying that $\exp(X)$ is homeomorphic to ($\approx$) the Hilbert cube $\mathrm{Q} = [-1,1]^\omega$ if and only if X is a Peano continuum, that is, it is compact, connected and locally connected. Later, Curtis [9] characterized non-compact metric spaces X for which $\exp(X)$ is homeomorphic to the Hilbert cube minus a point $\mathrm{Q}\setminus 0$ ($= \mathrm{Q}\setminus\{0\}$) or the pseudo-interior $\mathrm{s} = (-1,1)^\omega$ of Q.[1] In particular, $\mathrm{Bd}_H(\mathbb{R}^m) = \exp(\mathbb{R}^m) \approx \mathrm{Q}\setminus 0$. For more information concerning Vietoris hyperspaces, we refer to the book [13].

The aim of this work is to study topological types of some of the natural subspces of the Hausdorff hyperspace. We consider the following subspaces of $\mathrm{Bd}_H(X)$:

- $\mathrm{Nwd}(X)$ — all nowhere dense closed sets;
- $\mathrm{Perf}(X)$ — all perfect sets;[2]
- $\mathrm{Cantor}(X)$ — all compact sets homeomorphic to the Cantor set.

In case $X = \mathbb{R}^m$ with the standard metric, we can also consider the following subspace:

- $\mathfrak{N}(\mathbb{R}^m)$ — all closed sets of the Lebesgue measure zero.

We show that, in case $X = \mathbb{R}^m$, the above spaces are homeomorphic to the separable Hilbert space ℓ_2. Actually, we prove that if $\mathcal{F}$ is one of the above spaces then the pair $\langle \mathrm{Bd}_H(\mathbb{R}^m), \mathcal{F}\rangle$ is homeomorphic to $\langle \mathrm{Q}\setminus 0, \mathrm{s}\setminus 0\rangle$.

The completion of a metric space $X = \langle X, d\rangle$ is denoted by $\langle \tilde{X}, d\rangle$. Then $\mathrm{Bd}_H(X,d)$ can be identified with the subspace of $\mathrm{Bd}_H(\tilde{X},d)$, via the isometric embedding $A \mapsto \mathrm{cl}_{\tilde{X}} A$. Thus we shall often write $\mathrm{Bd}(X,d) \subseteq \mathrm{Bd}(\tilde{X},d)$, having in mind this identification. In this case, $\mathrm{Bd}(\tilde{X},d)$ is the completion of $\mathrm{Bd}(X,d)$. By such a reason, we also consider a dense subspace D of a metric space $X = \langle X, d\rangle$. For each $0 \leqslant k < m$, let

$$\nu^m_k = \{x = (x_i)^m_{i=1} \in \mathbb{R}^m : x_i \in \mathbb{R}\setminus\mathbb{Q} \text{ except for at most } k \text{ many } i\},$$

which is the universal space for completely metrizable subspaces in $\mathbb{R}^m$ of $\dim \leqslant k$. In case $2k+1 < m$, ν^m_k is homeomorphic to the k-dimensional

[1] It is well known that s is homeomorphic to the separable Hilbert space ℓ_2.

[2] I.e., completely metrizable closed sets which are dense in itself.

Nöbeling space ν_k^{2k+1}, which is the universal space for all separable completely metrizable spaces. Note that $\nu_0^m = (\mathbb{R} \setminus \mathbb{Q})^m \approx \mathbb{R} \setminus \mathbb{Q}$. We show that the pairs $\langle \mathrm{Bd}(\mathbb{R}), \mathrm{Bd}(\mathbb{R} \setminus \mathbb{Q})\rangle$ and $\langle \mathrm{Bd}(\mathbb{R}^m), \mathrm{Bd}(\nu_k^m)\rangle$, $0 \leqslant k < m-1$, are homeomorphic to $\langle \mathrm{Q} \setminus 0, \mathrm{s} \setminus 0\rangle$, so we have $\mathrm{Bd}_H(\nu_k^m) \approx \ell_2$ if $\langle m, k\rangle = \langle 1, 0\rangle$ or $0 \leqslant k < m-1$.

We also study the space $\mathrm{Cld}_H(\mathbb{R})$. It is very different from the hyperspace $\exp(\mathbb{R})$. It is not hard to see that $\mathrm{Cld}_H(\mathbb{R})$ has $2^{\aleph_0}$ many components, $\mathrm{Bd}(\mathbb{R})$ is the only separable one and any other component has weight $2^{\aleph_0}$. We show that a nonseparable component $\mathcal{H}$ of $\mathrm{Cld}_H(\mathbb{R})$ is homeomorphic to the Hilbert space $\ell_2(2^{\aleph_0})$ of weight $2^{\aleph_0}$ in case where $\mathcal{H} \not\ni \mathbb{R}, [0, \infty), (-\infty, 0]$. This is a partial answer (in case $n = 1$) of Problem 4 in [17].

1. Preliminaries

We use standard notation concerning sets and topology. For example, we denote by ω the set of all natural numbers. Given a set X, we denote by $[X]^{<\omega}$ the family of all finite subsets of X.

Given a metric space $X = \langle X, d\rangle$ and a set $A \subseteq X$, we denote by $\mathrm{B}(A, r)$ and $\overline{\mathrm{B}}(A, r)$ the open and the closed r-balls centered at A, that is,

$$\mathrm{B}(A, r) = \{x \in X\colon\ \mathrm{dist}(x, A) < r\} \quad \text{and}$$
$$\overline{\mathrm{B}}(A, r) = \{x \in X\colon\ \mathrm{dist}(x, A) \leqslant r\}.$$

The Hausdorff metric d_H on $\mathrm{Cld}(X)$ is defined as follows:

$$d_H(A, C) = \inf\{r > 0\colon\ A \subseteq \mathrm{B}(C, r) \text{ and } C \subseteq \mathrm{B}(A, r)\},$$

where d_H is actually a metric on $\mathrm{Bd}(X)$ but d_H is infinite-valued for $\mathrm{Cld}(X)$ if $\langle X, d\rangle$ is unbounded. The spaces $\mathrm{Cld}_H(X)$ and $\mathrm{Bd}_H(X)$ are sometimes denoted by $\mathrm{Cld}_H(X, d)$ and $\mathrm{Bd}_H(X, d)$, to emphasize the fact that they are determined by the metric on X. In fact, the metric $\varrho(x, y) = d(x, y)/(1 + d(x, y))$ induces the same topology on X as d but the Hausdorff metric ϱ_H induces a different one on $\mathrm{Cld}(X)$. On the other hand, the Hausdorff metric induced by the metric $\bar{d}(x, y) = \min\{d(x, y), 1\}$ is finite-valued and induces the same topology on $\mathrm{Cld}_H(X)$ as d_H; moreover $\mathrm{Cld}(X)$ is equal to $\mathrm{Bd}(X)$ as sets. Note that the subspace $\mathrm{Fin}(X) = [X]^{<\omega} \setminus \{\emptyset\}$ of $\mathrm{Bd}_H(X)$ of all nonempty finite subsets of X is dense in $\mathrm{Bd}_H(X)$ if and only if every bounded set in $X = \langle X, d\rangle$ is totally bounded.

Fact 1.1. *For a metric space $X = \langle X, d\rangle$, the following hold:*

(i) *If d is complete then $\langle \mathrm{Bd}(X, d), d_H\rangle$ is a complete metric space and the space $\mathrm{Cld}_H(X)$ is completely metrizable.*
(ii) *The space $\mathrm{Bd}_H(X, d)$ is separable if and only if every bounded set in X is totally bounded.*

We use the standard notation $\exp(X)$ for the Vietoris hyperspace of nonempty compact sets in X. Note that $\exp(X) \subseteq \mathrm{Bd}(X)$ for every metric space $X = \langle X, d\rangle$ and it is well known that the Hausdorff metric induces the Vietoris topology on $\exp(X)$. However, if closed bounded sets of X are not compact, then the space $\mathrm{Bd}_H(X)$ is very different from $\mathrm{Bd}_V(X)$ endowed with the Vietoris topology. We use the following notation:

$$A^- = \{C \in \mathrm{Cld}(X) \colon C \cap A \neq \emptyset\} \quad \text{and} \quad A^+ = \{C \in \mathrm{Cld}(X) \colon C \subseteq A\},$$

where $A \subseteq X$. When dealing with $\mathrm{Bd}(X)$ (or other subspace of $\mathrm{Cld}(X)$), we still write A^- and A^+ instead of $A^- \cap \mathrm{Bd}(X)$ and $A^+ \cap \mathrm{Bd}(X)$ respectively.

In the rest of this section, we shall give preliminary results of infinite-dimensional topology. For the details, we refer to the book [3]. We abbreviate "absolute neighborhood retract" to "ANR".

Let $X = \langle X, d\rangle$ be a metric space. It is said that a map $f \colon Y \to X$ can be *approximated* by maps in a class $\mathcal{F}$ of maps if for every map $\alpha \colon X \to (0,1)$ there exists a map $g \colon Y \to X$ which belongs to $\mathcal{F}$ and such that $d(f(y), g(y)) < \alpha(f(y))$ for every $y \in Y$. A closed subset $A \subseteq X$ is a Z-*set* in X if the identity map id_X of X can be approximated by maps $f \colon X \to X$ such that $f[X] \cap A = \emptyset$. Strengthening the last condition to $\mathrm{cl}_X(f[X]) \cap A = \emptyset$, we define the notion of a *strong* Z-*set*. In case X is locally compact, every Z-set in X is a strong Z-set. Moreover, it is well known that every Z-set in an ℓ_2-manifold is a strong Z-set. A countable union of (strong) Z-sets is called a (*strong*) Z_σ-*set*. We call X a (*strong*) Z_σ-*space* if it is a (strong) Z_σ-set in itself. An embedding $f \colon X \to Y$ is called a Z-*embedding* if $f[X]$ is a Z-set in Y.

It is said that $D \subseteq X$ is *homotopy dense* in X if there exists a homotopy $h \colon X \times [0,1] \to X$ such that $h_0 = \mathrm{id}$ and $h_t[X] \subseteq D$ for every $t > 0$, where $h_t(x) = h(x,t)$. The complement of a homotopy dense subset of X is said to be *homotopy negligible*. If $A \subseteq X$ is a homotopy negligible closed set then A is a Z-set in X.

