Generalized magnetic monopoles over contact manifolds

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A generalization of magnetic monopoles is given over an odd dimensional contact manifold and we discuss whether the Yang–Mills–Higgs functional attains at generalized monopoles the absolute minimal value, the topological invariant. © *1995 American Institute of Physics.*

I. REVIEW OF 3-DIM MAGNETIC MONOPOLES

First we recall basic facts on 3-dim Yang–Mills–Higgs fields and magnetic monopoles. Let $P \to M$ be a G-principal bundle over a complete open oriented 3-dim Riemannian manifold M (G is a compact semisimple group). Let (A,Φ) be a smooth connection on P and a smooth section of the adjoint bundle $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$, called a Higgs field. In what follows we call a pair (A,Φ) a configuration.

The Yang–Mills–Higgs functional $\mathscr{A}(A,\Phi)$ is defined as

$$\mathscr{A}(A,\Phi)=\frac{1}{2}\int_M \{|F_A|^2+|D_A\Phi|^2\}dv_g \tag{1}$$

We call a configuration Yang–Mills–Higgs field when the functional $\mathscr{A}$ is stationary at this configuration.

The Euler–Lagrange equations for the first variation of $\mathscr{A}$ are

$$d_A(*F_A)+*[\Phi,D_A\Phi]=0, \quad d_A(*D_A\Phi)=0. \tag{2}$$

Here $F_A = dA + \frac{1}{2}[A \wedge A]$ is the curvature form of A, and D_A, d_A are the covariant derivative and the covariant exterior derivative in the adjoint bundle $\mathfrak{g}_P$, respectively. Further $*$ denotes the Hodge star operator.

A configuration (A,Φ) satisfying the Bogomolny equation

$$*F_A = \pm D_A\Phi \tag{3}$$

is called a (magnetic) monopole. It is easily verified by using the Bianchi identity and the Ricci identity that a monopole satisfies the Euler–Lagrange equations and hence is Yang–Mills–Higgs.

We take the special 3-manifold $M=\mathbf{R}^3$, the Euclidean 3-space and for simplicity the gauge group $G=\mathrm{SU}(2)$.

We consider configurations satisfying the asymptotical decay conditions at infinity of $\mathbf{R}^3$;

$$|\Phi|(x)=m+O(1/r), \tag{4}$$

$$|F_A|(x), \quad |D_A\Phi|(x)=O(1/r^2). \tag{5}$$

(m is a constant, $r=|x|, x\in\mathbf{R}^3$ and $|\cdot|$ is the norm of the adjoint invariant inner product $(X,Y) = -\mathrm{tr}(XY)$ in the Lie algebra su(2)).

From the asymptotical conditions one gets a C^0 map Φ_∞ from the boundary of $\mathbf{R}^3$ at infinity, identified with a 2-sphere S^2 of radius 1, into a 2-sphere of radius m in su(2) by

$$\Phi_\infty(\hat{x})=\lim_{t\to\infty}\Phi(t\hat{x}), \quad \hat{x}\in S^2. \tag{6}$$

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The map Φ_∞ has the mapping degree $k\in\mathbf{N}$, called a monopole charge of (A,Φ). The following shows that the Yang–Mills–Higgs functional attains under the asymptotical decay conditions the minimum represented by the topological invariant.

Proposition 1 (Refs. 1,2,3): For any configuration (A,Φ) of monopole charge k

$$\mathscr{A}(A,\Phi)=4\pi|k|+\frac{1}{2}\int_M|F_A\mp *D_A\Phi|^2dv_g\geqslant 4\pi|k| \tag{7}$$

and the equality holds if and only if (A,Φ) is a monopole.

II. GENERALIZATION OF MAGNETIC MONOPOLES

Yang–Mills–Higgs fields can be defined over a complete open manifold of an arbitrary dimension. Indeed a Yang–Mills–Higgs field is defined, same as in the 3-dimensional case, as a stationary point of the Yang–Mills–Higgs functional (1). So the equations (2) are valid also as the Euler– Lagrange equations for arbitrary dimensional Yang–Mills–Higgs fields.