Fact 1.2. *For a closed set A in an ANR X, the following are equivalent:*

(a) *A is a* Z*-set in X;*
(b) *each map $f \colon [0,1]^n \to X$, $n \in \omega$, can be approximated by maps into $X \setminus A$;*
(c) *A is homotopy negligible in X.*

Fact 1.3. *Let D be a homotopy dense subset of an ANR X. Then the following hold:*

(i) *D is also an ANR.*
(ii) *A closed set $A \subseteq X$ is a* Z*-set in X if and only if $A \cap D$ is a* Z*-set in D.*
(iii) *If $A \subseteq X$ is a strong* Z*-set in X then $A \cap D$ is a strong* Z*-set in D.*

Proposition 1.4. *Assume that X is a homotopy dense subset of a* Q*-manifold M. Then X is an ANR and every* Z*-set in X is a strong* Z*-set. Furthermore, X is a strong Z_σ-space if and only if X is contained in a Z_σ-set in M.*

Proof. We verify only the "furthermore" statement. Assume $X \subseteq \bigcup_{n\in\omega} Z_n$, where each Z_n is a Z-set in M. Then each Z_n is a strong Z-set in M, because

M is locally compact, and therefore by Fact 1.3 (iii), each $Z_n \cap X$ is a strong Z-set in X. Conversely, if $X = \bigcup_{n\in\omega} X_n$, where each X_n is a (strong) Z-set in X, then by Fact 1.3 (ii), $Z_n = \operatorname{cl}_M X_n$ is a Z-set in M. Clearly, $X \subseteq \bigcup_{n\in\omega} Z_n$. □

Let $\mathcal{C}$ be a topological class of spaces, that is, if X is homeomorphic to some $Y \in \mathcal{C}$ then X also belongs to $\mathcal{C}$. It is said that $\mathcal{C}$ is *open* (resp. *closed*) hereditary if $X \in \mathcal{C}$ whenever X is an open (resp. closed) subspace of some $Y \in \mathcal{C}$. A space X is called *strongly $\mathcal{C}$-universal* if for every $Y \in \mathcal{C}$ and every closed subset $A \subseteq Y$, every map $f\colon Y \to X$ such that $f \restriction A$ is a Z-embedding can be approximated by Z-embeddings $g\colon X \to Y$ such that $g \restriction A = f \restriction A$. Similarly, one defines *$\mathcal{C}$-universality*, relaxing the above condition to the case $A = \emptyset$, that is, X is *$\mathcal{C}$-universal* if every map $f\colon Y \to X$ of $Y \in \mathcal{C}$ can be approximated by Z-embeddings.

Fact 1.5. *Let X be an ANR such that every* Z*-set in X is strong and let $\mathcal{C}$ be an open and closed hereditary topological class of spaces. If every open subspace $U \subseteq X$ is $\mathcal{C}$-universal then X is strongly $\mathcal{C}$-universal.*

Given a topological class $\mathcal{C}$ of spaces, we denote by $\sigma\mathcal{C}$ the class of all spaces of the form $X = \bigcup_{n\in\omega} X_n$, where each X_n is closed in X and $X_n \in \mathcal{C}$. Recall that X is a *$\mathcal{C}$-absorbing space* if $X \in \sigma\mathcal{C}$ is a strongly $\mathcal{C}$-universal ANR which is a strong Z_σ-space. In case $\mathcal{C}$ is closed hereditary, we can write $X = \bigcup_{n\in\omega} X_n$, where each X_n is a strong Z-set in X and $X_n \in \mathcal{C}$.

We shall denote by $\mathfrak{M}_0$ and $\mathfrak{M}_1$ the classes of all compact metrizable spaces and all Polish spaces[3] respectively. Let $\Sigma = \mathrm{Q} \setminus \mathrm{s}$ denote the pseudo-boundary[4] of Q.

Fact 1.6. *If X is an $\mathfrak{M}_0$-absorbing homotopy dense subspace of* Q*, then $\langle \mathrm{Q}, X\rangle \approx \langle \mathrm{Q}, \Sigma\rangle$. In case $X \subseteq \mathrm{Q} \setminus 0$, $\langle \mathrm{Q} \setminus 0, X\rangle \approx \langle \mathrm{Q} \setminus 0, \Sigma\rangle$.*

Fact 1.7. *Assume that X is a both homotopy dense and homotopy negligible subset of a Hilbert cube manifold M. If X is σ-compact then it is a strong Z_σ-space.*

Proof. Assume $X = \bigcup_{n\in\omega} K_n$, where each K_n is compact. Then each K_n is closed in M and therefore it is a strong Z-set by Fact 1.3 (iii). □

2. Borel classes of several Hausdorff hyperspaces

Let $\langle \tilde{X}, d\rangle$ denote the completion of $\langle X, d\rangle$. We identify $\mathrm{Bd}(X, d)$ with the subspace of $\mathrm{Bd}(\tilde{X}, d)$, via the isometric embedding $A \mapsto \operatorname{cl}_{\tilde{X}} A$. Then, $\langle \mathrm{Bd}(\tilde{X}), d_H\rangle$ is a completion of $\langle \mathrm{Bd}(X), d_H\rangle$. Moreover, it should be noticed that $A \in \mathrm{Bd}(\tilde{X}) \setminus \mathrm{Bd}(X)$ if and only if $A \neq \operatorname{cl}_{\tilde{X}}(A \cap X)$. Saint Raymond proved in [20, Théorème 1] that if X is the union of a Polish subset and a

[3] I.e., separable completely metrizable spaces.

[4] In some articles (e.g. [3]), Σ denotes the *radial interior* of Q, i.e., $\Sigma = \{x \in \mathrm{Q}\colon \sup_{n\in\omega} |x(n)| < 1\}$. However, there is an auto-homeomorphism of Q which maps the pseudoboundary onto the radial interior.

σ-compact subset then $\mathrm{Bd}_H(X)$ is $F_{\sigma\delta}$ (hence Borel) in $\mathrm{Bd}_H(\tilde{X})$.[5] In particular, we have the following:

Proposition 2.1. *If $X = \langle X, d\rangle$ is σ-compact then the space $\langle \mathrm{Bd}(X), d_H\rangle$ is $F_{\sigma\delta}$ in its completion $\langle \mathrm{Bd}(\tilde{X}), d_H\rangle$.*

Moreover, the following can be easily obtained by adjusting the proof of [20, Théorème 1]:[6]

Proposition 2.2. *If $X = \langle X, d\rangle$ is Polish (d is not necessarily complete) then the space $\langle \mathrm{Bd}(X), d_H\rangle$ is G_δ in its completion $\langle \mathrm{Bd}(\tilde{X}), d_H\rangle$.*

For the readers' convenience, direct short proofs of the above two propositions are given in the Appendix. Combining Fact 1.1 and Proposition 2.2, we have the following:

Corollary 2.3. *If $X = \langle X, d\rangle$ is Polish in which every bounded set is totally bounded, then the space $\mathrm{Bd}_H(X)$ is also Polish.*

Concerning the spaces $\mathrm{Nwd}(X)$ and $\mathrm{Perf}(X)$, we prove here the following:

Proposition 2.4. *For every separable metric space X, the space $\mathrm{Nwd}(X)$ is G_δ in $\mathrm{Bd}_H(X)$.*

Proof. Let $\{U_n : n \in \omega\}$ be a countable open base for X. For each $n \in \omega$, let

$$\mathcal{F}_n = \{A \in \mathrm{Bd}(X) : U_n \subseteq A\}.$$

Then each $\mathcal{F}_n$ is closed in $\mathrm{Bd}_H(X)$ and $\bigcup_{n\in\omega} \mathcal{F}_n = \mathrm{Bd}(X) \setminus \mathrm{Nwd}(X)$. $\square$

Proposition 2.5. *If X is locally compact then $\mathrm{Perf}(X)$ is G_δ in $\mathrm{Bd}_H(X)$.*

Proof. Let $\{U_n : n \in \omega\}$ enumerate an open base of X such that $\mathrm{cl}\, U_n$ is compact for every $n \in \omega$. Note that, by compactness, $(\mathrm{cl}\, U_n)^-$ is closed in $\mathrm{Bd}_H(X, d)$. For each $n, m \in \omega$ define

$$\Phi(n,m) = \{\langle k, l\rangle \in \omega^2 : U_k \cap U_l = \emptyset,\ U_k \cup U_l \subseteq \mathrm{B}(U_n, 1/m)\}.$$

We claim that

$$\mathrm{Bd}(X,d) \setminus \mathrm{Perf}(X) = \bigcup_{n,m\in\omega} \bigcap_{\langle k,l\rangle\in\Phi(n,m)} \Big((\mathrm{cl}\, U_n)^- \setminus (U_k^- \cap U_l^-)\Big).$$

The set on the right-hand side is F_σ, so this will finish the proof.

Note that a closed set in a Polish space is perfect if and only if it has no isolated points. If $A \in \mathrm{Bd}(X,d) \setminus \mathrm{Perf}(X)$ then there is $y \in A$ which is isolated in A. We can find $n, m \in \omega$ such that $y \in U_n$ and $\mathrm{B}(U_n, 1/m) \cap A = \{y\}$. Then $A \in (\mathrm{cl}\, U_n)^-$ and $A \notin U_k^- \cap U_l^-$ whenever $\langle k, l\rangle \in \Phi(n,m)$.

[5] In [20], X is assumed to be a subspace of a compact metric space, but the proof is valid without this assumption. Moreover, it is also proved in [20, Théorème 6] that if $\mathrm{Bd}_H(X)$ is absolutely Borel (i.e., Borel in its completion) then X is the union of a Polish subset and a σ-compact subset.

[6] A similar result was proved by Costantini [7] for the Wijsman topology.

Conversely, assume that there are $n, m \in \omega$ such that $A \in (\operatorname{cl} U_n)^-$ and $A \notin U_k^- \cap U_l^-$ for every $\langle k, l\rangle \in \Phi(n,m)$. Then $A \cap \mathrm{B}(U_n, 1/m) \neq \emptyset$ and the second condition says that $A \cap \mathrm{B}(U_n, 1/m)$ does not contain two points, so it is a singleton. Thus $A \notin \operatorname{Perf}(X)$. □

Replacing $(\operatorname{cl} U_n)^-$ by $\mathrm{B}(U_n, 1/m)^-$ in the formula from the proof above, we obtain the following:

Corollary 2.6. *The space* $\operatorname{Perf}(X)$ *is* $F_{\sigma\delta}$ *in* $\operatorname{Bd}_H(X)$ *if* X *is Polish.*

Since $\operatorname{Cantor}(\mathbb{R}^m) = \operatorname{Perf}(\mathbb{R}^m) \cap \operatorname{Nwd}(\mathbb{R}^m)$, the following is a combination of Propositions 2.4 and 2.5:

Corollary 2.7. *The space* $\operatorname{Cantor}(\mathbb{R}^m)$ *is* G_δ *in* $\operatorname{Bd}_H(\mathbb{R}^m)$.

Now, we shall prove the following:

Proposition 2.8. *The space* $\mathfrak{N}(\mathbb{R}^m)$ *is Polish.*

Proof. Let $\{I_n : n \in \omega\}$ enumerate all open rational cubes (i.e. products of rational intervals) in $\mathbb{R}^m$. Given $k \in \omega$, we define

$$S_k = \Big\{ s \in [\omega]^{<\omega} : \sum_{n \in s} |I_n| < 1/k \Big\},$$

where $|I|$ denotes the volume of the cube $I \subseteq \mathbb{R}^m$. We claim that

$$\mathfrak{N}(\mathbb{R}^m) = \bigcap_{k \in \omega} \bigcup_{s \in S_k} \Big(\bigcup_{n \in s} I_n \Big)^+.$$

Clearly, if A belongs to the right-hand side then for each $k \in \omega$ there is $s \subseteq \omega$ such that $A \subseteq \bigcup_{n \in s} I_n$ and $\sum_{n \in s} |I_n| < 1/k$; therefore A has Lebesgue measure zero.

Assume now A has Lebesgue measure zero and fix $k < \omega$. Then $A \subseteq \bigcup_{n \in \omega} J_n$, where each J_n is an open rational cube and $\sum_{n \in \omega} |J_n| < 1/k$. By compactness, $A \subseteq J_0 \cup \cdots \cup J_{l-1}$ for some m and $\{J_0, \ldots, J_{l-1}\} = \{I_n : n \in s\}$ for some $s \in S_k$. Thus $A \in \bigcup_{s \in S_k} (\bigcup_{n \in s} I_n)^+$. □

3. Almost convex metric spaces

Recall that a metric d on X is *almost convex* if for every $\alpha > 0$, $\beta > 0$ and for every $x, y \in X$ such that $d(x,y) < \alpha + \beta$, there exists $z \in X$ with $d(x,z) < \alpha$ and $d(z,y) < \beta$.

Fix a dense set X in a separable Banach space E. Let d denote the metric on X induced by the norm of E. Then $\langle X, d\rangle$ is an almost convex metric space and therefore by a result of [8] the space $\operatorname{Bd}(X,d)$ is an absolute retract. In case where X is G_δ, the space $\operatorname{Bd}(X,d)$ is completely metrizable by Proposition 2.2. If additionally E is finite-dimensional then $\operatorname{Bd}(X,d)$ is Polish by Corollary 2.3. In case where X is σ-compact, by Proposition 2.1, $\operatorname{Bd}(X,d)$ is absolutely $F_{\sigma\delta}$. It is natural to ask whether these spaces or their subspaces, discussed in §2, are homeomorphic to some standard spaces. Such standard spaces appear as homotopy dense subspaces of the Hilbert cube Q.

Let $\mathrm{UNb}(X,d)$ denote the family of all sets of the form $\overline{\mathrm{B}}(C,t)$, the closed t-neighborhood of $C \in \mathrm{Bd}(X,d)$, where $t > 0$.

Proposition 3.1. *If $\langle X,d\rangle$ is an almost convex metric space then the subspace $\mathrm{UNb}(X,d)$ is homotopy dense in $\mathrm{Bd}(X,d)$.*

Proof. Define a homotopy $h\colon \mathrm{Bd}(X,d)\times[0,1]\to \mathrm{Bd}(X,d)$ by the formula:

$$h(A,t) = \overline{\mathrm{B}}(A,t).$$

It suffices to verify the continuity of h with respect to Hausdorff metric topology. It has been checked in [8] that $d_H(\overline{\mathrm{B}}(A,t),\overline{\mathrm{B}}(A,s)) \leqslant |t-s|$. Thus we have

$$\begin{aligned} d_H(h(A,t),h(B,s)) &\leqslant d_H(h(A,t),h(A,s)) + d_H(h(A,s),h(B,s)) \\ &\leqslant |t-s| + d_H(h(A,s),h(B,s)). \end{aligned}$$

It remains to check that $d_H(\overline{\mathrm{B}}(A,s),\overline{\mathrm{B}}(B,s)) \leqslant d_H(A,B)$.

To complete the proof, we show the following:

$$r > d_H(A,B),\ \varepsilon > 0 \Longrightarrow r+\varepsilon \geqslant d_H(\overline{\mathrm{B}}(A,s),\overline{\mathrm{B}}(B,s)),$$

For this aim, it suffices to check that $\overline{\mathrm{B}}(A,s) \subseteq \mathrm{B}(\overline{\mathrm{B}}(B,s), r+\varepsilon)$; then by symmetry we shall also get $\overline{\mathrm{B}}(B,s) \subseteq \mathrm{B}(\overline{\mathrm{B}}(A,s), r+\varepsilon)$.

For each $x \in \overline{\mathrm{B}}(A,s)$, choose $a \in A$ such that $d(x,a) < s+\varepsilon$. There is $b \in B$ such that $d(a,b) < r$. Then we have $d(x,b) < s+r+\varepsilon$. Using the almost convexity of d, we can find y such that $d(b,y) < s$ and $d(y,x) < r+\varepsilon$. Then $y \in \mathrm{B}(B,s)$ and hence $x \in \mathrm{B}(y,r+\varepsilon) \subseteq \mathrm{B}(\overline{\mathrm{B}}(B,s),r+\varepsilon)$. □

Denote by $\mathrm{Reg}(X,d)$ the hyperspace of all nonempty bounded regularly closed subsets of a metric space $\langle X,d\rangle$. Clearly, $\mathrm{UNb}(X,d) \subseteq \mathrm{Reg}(X,d)$.

Corollary 3.2. *Let $\langle X,d\rangle$ be an almost convex metric space and $D \subseteq X$ a dense set. Then the spaces $\mathrm{Reg}(X,d)$ and $\mathrm{Bd}(D,d)$ are homotopy dense in $\mathrm{Bd}(X,d)$.*

Proof. Regarding $\mathrm{Bd}(D,d) \subseteq \mathrm{Bd}(X,d)$ via the embedding $A \mapsto \mathrm{cl}_X A$, we have $\mathrm{Reg}(X,d) \subseteq \mathrm{Bd}(D,d)$. This follows from the fact that $\mathrm{cl}(D\cap U) = \mathrm{cl}\,U$ for every open set $U \subseteq X$. Since $\mathrm{UNb}(X,d)$ is homotopy dense in $\mathrm{Bd}(X,d)$ by Proposition 3.1 and $\mathrm{UNb}(X,d) \subseteq \mathrm{Reg}(X,d)$, we have the result. □

4. Strict deformations

Assume we are looking at certain homotopy dense subspaces of the Hilbert cube Q. Let $X \supseteq X_0$ be such spaces. If $X_0 \approx \Sigma$ then, in order to conclude that $\langle \mathrm{Q}, X\rangle \approx \langle \mathrm{Q},\Sigma\rangle$, it suffices to check that X is a Z_σ-set in Q, by applying [6, Theorem 6.6]. However, to see that $X_0 \approx \Sigma$, we have to check that X_0 is strongly $\mathfrak{M}_0$-universal. Below is a tool which simplifies this step. To formulate it, we need some extra notions concerning homotopies.

A homotopy $\varphi\colon X\times[0,1]\to X$ is called a *strict deformation* if $\varphi_0 = \mathrm{id}$ and

$$\varphi(x,t) = \varphi(x',t') \wedge t > 0 \wedge t' > 0 \implies x = x'.$$

It is said that φ *omits* $A \subseteq X$ if $\varphi[X \times (0,1]] \cap A = \emptyset$. Finally, we say that a space X is *strictly homotopy dense* in Y if $X \subseteq Y$ and there exists a strict deformation which omits $Y \setminus X$ (so in particular X is homotopy dense in Y).

Lemma 4.1. *For every* Z*-set* A *in a* Q*-manifold* M*, there exists a strict deformation of* M *which omits* A*.*

Proof. Find a Z-embedding $f_0 \colon M \to M$ which is properly 2^{-2}-homotopic to the identity and so that $f_0[M] \cap A = \emptyset$. Further, find a Z-embedding $f_1 \colon M \to M$ which is properly 2^{-3}-homotopic to the identity and $f_1[M] \cap (f_0[M] \cup A) = \emptyset$. Continuing this way, we find Z-embeddings $f_n \colon M \to M$, $n \in \omega$, such that f_n is properly 2^{-n-2}-homotopic to the identity and

$$f_n[M] \cap (f_{n-1}[M] \cup \dots \cup f_0[M] \cup A) = \emptyset.$$

Then, we have proper $2^{-(n+1)}$-homotopies $g^n \colon M \times [0,1] \to M$, $n \in \omega$, such that $g^n_0 = f_n$ and $g^n_1 = f_{n+1}$. We can define a homotopy $g \colon M \times [0,1] \to M$ by $g(x,0) = x$ and

$$g(x,t) = g^n(x, 2 - 2^{n+1}t) \ \text{ for } 2^{-(n+1)} \leqslant t \leqslant 2^{-n},\ n \in \omega.$$

Note that $g_{2^{-n}} = f_n$ for each $n \in \omega$, each $g \restriction M \times [2^{-n-1}, 2^{-n}]$ is proper and 2^{-n-1}-close to the projection $\mathrm{pr}_M \colon M \times (0, 2^{-n}] \to M$. The continuity of g at $(x,0)$ is guaranteed by the last fact. Using the strong $\mathfrak{M}_0$-universality of M (see [3, Theorem 1.1.26]), we can inductively obtain $h_n \colon M \times [0,1] \to M$, $n \in \omega$, such that

(1) $h_n \restriction M \times [2^{-n-1}, 1]$ is a Z-embedding,
(2) $h_n \restriction M \times [2^{-n}, 1] = h_{n-1} \restriction M \times [2^{-n}, 1]$,
(3) $h_n \restriction M \times [0, 2^{-n-1}] = g \restriction M \times [0, 2^{-n-1}]$,
(4) $h_n \restriction M \times [2^{-n-1}, 2^{-n}]$ is 2^{-n-1}-close to $g \restriction M \times [2^{-n-1}, 2^{-n}]$, hence it is 2^{-n}-close to $\mathrm{pr}_M \colon M \times [2^{-n-1}, 2^{-n}] \to M$,
(5) $h_n[M \times [2^{-n-1}, 1]]$ is disjoint from A.