We consider a generalization of 3-dim monopole over a manifold of an arbitrary odd dimension admitting a special geometrical structure. Here a generalization should be canonically given in the sense that (i) the equation for generalized monopoles is a first order equation, like the Bogomolny equation (3) and (ii) the generalized monopoles reduce to the original 3-dim monopoles, when the manifold dimension is 3.

Let M be a complete open oriented Riemannian manifold of dimension $2n+1$. We call M a contact manifold if M has a 1-form η such that $2n+1$- form $\eta\wedge(d\eta)^n$ is nonzero over M. η is called a contact form.

Set $\omega=d\eta$. ω is a closed 2-form.

Definition: Let $P\to M$ be a G-principal bundle over a complete open contact manifold M. A configuration (A,Φ) on P is called a generalized monopole if (A,Φ) satisfies the generalized Bogomolny equations

$$*F_A=cD_A\Phi\wedge\omega^{n-1},\quad *D_A\Phi=cF_A\wedge\omega^{n-1}. \tag{8}$$

(c is a constant). It is clear that when $\dim M=3$ (8) reduces to the single equation (3) which is free from any contact form on M.

Proposition 2: A generalized monopole is Yang–Mills–Higgs.

Proof: It suffices to check (2). Set $\Omega=\omega^{n-1}$. Then

$$\begin{aligned}d_A(*F_A)+*[\Phi,D_A\Phi]&=cd_A(D_A\Phi\wedge\Omega)+[\Phi,cF_A\wedge\Omega]\\&=cd_AD_A\Phi\wedge\Omega-cD_A\Phi\wedge d\Omega+c[\Phi,F_A\wedge\Omega]\\&=c[F_A,\Phi]\wedge\Omega+c[\Phi,F_A]\wedge\Omega\\&=0.\end{aligned}$$

Similarly

$$d_A(*D_A\Phi)=d_A(cF_A\wedge\Omega)=c(d_AF_A\wedge\Omega+F_A\wedge d\Omega)=0.$$

So any generalized monopole is Yang–Mills–Higgs.

The possible values the constant c of (8) takes depend only on the dimension of M, as will be shown in 3.

We consider next the question whether the thus defined generalized monopoles take the absolute minimal value for the functional $\mathscr{A}$.

Same as before, we assume that the structure group G is SU(2).

It is easy to show the following identity:

$$\begin{aligned}&\{|F_A - c*(D_A\Phi\wedge\Omega)|^2 + |D_A\Phi - c*(F_A\wedge\Omega)|^2\}dv\\&\quad=\{|F_A|^2+|D_A\Phi|^2+c^2|F_A\wedge\Omega|^2+c^2|D_A\Phi\wedge\Omega|^2\}dv-4c((-\mathrm{tr})(F_A\wedge D_A\Phi))\wedge\Omega.\end{aligned} \tag{9}$$

The $2n+1$-form in the last term is an exact form, namely,

$$(-\mathrm{tr}(F_A\wedge D_A\Phi))\wedge\Omega = d\Theta, \tag{10}$$

where Θ is a $2n$-form given by $\Theta = -\mathrm{tr}(F_A\cdot\Phi)\wedge\Omega$.

The integral $\int_M(-\mathrm{tr}(F_A\wedge D_A\Phi))\wedge\Omega = \int_M d\Theta$ is shown to be a topological invariant determined by the Higgs field Φ_∞ at infinity, provided certain asymptotical conditions, one on M and another on configurations, are fulfilled. Suppose (☆1) there is $a_o>0$ such that the distance function from a point $o\in M$ $x\in M\mapsto d(x,o)$ has non zero gradient vector for all x of $d(x,o)\geqslant a_o$ and hence for all sufficiently large a $\partial M_a=\{x;d(x,o)=a\}$ is a smooth hypersurface in M smoothly parametrized by a, (☆2) (A,Φ) is of finite $\mathscr{A}(A,\Phi)$ and satisfies with respect to their restriction to ∂M_a

$$|\Phi|(x)=m+O(1/a), \quad |D_A\Phi|(x)=O(1/a^2)$$

$(a=d(x,o))$.

Integrating (9) over M, we get

$$\begin{aligned}&\int_M\{|F_A|^2+|D_A\Phi|^2+c^2|F_A\wedge\Omega|^2+c^2|D_A\Phi\wedge\Omega|^2\}\\&\quad=\int_M\{|F_A-c*(D_A\Phi\wedge\Omega)|^2+|D_A\Phi-c*(F_A\wedge\Omega)|^2\}+4c\int_M d\Theta.\end{aligned} \tag{11}$$

Here $\int_M d\Theta = \lim_{a\to\infty}\int_{d(x,o)\leqslant a}d\Theta$ and the integral $\int_{d(x,o)\leqslant a}d\Theta$ reduces by Stoke's theorem to the hypersurface integral $\int_{\partial M_a}\Theta$ to which we are able to use the conditions (☆1),(☆2) and apply the argument given in Horváthy and Rawnsley[4] and II.5, Jaffe and Taubes.[1] Therefore $\int_M d\Theta$ turns out to be a topological invariant of the Higgs field Φ_∞ at infinity, which we denote by $p(\Phi_\infty)$.

Proposition 3: Let M be a complete open oriented Riemannian $2n+1$-dim manifold having a contact form η. Then the following inequality holds for any configuration (A,Φ) under the conditions (☆1),(☆2).

$$\int_M\{|F_A|^2+|D_A\Phi|^2+c^2|F_A\wedge\Omega|^2+c^2|D_A\Phi\wedge\Omega|^2\}\geqslant 4cp(\Phi_\infty) \tag{12}$$

and the equality holds if and only if (A,Φ) is a generalized monopole.

Proof: The inequality clearly follows from (11).

Suppose that the equality in (12) holds. Then

$$F_A=c*(D_A\Phi\wedge\Omega), \quad D_A\Phi=c*(F_A\wedge\Omega).$$

Since $\dim M$ is odd, the star operator $*$ for 1- and 2-forms satisfies $* \circ * = id$ so that the above equations are just (8).

III. CONTACT MANIFOLDS AND ASSOCIATED CONTACT METRICS

A contact manifold M with a contact form η is endowed with a metric g associated to the contact form (see Proposition in Sec. 3, Ref. 5).

In fact, the contact form η yields on M a contravariant vector field ξ and a tensor field φ of type $(1,1)$ satisfying

$$\eta(\xi)=1, \quad \varphi(\varphi(X))=-X+\eta(X)\xi \tag{13}$$

(X is an arbitrary tangent vector).

Then M has a metric g which is compatible with the η, namely,

$$g(\varphi(X),\varphi(Y))=g(X,Y)-\eta(X)\eta(Y) \tag{14}$$

and further satisfies

$$d\eta(X,Y)=g(X,\varphi(Y)). \tag{15}$$

We call such a metric g an associated contact metric.

Example: $\mathbf{R}^{2n+1}$ is a contact manifold with a contact form $\eta=dz-\Sigma_i y^i dx^i$ in terms of the Cartesian coordinates $\{x^i,y^i,z\}$. $\omega=d\eta=\Sigma dx^i\wedge dy^i$. The metric $g=\eta\otimes\eta+\Sigma\{(dx^i)^2+(dy^i)^2\}$ is an associated complete contact metric on $\mathbf{R}^{2n+1}$.

We remark that the Euclidean metric for $2n+1>3$ cannot be associated to any contact form (see Theorem, Chap VI, Ref. 5). This fact may be consistent with that any Yang–Mills–Higgs field of $\mathscr{A}<\infty$ on the Euclidean space $\mathbf{R}^{\ell}$, $\ell>3$, turns out trivial, namely, a flat connection with a covariant constant Higgs field(see the argument of the stress tensor in Chap. II, Ref. 1).

Suppose that an open contact manifold M with a contact form η admits a complete associated contact metric g.

To investigate the absolute minimal value of the functional $\mathscr{A}$ on M we define the operator over p-forms on M

$$*\circ L:\Lambda^p(M)\rightarrow\Lambda^{3-p}(M), \tag{16}$$

where $L:\Lambda^p(M)\rightarrow\Lambda^{2n+p-2}(M)$ is the exterior multiplication by the $2n-2$-form Ω. So one defines an endomorphism of $(\Lambda^1\oplus\Lambda^2)(M)$

$$(*\circ L)(\alpha,\beta)=((*\circ L)(\beta),(*\circ L)(\alpha)), \quad (\alpha,\beta)\in(\Lambda^1\oplus\Lambda^2)(M). \tag{17}$$

As is easily shown, this endomorphism is self adjoint with respect to the naturally defined metric on $(\Lambda^1\oplus\Lambda^2)(M)$.