Finally, the limit $h = \lim_{n\to\infty} h_n$ is the desired one. □

Theorem 4.2. *Assume that* X *is a* Z_σ*-subset of a* Q*-manifold* M *which is strictly homotopy dense in* M*. Then* X *is an* $\mathfrak{M}_0$*-absorbing space. In particular, if* $M \approx \mathrm{Q}$ *then* $\langle M, X\rangle \approx \langle \mathrm{Q}, \Sigma\rangle$ *and if* $M \approx \mathrm{Q} \setminus 0$ *then* $\langle M, X\rangle \approx \langle \mathrm{Q} \setminus 0, \Sigma\rangle$*.*

Proof. The assumption says in particular that X is homotopy dense in M, so it follows from Proposition 1.4 that X is an ANR being a strong Z_σ-space. It remains to check that X is strongly $\mathfrak{M}_0$-universal. For the additional statement, we can just apply Fact 1.6.

Fix a map $f \colon A \to X$ of a compact metric space such that $f \restriction B$ is a Z-embedding, where $B \subseteq A$ is closed. Note that every compact subset of X is a Z-set in M, hence it is a Z-set in X by Fact 1.3 (ii), so we just have to preserve $f \restriction B$, not worrying about Z-sets. We assume that A is endowed with the metric such that $\operatorname{diam}(A) \leqslant 1$. Fix $\varepsilon > 0$. Using the strong $\mathfrak{M}_0$-universality of M (see [3, Theorem 1.1.26]), we can find a Z-embedding $g\colon A \to M$ which is $\varepsilon/2$-close to f and such that $g[A \setminus B] \cap X = \emptyset$ (here we use the fact that X is a Z_σ-set in M and also that $f[B]$ is a Z-set in M).

By Lemma 4.1, we have a strict deformation $\varphi\colon M \times [0,1] \to M$ which omits $f[B]$. Fix a metric d for M and choose a map $\gamma\colon A \to [0,1]$ so that $\gamma^{-1}(0) = B$ and

$$d(g(a), \varphi(g(a), \gamma(a))) < \varepsilon/4 \text{ for every } a \in A.$$

On the other hand, by the assumption, there is a strict deformation $\psi\colon M \times [0,1] \to M$ which omits $M \setminus X$. Define $h\colon A \to X$ by setting

$$h(a) = \psi(\varphi(g(a), \gamma(a)), \delta(a)),$$

where $\delta\colon A \to [0,1]$ is a map chosen so that $B = \delta^{-1}(0)$ and

$$d(h(a), \varphi(g(a), \gamma(a))) < \min\{\varepsilon/4,\ \operatorname{dist}(\varphi(g(a), \gamma(a)), f[B])\}.$$

This ensures us that h is $\varepsilon/2$-close to g and that $h(a) \notin f[B]$ whenever $a \in A \setminus B$. Then h is a map which is ε-close to f and $h[A] \subseteq X$. Furthermore, $h \restriction B = g \restriction B = f \restriction B$. It remains to check that h is one-to-one (then it is a Z-embedding, since every compact set in X is a Z-set).

Suppose $h(a) = h(a')$. If $a, a' \in B$ then $g(a) = g(a')$ and consequently $a = a'$. When $a, a' \in A \setminus B$, since ψ and φ are strict deformations, $g(a) = g(a')$ and hence $a = a'$. In case $a \in B$ and $a' \notin B$, we have $h(a) = g(a) = f(a) \in f[B]$ but $h(a') \notin f[B]$ because φ omits $f[B]$. Thus, this case does not occur. □

5. Pseudo-interiors of $\mathrm{Bd}_H(\mathbb{R}^m)$

Throughout this section, $m > 0$ is a fixed natural number. A particular case of a well known theorem of Curtis [9] says that $\mathrm{Bd}_H(\mathbb{R}^m) = \exp(\mathbb{R}^m)$ is homeomorphic to $\mathrm{Q} \setminus 0$. We shall consider the standard (convex) Euclidean metric d on $\mathbb{R}^m$. In this section, we investigate various G_δ subspaces of $\mathrm{Bd}_H(\mathbb{R}^m)$. The main result of this section is the following:

Theorem 5.1. *Let $\mathcal{F} \subseteq \mathrm{Bd}_H(\mathbb{R}^m)$ be one of the subspaces below:*

$$\mathrm{Nwd}(\mathbb{R}^m),\ \mathrm{Perf}(\mathbb{R}^m),\ \mathrm{Cantor}(\mathbb{R}^m),\ \mathfrak{N}(\mathbb{R}^m),\ \mathrm{Bd}(D),$$

where D is a dense G_δ set in $\mathbb{R}^m$ such that $\mathbb{R}^m \setminus D$ is also dense in $\mathbb{R}^m$ and in case $m > 1$ it is assumed that $D = p[D] \times \mathbb{R}$, where $p : \mathbb{R}^m \to \mathbb{R}^{m-1}$ is the projection onto the first $m-1$ coordinates. Then the pair $\langle \mathrm{Bd}(\mathbb{R}^m), \mathcal{F} \rangle$ is homeomorphic to $\langle \mathrm{Q} \setminus 0, \mathrm{s} \setminus 0 \rangle$.

Applying Theorem 5.1 above, we have

Corollary 5.2. *Suppose $\langle m, k \rangle = \langle 1, 0 \rangle$ or $0 \leqslant k < m - 1$. Then,*

$$\langle \mathrm{Bd}(\mathbb{R}^m), \mathrm{Bd}(\nu_k^m) \rangle \approx \langle \mathrm{Q} \setminus 0, \mathrm{s} \setminus 0 \rangle.$$

Consequently, $\mathrm{Bd}_H(\nu_k^m) \approx \ell_2$.

Proof. As a direct consequence of Theorem 5.1, we have

$$\langle \mathrm{Bd}(\mathbb{R}), \mathrm{Bd}(\nu_0^1)\rangle = \langle \mathrm{Bd}(\mathbb{R}), \mathrm{Bd}(\mathbb{R}\setminus\mathbb{Q})\rangle \approx \langle \mathrm{Q}\setminus 0, \mathrm{s}\setminus 0\rangle.$$

For each $0 \leqslant k < m-1$, observe that $\mathbb{R}^m \setminus (\nu_k^{m-1}\times\mathbb{R}) = (\mathbb{R}^{m-1}\setminus\nu_k^{m-1})\times\mathbb{R} \subseteq \mathbb{R}^m \setminus \nu_k^m$. Thus, it follows that

$$\mathrm{Bd}(\mathbb{R}^m)\setminus \mathrm{Bd}(\nu_k^{m-1}\times\mathbb{R}) \subseteq \mathrm{Bd}(\mathbb{R}^m)\setminus\mathrm{Bd}(\nu_k^m).$$

By Proposition 2.2 and Corollary 3.2, $\mathrm{Bd}(\nu_k^m)$ is a homotopy dense G_δ set in $\mathrm{Bd}_H(\mathbb{R}^m)$, which implies that $\mathrm{Bd}(\mathbb{R}^m)\setminus\mathrm{Bd}(\nu_k^m)$ is a Z_σ-set in $\mathrm{Bd}(\mathbb{R}^m)$. On the other hand, we can apply Theorem 5.1 to obtain

$$\langle \mathrm{Bd}(\mathbb{R}^m), \mathrm{Bd}(\mathbb{R}^m)\setminus\mathrm{Bd}(\nu_k^{m-1}\times\mathbb{R})\rangle \approx \langle \mathrm{Q}\setminus 0, \Sigma\rangle.$$

Then, it follows from Theorem 6.6 in [6] that

$$\langle \mathrm{Bd}(\mathbb{R}^m), \mathrm{Bd}(\mathbb{R}^m)\setminus\mathrm{Bd}(\nu_k^m)\rangle \approx \langle \mathrm{Q}\setminus 0, \Sigma\rangle.$$

Thus, we have the result. □

The conclusion of Theorem 5.1 is equivalent to

$$\langle \mathrm{Bd}_H(\mathbb{R}^m), \mathrm{Bd}_H(\mathbb{R}^m)\setminus\mathcal{F}\rangle \approx \langle \mathrm{Q}\setminus 0, \Sigma\rangle.$$

We saw in §2 that the subspace $\mathcal{F}\subseteq\mathrm{Bd}(\mathbb{R}^m)$ in Theorem 5.1 is G_δ, that is, $\mathrm{Bd}_H(\mathbb{R}^m)\setminus\mathcal{F}$ is F_σ in $\mathrm{Bd}_H(\mathbb{R}^m)$. If $\mathcal{F}$ contains a homotopy dense subset of $\mathrm{Bd}_H(\mathbb{R}^m)$ then the complement $\mathrm{Bd}_H(\mathbb{R}^m)\setminus\mathcal{F}$ is a Z_σ-set. Thus, in order to apply Theorem 4.2 to obtain the result, it suffices to show that $\mathcal{F}$ contains a homotopy dense subset of $\mathrm{Bd}_H(\mathbb{R}^m)$ and the complement $\mathrm{Bd}_H(\mathbb{R}^m)\setminus\mathcal{F}$ contains a strictly homotopy dense subset of $\mathrm{Bd}_H(\mathbb{R}^m)$. Observe that

$$\mathrm{Fin}(\mathbb{R}^m)\subseteq\mathfrak{N}(\mathbb{R}^m)\subseteq\mathrm{Nwd}(\mathbb{R}^m) \quad\text{and}\quad \mathrm{Cantor}(\mathbb{R}^m)\subseteq\mathrm{Perf}(\mathbb{R}^m).$$

As a special case of a well known result due to Curtis and Nguyen To Nhu [11], we have

$$\langle \mathrm{Bd}_H(\mathbb{R}^m), \mathrm{Fin}(\mathbb{R}^m)\rangle = \langle \exp(\mathbb{R}^m), \mathrm{Fin}(\mathbb{R}^m)\rangle \approx \langle \mathrm{Q}\setminus 0, \mathrm{Q}_f\setminus 0\rangle,$$

where Q_f denotes the subspace of Q consisting of all eventually zero sequences, which is homotopy dense in Q. This fact implies the following:

Lemma 5.3. *The subspace* $\mathrm{Fin}(\mathbb{R}^m)$ *is homotopy dense in* $\mathrm{Bd}_H(\mathbb{R}^m)$*.*

Using Lemma 5.3 above, we can easily show the following:

Lemma 5.4. *The space* $\mathrm{Cantor}(\mathbb{R}^m)$ *is homotopy dense in* $\mathrm{Bd}_H(\mathbb{R}^m)$*.*