Lemma: Let (M,η,g) be a contact manifold of dimension $2n+1>3$ with an associated contact metric.

Then $(*\circ L)^2$ has the eigenvalues $0,\{(n-1)!\}^2,n\{(n-1)!\}^2$ so that the endomorphism $*\circ L$ has eigenvalues 0, $\pm(n-1)!$ and $\pm(n-1)!\sqrt{n}$. Furthermore for any $(\alpha,\beta)\in(\Lambda^1\oplus\Lambda^2)(M)$

$$|\alpha\wedge\Omega|^2+|\beta\wedge\Omega|^2\leqslant n\{(n-1)!\}^2(|\alpha|^2+|\beta|^2). \tag{18}$$

Here the equality holds in (18) if and only if (α,β) is in the eigenspace belonging to eigenvalue $n\{(n-1)!\}^2$.

Proof: The first part of the lemma is an elementary exercise in Grassmannian algebra, provided we use an associated local orthonormal basis $\{\xi,e_{2i-1},e_{2i}=\varphi(e_{2i-1})\}$ and its dual basis $\{\eta,\theta^{2i-1},\theta^{2i}\}$.

To see (18) we write

$$|\alpha\wedge\Omega|^2+|\beta\wedge\Omega|^2=|(*\circ L)\alpha|^2+|(*\circ L)\beta|^2$$

from which the desired inequality is available.

Remark: The eigenspaces of $*\circ L$ have the dimension $(n-1)(2n+1)$ for eigenvalue 0, $2n$ for eigenvalue $\pm(n-1)!$ and 1 for $\pm\ (n-1)!\sqrt{n}$, respectively.

In fact, $(\eta,\pm\omega)$ give the eigenvectors of eigenvalues $\pm(n-1)!\sqrt{n}$ and $\{(\theta^{2i-1},\pm\theta^{2i}\wedge\eta),(\theta^{2i},\pm\eta\wedge\theta^{2i-1})\}_{i=1,\dots,n}$ form bases of the eigenspaces of eigenvalues $\pm(n-1)!$.

Moreover the eigenspace of zero eigenvalue, identified with Ker$L$ in $\Lambda^2(M)$, has the following basis

$$\theta^1\wedge\theta^2-\theta^{2i-1}\wedge\theta^{2i},\quad 2\leqslant i\leqslant n,$$

$$\theta^{2i-1}\wedge\theta^{2j-1},\quad \theta^{2i-1}\wedge\theta^{2j},\theta^{2i}\wedge\theta^{2j-1},\quad \theta^{2i}\wedge\theta^{2j},\quad 1\leqslant i<j\leqslant n.$$

Note for M of dimension 3 $*\circ L$ has only eigenvalues ± 1. $(\eta,\pm\omega=\pm\theta^1\wedge\theta^2)$, $(\theta^1,\pm\theta^2\wedge\eta)$ and $(\theta^2,\pm\eta\wedge\theta^1)$ form bases of the eigenspaces of eigenvalues ± 1, respectively.

Further the generalized Bogomolny equations (8) can be written as

$$(D_A\Phi,F_A)=c(*\circ L)(D_A\Phi,F_A) \tag{19}$$

in terms of the endomorphism $*\circ L$. So, a generalized monopole (A,Φ) must belong to the eigenspace of $*\circ L$ with eigenvalue c^{-1} and hence the possible values of the constant c in (8) are $\pm 1/(n-1)!$ and $\pm 1/(n-1)!\sqrt{n}$.

The following is an immediate consequence of Proposition 3 and the above lemma.

Proposition 4: Under the conditions same as in Proposition 3

$$\mathscr{A}(A,\Phi)\geqslant\frac{2c}{1+a_n^2c^2}\,p(\Phi_\infty) \tag{20}$$

$(a_n=(n-1)!\sqrt{n})$. Here the equality holds if and only if a configuration (A,Φ) is a generalized monopole of the constant $c=\pm a_n^{-1}$.

Proof: Although we have shown the proposition, we will give another inequality on $\mathscr{A}$ which is quite parallel to that for the generalized (anti-)self-dual connections on a quaternionic Kähler manifold.[6]

Decompose the adjoint bundle valued forms $\Xi=(D_A\Phi,F_A)$ as the sum

$$\Xi=\Xi_1+\Xi_2+\Xi_{-1}+\Xi_{-2}+\Xi_0,$$

where $\Xi_{\pm 1},\Xi_{\pm 2}$ and Ξ_0 are the components of Ξ corresponding to eigenvalues $\pm(n-1)!\sqrt{n}$, $\pm(n-1)!$ and 0, respectively.

The topological invariant $p(\Phi_\infty)$ is then represented as

$$p(\Phi_\infty)=\tfrac{1}{2}\{a_n|\Xi_1|^2+b_n|\Xi_2|^2-a_n|\Xi_{-1}|^2-b_n|\Xi_{-2}|^2\}, \tag{21}$$

$(b_n=(n-1)!<a_n)$ because by using the inner products we can write

$$(-\mathrm{tr}(F_A\wedge D_A\Phi))\wedge\Omega=(F_A,*\circ L(D_A\Phi))=(D_A\Phi,*\circ L(F_A)). \tag{22}$$

On the other hand the functional $\mathscr{A}$ has the form

$$\mathscr{A}(A,\Phi)=\tfrac{1}{2}\{|\Xi_1|^2+|\Xi_2|^2+|\Xi_{-1}|^2+|\Xi_{-2}|^2+|\Xi_0|^2\} \tag{23}$$

so that from (21)

$$\mathscr{A}(A,\Phi)=\frac{1}{a_n}p(\Phi_\infty)+\frac{1}{2}\left\{\left(1-\frac{b_n}{a_n}\right)|\Xi_2|^2+2|\Xi_{-1}|^2+\left(1+\frac{b_n}{a_n}\right)|\Xi_{-2}|^2+|\Xi_0|^2\right\} \tag{24}$$

from which it follows that a c-generalized monopole, $c=1/a_n$, minimizes the functional $\mathscr{A}$. Similarly, a c-generalized monopole, $c=-1/a_n$, also minimizes.

In spite of the above characterization of $\pm a_n^{-1}$-generalized monopole the following proposition shows that when $\dim M>3$ there do not exist any $\pm$ $(1/a_n)$-generalized monopoles with nonzero topological invariant satisfying (☆2).

Proposition 5: Let (A,Φ) be a $\pm(1/a_n)$-generalized monopole satisfying (☆2). If $\dim M>3$, then (A,Φ) must be a trivial configuration, that is, $F_A=0$ and $D_A\Phi=0$, and hence the topological invariant $p(\Phi_\infty)=0$.

Proof: Since the eigenspaces of eigenvalues $\pm(1/a_n)$ are $\mathbf{R}(\pm\eta,\omega)$, it holds $(D_A\Phi,F_A)=\Psi\otimes(\pm\eta,\omega)$ for some section Ψ of $\mathfrak{g}_P$. It follows from the Bianchi identity that $D_A\Psi=0$. So the norm of $D_A\Phi=\pm\Psi\otimes\eta$ is constant which must be zero from (☆2). Therefore $\Psi=0$, i.e., the pair $(D_A\Phi,F_A)=0$. The invariant $p(\Phi_\infty)=\int_M(-\mathrm{tr}(F_A\wedge D_A\Phi))\wedge\Omega$ now becomes zero.

IV. HERMITIAN GEOMETRY OF GENERALIZED MONOPOLE

Let M be a contact manifold with the respective tensor fields η,ξ,φ,g defined at 3. The product manifold $M\times\mathbf{R}$ (or $M\times S^1$) then has the almost Hermitian structure, that is, admits an almost complex structure J,

$$J\left(X+f\frac{d}{dt}\right)=\varphi(X)-f\xi+\eta(X)\frac{d}{dt} \tag{25}$$

and a Hermitian metric $\bar{g}$,

$$\bar{g}\left(X+f_1\frac{d}{dt},Y+f_2\frac{d}{dt}\right)=g(X,Y)+f_1\cdot f_2. \tag{26}$$

Every configuration (A,Φ) on M is then regarded as a time-independent connection $\mathbf{A}=A+\Phi dt$ on $M\times\mathbf{R}$.