Proof. Let h be a homotopy of $\mathrm{Bd}_H(\mathbb{R}^m)$ which witnesses that $\mathrm{Fin}(\mathbb{R}^m)$ is homotopy dense, i.e., $h(A,t)$ is a finite set for every $t>0$. Choose a Cantor set $C\subseteq[0,1]^m$ with $0\in C$ and define a homotopy $\varphi\colon \mathrm{Bd}_H(\mathbb{R}^m)\times[0,1]\to\mathrm{Bd}_H(\mathbb{R}^m)$ by

$$\varphi(A,t) = h(A,t) + tC.$$

Then $\varphi_0 = \mathrm{id}$ and $\varphi(A,t)\in\mathrm{Cantor}(\mathbb{R}^m)$ for every $t>0$ because a finite union of Cantor sets is a Cantor set. □

Concerning the space $\mathrm{Bd}(D)$ in Theorem 5.1, we have shown in Corollary 3.2 that it is homotopy dense in $\mathrm{Bd}_H(\mathbb{R}^m)$. Thus, to complete the proof of Theorem 5.1, it remains to show the following:

Lemma 5.5. *Under the same assumption as Theorem 5.1, each of the following spaces are strictly homotopy dense in* $\mathrm{Bd}_H(\mathbb{R}^m)$*:*

$$\mathrm{Bd}(\mathbb{R}^m) \setminus \mathrm{Nwd}(\mathbb{R}^m),\ \mathrm{Bd}(\mathbb{R}^m) \setminus \mathrm{Perf}(\mathbb{R}^m),\ \mathrm{Bd}(\mathbb{R}^m) \setminus \mathrm{Bd}(D).$$

First, we show the following lemma, which also gives a direct proof of Lemma 5.3:

Lemma 5.6. *For* $D \subseteq \mathbb{R}^m$*, if* $\mathbb{R}^m \setminus D$ *is dense in* $\mathbb{R}^m$ *then* $\mathrm{Fin}(\mathbb{R}^m) \setminus \mathrm{Bd}(D)$ *is homotopy dense in* $\mathrm{Bd}_H(\mathbb{R}^m)$.

Proof. Let $\mathcal{H} = \mathrm{Fin}(\mathbb{R}^m) \setminus \mathrm{Bd}(D)$, that is, $\mathcal{H}$ consists of all nonempty finite sets $A \subseteq \mathbb{R}^m$ such that $A \setminus D \neq \emptyset$. Then $\mathcal{H}$ is dense in $\mathrm{Bd}_H(\mathbb{R}^m)$. Moreover, $\mathcal{H}$ is closed under finite unions, i.e., $A \cup B \in \mathcal{H}$ whenever $A, B \in \mathcal{H}$. Recall that $\langle \mathrm{Bd}_H(\mathbb{R}^m), \cup \rangle$ is a Lawson semilattice (see [18]), that is, the union operator $\langle A, B\rangle \mapsto A \cup B$ is continuous and $\mathrm{Bd}_H(\mathbb{R}^m)$ has an open base consisting of subsemilattices; namely, every open ball with respect to the Hausdorff metric is a subsemilattice of $\langle \mathrm{Bd}_H(\mathbb{R}^m), \cup \rangle$. By virtue of [16, Theorem 5.1], it suffices to show that $\mathcal{H}$ is relatively LC^0 in $\mathrm{Bd}_H(\mathbb{R}^m)$. Recall that a subspace Y of a space X is *relatively* LC^0 *in* X if every neighborhood U of each $x \in X$ contains a neighborhood V of x in X such that every $a, b \in V \cap Y$ can be joined by a path in $U \cap Y$.

Fix $A \in \mathrm{Bd}_H(\mathbb{R}^m)$ and $\varepsilon > 0$. For each $A_0, A_1 \in \mathrm{B}_{d_H}(A, \varepsilon/2) \cap \mathcal{H}$, we describe how to construct a path in $\mathrm{B}_{d_H}(A, \varepsilon) \cap \mathcal{H}$ which joins A_0 to $A_0 \cup A_1$. Let $A_1 = \{p_0, \ldots, p_{n-1}\}$. For each $i < n$, find $q_i \in A_0$ such that $\|p_i - q_i\| < \varepsilon/2$, and define

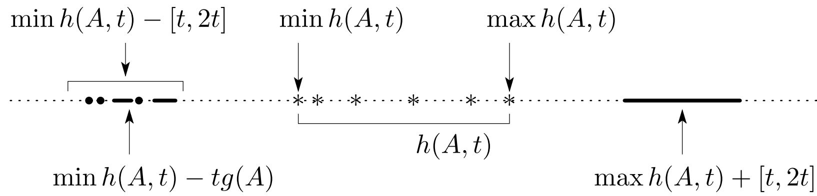
$$h(t) = A_0 \cup \{(1-t)q_i + tp_i \colon i < n\} \quad \text{for each } t \in [0,1].$$

Then $h(t) \in \mathcal{H}$ because $A_0 \subseteq h(t) \in \mathrm{Fin}(\mathbb{R}^m)$. Further, $d_H(A_0, h(t)) < \varepsilon/2$, that is, $h(t) \in \mathrm{B}_{d_H}(A, \varepsilon)$. Finally, $h(0) = A_0$ and $h(1) = A_0 \cup A_1$. By the same argument, we can construct a path in $\mathrm{B}_{d_H}(A, \varepsilon) \cap \mathcal{H}$ which joins $A_0 \cup A_1$ to A_1. □

Proof of Lemma 5.5. First, we show the case $m = 1$. It suffices to construct a strict deformation $\varphi\colon \mathrm{Bd}_H(\mathbb{R}) \times [0,1] \to \mathrm{Bd}_H(\mathbb{R})$ which omits $\mathrm{Nwd}(\mathbb{R}) \cup \mathrm{Perf}(\mathbb{R}) \cup \mathrm{Bd}(D)$. Let h be a homotopy of $\mathrm{Bd}(\mathbb{R})$ which witnesses that $\mathrm{Fin}(\mathbb{R}) \setminus \mathrm{Bd}(D)$ is homotopy dense (Lemma 5.6). Since $\mathrm{Bd}_H([1,2]) \approx \mathrm{Q}$, we have an embedding $g : \mathrm{Bd}_H(\mathbb{R}) \to \mathrm{Bd}_H([1,2])$. The desired φ can be defined as follows:

$$\varphi(A,t) = h(A,t) \cup \{\max h(A,t) + [t, 2t],\ \min h(A,t) - tg(A)\}.$$

For each $t > 0$, it is clear that $\varphi(A,t) \notin \mathrm{Nwd}(\mathbb{R}) \cup \mathrm{Perf}(\mathbb{R})$. Since $h(A,t)$ contains an isolated point from $\mathbb{R} \setminus D$ which remains to be isolated in $\varphi(A,t)$, we see that $\varphi(A,t) \notin \mathrm{Bd}(D)$. Given $\varphi(A,t)$ for $t > 0$, we can reconstruct t

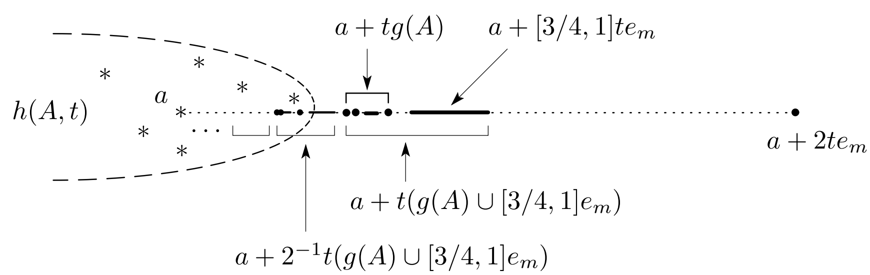


as the length of the interval $J \subseteq \varphi(A,t)$ with $\max J = \max \varphi(A,t)$. Consequently, $g(A)$ can be reconstructed from $\varphi(A,t)$. Thus, φ is a strict deformation.

Next, we show the case $m > 1$. To see that $\mathrm{Bd}(\mathbb{R}^m) \setminus \mathrm{Perf}(\mathbb{R}^m)$ and $\mathrm{Bd}(\mathbb{R}^m) \setminus \mathrm{Bd}(D)$ are strictly homotopy dense in $\mathrm{Bd}_H(\mathbb{R}^m)$, we shall construct a strict deformation $\varphi\colon \mathrm{Bd}_H(\mathbb{R}^m) \times [0,1] \to \mathrm{Bd}_H(\mathbb{R}^m)$ which omits $\mathrm{Perf}(\mathbb{R}^m) \cup \mathrm{Bd}(D)$. Recall $p : \mathbb{R}^m \to \mathbb{R}^{m-1}$ is the projection onto the first $m-1$ coordinates. Note that $p[D]$ is a dense G_δ set in $\mathbb{R}^{m-1}$ and $\mathbb{R}^{m-1} \setminus p[D]$ is also dense in $\mathbb{R}^{m-1}$. Let $e_m = \langle 0, 0, \dots, 0, 1\rangle \in \mathbb{R}^m$.

Since $\mathbb{R}^m \setminus (p[D] \times \mathbb{R})$ is dense in $\mathbb{R}^m$, it follows from Lemma 5.6 that $\mathrm{Fin}(\mathbb{R}^m) \setminus \mathrm{Bd}(p[D] \times \mathbb{R})$ is homotopy dense in $\mathrm{Bd}_H(\mathbb{R}^m)$. Let h be a homotopy of $\mathrm{Bd}(\mathbb{R}^m)$ which witnesses this, i.e., for $t > 0$, $h(A,t)$ is finite and $p[h(A,t)] \not\subseteq p[D]$. Since $\mathrm{Bd}_H([3/5, 2/3]) \approx \mathrm{Q}$, we have an embedding $g : \mathrm{Bd}_H(\mathbb{R}^m) \to \mathrm{Bd}_H([3/5, 2/3])$. The desired φ can be defined as follows:

$$\varphi(A,t) = h(A,t) + t\left(\bigcup_{i\in\omega} 2^{-i}(g(A) \cup [3/4, 1])e_m \cup \{2e_m\}\right).$$



For each $t > 0$, $\varphi(A,t)$ has an isolated point because $\max \mathrm{pr}_m[\varphi(A,t)]$ is attained by an isolated point of $\varphi(A,t)$, where pr_m denotes the projection onto the m-th coordinate. Hence, $\varphi(A,t) \notin \mathrm{Perf}(\mathbb{R}^m)$. Since $p[\varphi(A,t)] = p[h(A,t)]$ is finite and contains a point of $\mathbb{R}^{m-1} \setminus p[D]$, it follows that $\mathrm{cl}(\varphi(A,t) \cap (p[D] \times \mathbb{R})) \neq \varphi(A,t)$, which means $\varphi(A,t) \notin \mathrm{Bd}(p[D] \times \mathbb{R})$.

Given $\varphi(A,t)$ for $t > 0$, we can find t as the distance from $\max \mathrm{pr}_m[\varphi(A,t)]$ to the interior of $\mathrm{pr}_m[\varphi(A,t)]$. Let $a_0 \in \varphi(A,t)$ be such that

$$\mathrm{pr}_m(a_0) = \min \mathrm{pr}_m[\varphi(A,t)] = \min \mathrm{pr}_m[h(A,t)].$$

Then, for sufficiently large i,

$$(a_0 + 2^{-i}t(g(A) \cup [3/4, 1])e_m) \cap h(A,t) = \emptyset.$$

Thus, we can reconstruct $2^{-i}tg(A)$ and consequently also $g(A)$ from $\varphi(A,t)$. This shows that φ is a strict deformation.

For $\mathrm{Bd}(\mathbb{R}^m) \setminus \mathrm{Nwd}(\mathbb{R}^m)$, we define a homotopy $\psi\colon \mathrm{Bd}_H(\mathbb{R}^m) \times [0,1] \to \mathrm{Bd}_H(\mathbb{R}^m)$ as follows:

$$\psi(A,t) = h(A,t) + t\left(\bigcup_{i\in\omega} 2^{-i}(g(A) \cup [3/4,1])e_m \cup \overline{\mathrm{B}}(2e_m, 1/2)\right).$$

In other wards, replacing the points $a + 2te_m \in \varphi(A,t)$, $a \in h(A,t)$, by the closed balls

$$a + t\overline{\mathrm{B}}(2e_m, 1/2) = \overline{\mathrm{B}}(a + 2te_m, t/2),\ a \in h(A,t),$$

we can obtain $\psi(A,t)$ from $\varphi(A,t)$. Evidently ψ omits $\mathrm{Nwd}(\mathbb{R}^m)$. Given $\psi(A,t)$ for $t > 0$, let $a_0 \in \psi(A,t)$ be such that

$$\mathrm{pr}_m(a_0) = \min \mathrm{pr}_m[\psi(A,t)] = \min \mathrm{pr}_m[h(A,t)].$$

Then we can get t as the diameter of the ball $\overline{\mathrm{B}}(a_0 + 2te_m, t/2)$ (which is equal to $2/3$ of the distance from a_0 to this ball). Now, by the same arguments as for φ, we can reconstruct $g(A)$ from $\psi(A,t)$. Thus, ψ is a strict deformation. □

Let us note that the subspace $\mathrm{UNb}(\mathbb{R}) \cup \mathrm{Fin}(\mathbb{R})$ is actually equal to the space $\mathrm{Pol}(\mathbb{R})$ consisting of all compact polyhedra in $\mathbb{R}$. It follows from the result of [21] that the pair $\langle \exp(\mathbb{R}), \mathrm{Pol}(\mathbb{R})\rangle$ is homeomorphic to $\langle \mathrm{Q}, \mathrm{Q}_f\rangle$.

6. Nonseparable components of $\mathrm{Cld}_H(\mathbb{R})$

In this section, we consider the space $\mathrm{Cld}_H(\mathbb{R})$ of all nonempty closed subsets of $\mathbb{R}$. We shall also consider its natural subspaces, using the same notation as before, but having in mind the new setting. For example, $\mathrm{Perf}(\mathbb{R})$ and $\mathrm{Nwd}(\mathbb{R})$ will denote the subspace of $\mathrm{Cld}(\mathbb{R})$ consisting of all perfect closed subsets of $\mathbb{R}$ and all closed sets with no interior points, respectively. Now $\mathrm{Perf}(\mathbb{R}) \cap \mathrm{Nwd}(\mathbb{R})$ consists of all nonempty closed (possibly unbounded) subsets of $\mathbb{R}$ which have neither isolated points nor interior points. In the new setting, we have

$$\mathrm{Cantor}(\mathbb{R}) = \mathrm{Perf}(\mathbb{R}) \cap \mathrm{Nwd}(\mathbb{R}) \cap \mathrm{Bd}(\mathbb{R}).$$

As shown in [17, Proposition 7.2], $\mathrm{Cld}_H(\mathbb{R})$ has $2^{\aleph_0}$ many components, $\mathrm{Bd}(\mathbb{R})$ is the only separable one and any other component has weight $2^{\aleph_0}$. The following is the main theorem in this section:

Theorem 6.1. *Let $\mathcal{H}$ be a nonseparable component of* $\mathrm{Cld}_H(\mathbb{R})$ *which does not contain* $\mathbb{R}$, $[0,+\infty)$, $(-\infty,0]$. *Then $\mathcal{H} \approx \ell_2(2^{\aleph_0})$.*

We shall say that a set $A \subseteq \mathbb{R}$ *has infinite uniform gaps* if there are $\delta > 0$ and pairwise disjoint open intervals $I_0, I_1, \dots$ such that $\mathrm{diam}\, I_n \geqslant \delta$, $A \cap I_n = \emptyset$ and $\mathrm{bd}\, I_n \subseteq A$ for every $n \in \omega$. Define

$$\mathcal{V} = \{A \in \mathrm{Cld}(\mathbb{R})\colon A \text{ has infinite uniform gaps }\}.$$

Clearly, $\mathcal{V}$ is open in $\mathrm{Cld}_H(\mathbb{R})$ and $\mathcal{V} \cap \mathrm{Bd}(\mathbb{R}) = \emptyset$. For each $A \in \mathrm{Cld}(\mathbb{R}) \setminus \mathrm{Bd}(\mathbb{R})$ and $\varepsilon > 0$, let $D \subseteq A$ be a maximal ε-discrete subset. Then $D \in \mathcal{V}$ and $d_H(A, D) \leqslant \varepsilon$ because $D \subseteq A \subseteq \mathrm{B}(D, \varepsilon)$. Thus, $\mathcal{V}$ is dense in $\mathrm{Cld}_H(\mathbb{R}) \setminus \mathrm{Bd}(\mathbb{R})$.

If $\mathcal{H}$ is a nonseparable component of $\mathrm{Cld}_H(\mathbb{R})$ and $\mathbb{R}, [0, +\infty), (-\infty, 0] \notin \mathcal{H}$ then $\mathcal{H} \subseteq \mathcal{V}$. Indeed, each $A \in \mathcal{H}$ is unbounded and every component of $\mathbb{R} \setminus A$ is an open interval. Let $\mathcal{J}$ be the set of all bounded component of $\mathbb{R} \setminus A$. Assume that $\{\operatorname{diam} I \colon I \in \mathcal{J}\}$ is bounded. When A is bounded below (or bounded above), $d_H(A, [0, \infty)) < \infty$ (or $d_H(A, (-\infty, 0]) < \infty$), which implies $[0, +\infty) \in \mathcal{H}$ (or $(-\infty, 0] \in \mathcal{H}$). When A is not bounded below nor above, $d_H(A, \mathbb{R}) < \infty$, which implies $\mathbb{R} \in \mathcal{H}$. Therefore, $\{\operatorname{diam} I \colon I \in \mathcal{J}\}$ is unbounded. In particular, A has infinite uniform gaps.

Due to Theorem A in [17], every component of $\mathrm{Cld}_H(\mathbb{R})$ is an AR, hence it is contractible. Since a contractible $\ell_2(2^{\aleph_0})$-manifold is homeomorphic to $\ell_2(2^{\aleph_0})$, Theorem 6.1 above follows from the following theorem:

Theorem 6.2. *The open dense subset $\mathcal{V}$ of $\mathrm{Cld}_H(\mathbb{R})$ is an $\ell_2(2^{\aleph_0})$-manifold.*

Proof. It suffices to show that each $A_0 \in \mathcal{V}$ has an open neighborhood $\mathcal{U} \subseteq \mathcal{V}$ which is an $\ell_2(2^{\aleph_0})$-manifold. In this case, $\mathcal{U}$ is a completely metrizable ANR because it is an open set in a completely metrizable ANR $\mathrm{Cld}_H(\mathbb{R})$. Due to Toruńczyk characterization of $\ell_2(2^{\aleph_0})$-manifold [22] (cf. [23]), we have to show that $\mathcal{U}$ has the following two properties:

(i) For each maps $f : [0,1]^n \times 2^\omega \to \mathcal{U}$ and $\alpha : \mathcal{U} \to (0,1)$, there exists a map $g : [0,1]^n \times 2^\omega \to \mathcal{U}$ such that $d_H(g(z), f(z)) < \alpha(f(z))$ for each $z \in [0,1]^n \times 2^\omega$ and $\{g[[0,1]^n \times \{x\}] : x \in 2^\omega\}$ is discrete in $\mathcal{U}$;

(ii) For any finite-dimensional simplicial complexes K_n, $n \in \omega$, with $\operatorname{card} K_n \leqslant 2^{\aleph_0}$, for every maps $f : \bigoplus_{n\in\omega} |K_n| \to \mathcal{U}$ and $\alpha : \mathcal{U} \to (0,1)$, there exists a map $g : \bigoplus_{n\in\omega} |K_n| \to \mathcal{U}$ such that $d_H(g(z), f(z)) < \alpha(f(z))$ for each $z \in \bigoplus_{n\in\omega} |K_n|$ and $\{g[|K_n|] : n \in \omega\}$ is discrete in $\mathcal{U}$.

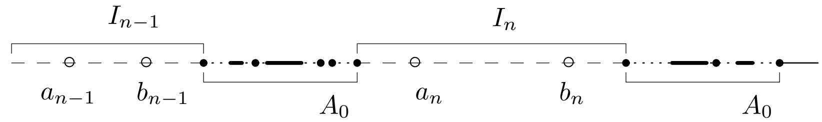
In the above, 2^ω is the discrete space of all functions of ω to $2 = \{0,1\}$. To this end, it suffices to prove the following:

- For each map $\alpha : \mathcal{U} \to (0,1)$, there exist maps $f_x : \mathcal{U} \to \mathcal{U}$, $x \in 2^\omega$, such that $d_H(f_x(A), A) < \alpha(A)$ for every $A \in \mathcal{U}$ and $\{f_x[\mathcal{U}] : x \in 2^\omega\}$ is discrete.

Fix $A_0 \in \mathcal{V}$ and choose open intervals $I_0, I_1, \ldots$ such that $\operatorname{diam} I_n \geqslant \delta$, $A_0 \cap I_n = \emptyset$ and $\operatorname{bd} I_n \subseteq A_0$ (i.e., $\inf I_n,\ \sup I_n \in A_0$) for every $n \in \omega$. Taking a subsequence if necessary, we may assume that either $\sup I_n < \inf I_{n+1}$ for every $n \in \omega$ or $\inf I_n > \sup I_{n+1}$ for every $n \in \omega$. Because of similarity, we may assume that the first possibility occurs.

Choose intervals $[a_n, b_n] \subseteq I_n$, $n \in \omega$, so that $b_n - a_n > \delta/4$,

$$\inf_{n\in\omega} \operatorname{dist}(a_n, \mathbb{R} \setminus I_n) = \inf_{n\in\omega} (a_n - \inf I_n) > \delta/4 \text{ and}$$
$$\inf_{n\in\omega} \operatorname{dist}(b_n, \mathbb{R} \setminus I_n) = \inf_{n\in\omega} (\sup I_n - b_n) > \delta/4.$$



Observe that if $A \in \mathrm{Cld}_H(\mathbb{R})$ and $d_H(A, A_0) < \delta/4$ then $A \cap (b_{n-1}, a_n) \neq \emptyset$ for every $n \in \omega$, where $b_{-1} = -\infty$. For each $A \in \mathrm{Cld}_H(\mathbb{R})$ with $d_H(A, A_0) < \delta/4$, we can define

$$r_n(A) = \max(A \cap (b_{n-1}, a_n)), \ n \in \omega.$$

For each $A, A' \in \mathrm{Cld}_H(\mathbb{R})$ with $d_H(A, A_0), d_H(A', A_0) < \delta/4$, we have

$$|r_n(A) - r_n(A')| \leqslant d_H(A, A').$$

Indeed, without loss of generality, we may assume that $r_n(A) < r_n(A')$. Then, the open interval $(r_n(A), b_n)$ contains no points of A and $r_n(A') \in (r_n(A), b_n)$. Since $b_n - r_n(A') > \delta/2$ and

$$r_n(A') - r_n(A) \leqslant |r_n(A') - r_n(A_0)| + |r_n(A) - r_n(A_0)| < \delta/2,$$

we have $|r_n(A') - r_n(A)| \leqslant d_H(A, A')$. Then, it follows that

$$\begin{aligned}\inf_{n\in\omega}(a_n - r_n(A)) - d_H(A, A') &\leqslant \inf_{n\in\omega}(a_n - r_n(A')) \\ &\leqslant \inf_{n\in\omega}(a_n - r_n(A)) + d_H(A, A').\end{aligned}$$

This means that $A \mapsto \inf_{n\in\omega}(a_n - r_n(A))$ is continuous. Since $r_n(A_0) = \inf I_n$, we have $\inf_{n\in\omega}(a_n - r_n(A_0)) > \delta/4$. Thus, A_0 has the following open neighborhood:

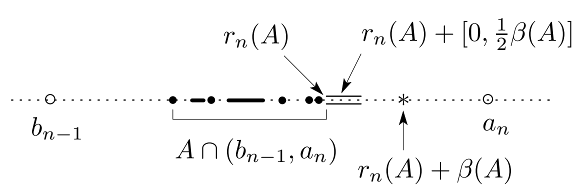
$$\mathcal{U} = \{A \in \mathrm{Cld}_H(\mathbb{R})\colon d_H(A, A_0) < \delta/4, \ \inf_{n\in\omega}(a_n - r_n(A)) > \delta/4\} \subseteq \mathcal{V}.$$

Now, for each map $\alpha\colon \mathcal{U} \to (0,1)$, we define a map $\beta : \mathcal{U} \to (0,1)$ as follows:

$$\beta(A) = \min\left\{\tfrac{1}{2}\alpha(A), \ \tfrac{1}{4}\delta - d_H(A, A_0), \ \inf_{n\in\omega}(a_n - r_n(A)) - \tfrac{1}{4}\delta\right\}.$$

Given a sequence $x = (x(n))_{n\in\omega} \in 2^\omega$, let

$$f_x(A) = A \cup \bigcup_{n\in\omega}\big(r_n(A) + \big([0, \tfrac{1}{2}\beta(A)] \cup \{\beta(A)\cdot x(n)\}\big)\big).$$



This defines a map $f_x\colon \mathcal{U} \to \mathcal{U}$ which is α-close to id. We claim that if $x \neq y \in 2^\omega$ then

$$d_H(f_x(A), f_y(A')) \geqslant \min\left\{\tfrac{1}{4}\beta(A), \tfrac{1}{4}\beta(A')\right\} \text{ for every } A, A' \in \mathcal{U}.$$

Indeed, assume that $x(n) = 1$, $y(n) = 0$ and let $s = \min\{\frac{1}{4}\beta(A), \frac{1}{4}\beta(A')\}$. Then

(1) $\max(f_x(A) \cap (b_{n-1}, a_n)) = r_n(A) + \beta(A)$;
(2) $f_x(A)$ has no points in the open interval $(r_n(A)+\frac{1}{2}\beta(A), r_n(A)+\beta(A))$;
(3) $\max(f_y(A') \cap (b_{n-1}, a_n)) = r_n(A') + \frac{1}{2}\beta(A')$;
(4) $[r_n(A'), r_n(A') + \beta(A')/2] \subseteq f_y(A')$.

In case $r_n(A') + \frac{1}{2}\beta(A') \geqslant r_n(A) + \beta(A) + s$ or $r_n(A') + \frac{1}{2}\beta(A') \leqslant r_n(A) + \beta(A) - s$, we have

$$d_H(f_x(A) \cap (b_{n-1}, a_n), f_y(A') \cap (b_{n-1}, a_n)) \geqslant s.$$

In case $r_n(A) + \beta(A) - s < r_n(A') + \frac{1}{2}\beta(A') \leqslant r_n(A) + \beta(A) + s$, since $2s \leqslant \frac{1}{2}\beta(A')$, we have $r_n(A') < r_n(A)+\beta(A)-s$, hence $r_n(A)+\beta(A)-s \in f_y(A')$. Thus, it follows that

$$d_H(f_x(A) \cap (b_{n-1}, a_n), f_y(A') \cap (b_{n-1}, a_n)) \geqslant s.$$

Finally, we show that $\{f_x[\mathcal{U}] : x \in 2^\omega\}$ is a discrete collection of $\mathcal{U}$. If not, we have A, $A_i \in \mathcal{U}$ and $x_i \in 2^\omega$, $i \in \omega$, such that $x_i \neq x_j$ if $i \neq j$, and $f_{x_i}(A_i) \to A$ $(i \to \infty)$. Then $c = \inf_{i\in\omega} \beta(A_i) = 0$. Indeed, otherwise we could find $i < j$ such that

$$d_H(f_{x_i}(A_i), A),\ d_H(f_{x_j}(A_j), A) < c/10$$

and $\beta(A_i),\ \beta(A_j) > 4c/5$. It follows that $d_H(f_{x_i}(A_i), f_{x_j}(A_j)) < c/5$, but

$$d_H(f_{x_i}(A_i), f_{x_j}(A_j)) \geqslant \min\{\beta(A)/4, \beta(A')/4\} > c/5,$$

which is a contradiction. Thus, $\inf_{i\in\omega} \beta(A_i) = 0$. Taking a subsequence, we may assume that $\lim_{i\to\infty} \beta(A_i) = 0$. Then $A_i \to A$ $(i \to \infty)$ because $d_H(f_{x_i}(A_i), A_i) \leqslant \beta(A_i)$. It follows that $\beta(A) = 0$, which is a contradiction. This completes the proof. □

Let $\mathcal{D}(X)$ be the subspace of $\mathrm{Cld}_H(X)$ consisting of all discrete sets in X. It follows from the result of [4] that $\mathcal{D}(X)$ is homotopy dense in $\mathrm{Cld}_H(X)$ for every almost convex metric space X. By the same proof, Lemma 5.6 can be extended to $\mathrm{Cld}_H(\mathbb{R}^m)$.

Proposition 6.3. *Assume $D \subseteq \mathbb{R}^m$ is such that $\mathbb{R}^m \setminus D$ is dense. Then $\mathcal{D}(\mathbb{R}^m) \setminus \mathrm{Cld}(D)$ is homotopy dense in $\mathrm{Cld}_H(\mathbb{R}^m)$.*

Now, we consider the subspaces $\mathfrak{N}(\mathbb{R})$, $\mathrm{Nwd}(\mathbb{R})$, $\mathrm{Perf}(\mathbb{R})$ and $\mathrm{Cld}(\mathbb{R} \setminus \mathbb{Q})$ of $\mathrm{Cld}_H(\mathbb{R})$. Similarly to $\mathrm{Bd}_H(\mathbb{R})$, the following can be shown:

Proposition 6.4. *The sets $\mathrm{Cld}(\mathbb{R}) \setminus \mathfrak{N}(\mathbb{R})$, $\mathrm{Cld}(\mathbb{R}) \setminus \mathrm{Nwd}(\mathbb{R})$, $\mathrm{Cld}(\mathbb{R}) \setminus \mathrm{Perf}(\mathbb{R})$ and $\mathrm{Cld}(\mathbb{R}) \setminus \mathrm{Cld}(\mathbb{R} \setminus \mathbb{Q})$ are Z_σ-sets in the space $\mathrm{Cld}_H(\mathbb{R})$.*

Due to Negligibility Theorem ([1], [12]) if M is an $\ell_2(2^{\aleph_0})$-manifold and A is a Z_σ-set in M then $M \setminus A \approx M$. Thus, combining Proposition 6.4 and Theorem 6.1, we have the following:

Corollary 6.5. *Let $\mathcal{H}$ be a nonseparable component of $\mathrm{Cld}_H(\mathbb{R})$ which does not contain $\mathbb{R}$, $[0,+\infty)$, $(-\infty,0]$. Then $\mathcal{H}\cap\mathfrak{N}(\mathbb{R})$, $\mathcal{H}\cap\mathrm{Nwd}(\mathbb{R})$, $\mathcal{H}\cap\mathrm{Perf}(\mathbb{R})$ and $\mathcal{H}\cap\mathrm{Cld}(\mathbb{R}\setminus\mathbb{Q})$ are homeomorphic to $\ell_2(2^{\aleph_0})$.*

7. Open problems

The following questions are left open.

Question 1. In case $m > 1$, under the only assumption that $D \subseteq \mathbb{R}^m$ is a dense G_δ set and $\mathbb{R}^m \setminus D$ is also dense in $\mathbb{R}^m$, is the pair $\langle \mathrm{Bd}(\mathbb{R}^m), \mathrm{Bd}(D)\rangle$ homeomorphic to $\langle \mathrm{Q}\setminus 0, \mathrm{s}\setminus 0\rangle$? In particular, is the pair $\langle \mathrm{Bd}(\mathbb{R}^m), \mathrm{Bd}(\nu^m_{m-1})\rangle$ homeomorphic to $\langle \mathrm{Q}\setminus 0, \mathrm{s}\setminus 0\rangle$?

Question 2. Does Theorem 6.1 hold even if $\mathcal{H}$ contains $\mathbb{R}$, $[0,\infty)$ or $(-\infty,0]$?

Question 3. For $m > 1$, is $\mathrm{Cld}_H(\mathbb{R}^m) \setminus \mathrm{Bd}(\mathbb{R}^m)$ an $\ell_2(2^{\aleph_0})$-manifold?

8. Appendix

For the convenience of readers, we give short and straightforward proofs of Propositions 2.1 and 2.2.

Proposition 8.1 (2.1)**.** *If $\langle X, d\rangle$ is σ-compact then the space $\langle \mathrm{Bd}(X), d_H\rangle$ is $F_{\sigma\delta}$ in its completion $\langle \mathrm{Bd}(\tilde{X}), d_H\rangle$.*

Proof. Fix a countable open base $\{U_n \colon n \in \omega\}$ for $\tilde{X}$. Since $U_n \cap X$ is F_σ, we have $U_n \cap X = \bigcup_{k\in\omega} K^n_k$, where each K^n_k is compact. Observe that, by compactness, the sets $(\tilde{X}\setminus K^n_k)^+$ are open in the Hausdorff metric topology. We claim that

$$\mathrm{Bd}(\tilde{X}) \setminus \mathrm{Bd}(X) = \bigcup_{n\in\omega}\Big(U^-_n \cap \bigcap_{k\in\omega}(\tilde{X}\setminus K^n_k)^+\Big),$$

which shows that $\mathrm{Bd}(\tilde{X}) \setminus \mathrm{Bd}(X)$ is a countable union of G_δ sets. This is what we want to prove.

Assume $A \in \mathrm{Bd}(\tilde{X})\setminus\mathrm{Bd}(X)$, that is, $A \neq \mathrm{cl}_{\tilde{X}}(A\cap X)$. Then there is $n \in \omega$ such that $U_n \cap A \neq \emptyset$ and $U_n \cap A \cap X = \emptyset$, which means that $A \in U^-_n$ and $A \in (\tilde{X}\setminus K^n_k)^+$ for every $k \in \omega$. Conversely, if $A \in U^-_n \cap \bigcap_{k\in\omega}(\tilde{X}\setminus K^n_k)^+$ then $U_n \cap A \neq \emptyset$ and $U_n \cap A \cap X = \emptyset$, so $A \neq \mathrm{cl}_{\tilde{X}}(A\cap X)$. □

Proposition 8.2 (2.2)**.** *If $\langle X, d\rangle$ is Polish then the space $\langle \mathrm{Bd}(X), d_H\rangle$ is G_δ in its completion $\langle \mathrm{Bd}(\tilde{X}), d_H\rangle$.*

Proof. Let $\{W_n \colon n \in \omega\}$ be a family of open subsets of $\tilde{X}$ such that $X = \bigcap_{n\in\omega} W_n$. Fix a countable open base $\{V_n \colon n \in \omega\}$ for $\tilde{X}$. We claim that

$$(*)\qquad \mathrm{Bd}(\tilde{X}) \setminus \mathrm{Bd}(X) = \bigcup_{n\in\omega}\bigcup_{k\in\omega}\Big(V^-_n \setminus (V_n \cap W_k)^-\Big).$$

As V^- is open in the metric space $\langle \mathrm{Bd}(\tilde{X}, d), d_H\rangle$ whenever $V \subseteq \tilde{X}$ is open, it follows that V_n^- is F_σ and therefore the set on the right-hand side of $(*)$ is F_σ in $\mathrm{Bd}_H(\tilde{X})$. It remains to prove $(*)$.

If $A \in V_n^- \setminus (V_n \cap W_k)^-$ then we have $x \in V_n \cap A$. Since $V_n \cap (A \cap X) = \emptyset$, it follows that $x \notin \mathrm{cl}_{\tilde{X}}(A \cap X)$. Thus $A \notin \mathrm{Bd}(X)$. Now assume $A \in \mathrm{Bd}(\tilde{X}) \setminus \mathrm{Bd}(X)$, that is, $A \neq \mathrm{cl}_{\tilde{X}}(A \cap X)$. Then there exists an open set $U \subseteq \tilde{X}$ such that $U \cap A \neq \emptyset$ and $U \cap A \cap X = \emptyset$. Hence $\bigcap_{k\in\omega} A \cap U \cap W_k = \emptyset$. Note that $A \cap U$ is a Baire space because of the completeness of $\langle \tilde{X}, d\rangle$. Thus, by the Baire Category Theorem, there exists $k \in \omega$ such that $A \cap U \cap W_k$ is not dense in $A \cap U$. Find a basic open set $V_n \subseteq U$ such that $V_n \cap A \neq \emptyset$ and $V_n \cap A \cap W_k = \emptyset$. Then $A \in V_n^- \setminus (V_n \cap W_k)^-$. □

Let $\mathfrak{B}(X)$ denote the Borel field on a topological space X. Given $\mathfrak{H} \subseteq \mathrm{Cld}(X)$, the *Effros σ-algebra* $\mathfrak{E}(\mathfrak{H})$ is the σ-algebra generated by

$$\{U^- \cap \mathfrak{H} \colon U \text{ is open in } X\}.$$

It is well known that $\mathfrak{E}(\exp(X)) = \mathfrak{B}(\exp(X))$ for every separable metric space X (see [5, Theorem 6.5.15]).[7] Whenever X is a separable metric space in which every bounded set is totally bounded, we can regard $\mathrm{Bd}_H(X) \subseteq \exp(\tilde{X})$ by the identification as in §2, where $\tilde{X}$ is the completion of X. Then, we have not only $\mathfrak{E}(\mathrm{Bd}(X)) = \mathfrak{B}(\mathrm{Bd}_H(X))$ but also $\mathfrak{E}(\mathfrak{H}) = \mathfrak{B}(\mathfrak{H})$ for $\mathfrak{H} \subseteq \mathrm{Bd}_H(X)$. This implies that $\mathfrak{E}(\mathfrak{H})$ is standard if $\mathfrak{H}$ is absolutely Borel (cf. [15, 12.B]). The results in §2 provide such hyperspaces $\mathfrak{H}$.

In relation to the results above, we can prove the following:

Proposition 8.3. *Let $X = \langle X, d\rangle$ be an analytic metric space in which bounded sets are totally bounded. Then, the space $\mathrm{Bd}_H(X)$ is analytic.*

Proof. The completion $\langle \tilde{X}, d\rangle$ of $\langle X, d\rangle$ is a Polish space in which closed bounded sets are compact. Then $\mathrm{Bd}_H(\tilde{X}, d) = \exp(\tilde{X})$ is Polish. Fix a countable open base $\{U_n \colon n \in \omega\}$ for $\tilde{X}$. Since X is analytic, there exists a tree $\{X_s \colon s \in \omega^{<\omega}\}$ of closed subsets of $\tilde{X}$ such that $X = \bigcup_{f\in\omega^\omega} \bigcap_{n\in\omega} X_{f\restriction n}$, which is the result of the Suslin operation on the family $\{X_s \colon s \in \omega^{<\omega}\}$ (e.g. see [14, Lemma 11.7]). We may assume that $X_s \supseteq X_t$ whenever $s \subseteq t$. Let $W_s = \mathrm{B}(X_s, 2^{-|s|})$, where $|s|$ denotes the length of the sequence s. Then $\mathrm{cl}\, W_s \supseteq \mathrm{cl}\, W_t$ whenever $s \subseteq t$. Moreover, $\bigcap_{n\in\omega} X_{f\restriction n} = \bigcap_{n\in\omega} \mathrm{cl}\, W_{f\restriction n}$ for each $f \in \omega^\omega$. We claim that

$$(\sharp) \qquad \mathrm{Bd}(X, d) = \bigcap_{k\in\omega} \bigcup_{f\in\omega^\omega} \bigcap_{n\in\omega} \Big((\mathrm{Bd}(\tilde{X}, d) \setminus U_k^-) \cup (U_k \cap W_{f\restriction n})^- \Big),$$

where, as usual, we regard $\mathrm{Bd}(X, d) \subseteq \mathrm{Bd}(\tilde{X}, d)$, via the embedding $A \mapsto \mathrm{cl}_{\tilde{X}} A$. The above formula $(\sharp)$ shows that $\mathrm{Bd}(X, d)$ can be obtained from $\mathrm{Bd}(\tilde{X}, d)$ by using the Suslin operation and countable intersection, which shows that it is analytic. It remains to prove $(\sharp)$.

[7] $\mathfrak{E}(\mathrm{Cld}(X)) = \mathfrak{B}(\mathrm{Cld}_H(X))$ for every totally bounded separable metric space X (cf. [5, Hess' Theorem 6.5.14 with Theorem 3.2.3]).

Fix $A \in \mathrm{Bd}(\tilde{X}, d) \setminus \mathrm{Bd}(X, d)$. Then $A \neq \mathrm{cl}(A \cap X)$ and hence there exists $k \in \omega$ such that $A \in U_k^-$ and $\mathrm{cl}\, U_k \cap A \cap X = \emptyset$. Then $A \notin \mathrm{Bd}(\tilde{X}, d) \setminus U_k^-$. For each $f \in \omega^\omega$, we have

$$A \cap \mathrm{cl}\, U_k \cap \bigcap_{n \in \omega} \mathrm{cl}\, W_{f \restriction n} = A \cap \mathrm{cl}\, U_k \cap \bigcap_{n \in \omega} X_{f \restriction n} = \emptyset.$$

By compactness, there is $n \in \omega$ such that $A \cap \mathrm{cl}\, U_k \cap \mathrm{cl}\, W_{f \restriction n} = \emptyset$, hence $A \notin (U_k \cap W_{f \restriction n})^-$.

Now assume that $A \in \mathrm{Bd}(\tilde{X}, d)$ does not belong to the right-hand side of ($\sharp$), that is, there exists $k \in \omega$ such that $A \in U_k^-$ and for every $f \in \omega^\omega$ there is $n \in \omega$ with $A \notin (U_k \cap W_{f \restriction n})^-$. In particular, $A \cap U_k \cap \bigcap_{n \in \omega} X_{f \restriction n} = \emptyset$ for every $f \in \omega^\omega$ and consequently $U_k \cap A \cap X = \emptyset$. On the other hand, $A \cap U_k \neq \emptyset$. Thus it follows that $A \neq \mathrm{cl}_{\tilde{X}}(A \cap X)$, which means $A \notin \mathrm{Bd}(X, d)$. $\square$

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