At each point of $M\times\mathbf{R}$ the space of 2-forms $\Lambda^2(M\times\mathbf{R})$ can be identified as

$$\Lambda^2(M\times\mathbf{R})\cong(\Lambda^1\oplus\Lambda^2)(M) \tag{27}$$

by

$$\alpha\wedge dt+\beta\to(\alpha,\beta). \tag{28}$$

Then $(\alpha,\beta)\in(\Lambda^1\oplus\Lambda^2)(M)$ satisfies $(\alpha,\beta)=c(*\circ L)(\alpha,\beta)$, if and only if

$$☆(\alpha\wedge dt+\beta)=c(\alpha\wedge dt+\beta)\wedge\Omega, \tag{29}$$

where ☆ is the Hodge star operator on $M\times\mathbf{R}$ and the $(2n-2)$-form $\Omega=\omega^{n-1}$ is considered as a form over $M\times\mathbf{R}$. This is directly derived from the following

$$\star:\Lambda^2(M\times\mathbf{R})\to\Lambda^{2n}(M\times\mathbf{R});\quad \star(\alpha\wedge dt)=*\alpha,\quad \star\beta=(*\beta)\wedge dt \tag{30}$$

Since the curvature form $F_{\mathbf{A}}$ of $\mathbf{A}$ is

$$F_{\mathbf{A}}=D_A\Phi\wedge dt+F_A, \tag{31}$$

by using (29) and (30) we have obviously

Proposition 6: Let (A,Φ) be a configuration on M.

(i) (A,Φ) is a Yang–Mills–Higgs field on M if and only if $\mathbf{A}$ is a Yang–Mills connection on the almost Hermitian manifold $M\times\mathbf{R}$.

(ii) (A,Φ) is a generalized monopole with constant c if and only if $\mathbf{A}$ satisfies the equation

$$\star F_{\mathbf{A}}=cF_{\mathbf{A}}\wedge\omega^{n-1}. \tag{32}$$

The equation (32) is quite similar to the Kähler manifold version of (anti-)self-dual equation,[7,8] whereas in our case ω is a degenerate 2-form. When $\dim M=3$ the proposition gives us the classical observation given in Manton[9] that (A,Φ) is a Yang–Mills–Higgs field (a monopole) if and only if $\mathbf{A}$ is a Yang–Mills connection (an instanton).

Let $\tilde{\omega}=d(e^{-t}\eta)$ be an exact 2-form on $M\times\mathbf{R}$. Then $\tilde{\omega}=e^{-t}(\eta\wedge dt+\omega)$ is the fundamental form of the Hermitian metric $e^{-t}\bar{g}$. Note that $\tilde{\omega}^{n-1}=e^{-(n-1)t}(\omega^{n-1}+(n-1)\omega^{n-2}\wedge\eta\wedge dt)$.

Since for a $\bar{g}$-orthonormal basis $\{\theta^i,\eta,dt\}$ on $M\times\mathbf{R}$ we have

$$(\theta^{2i-1}+\sqrt{-1}\,\theta^{2i})\wedge(\eta+\sqrt{-1}\,dt)=-(\theta^{2i}\wedge dt+\eta\wedge\theta^{2i-1})+\sqrt{-1}(\theta^{2i-1}\wedge dt+\theta^{2i}\wedge\eta)$$

and

$$(\theta^{2i-1}-\sqrt{-1}\,\theta^{2i})\wedge(\eta+\sqrt{-1}\,dt)=(\theta^{2i}\wedge dt+\eta\wedge\theta^{2i-1})+\sqrt{-1}(\theta^{2i-1}\wedge dt+\theta^{2i}\wedge\eta)$$

so that (28) gives from the remark in 3 a characterization of $\pm\{(n-1)!\}^{-1}$-generalized monopole in terms of Hermitian geometry[8,10] as

Proposition 7: (i) (A,Φ) is a $\pm\{(n-1)!\}^{-1}$-generalized monopole on M if and only if $F_{\mathbf{A}}$ of $\mathbf{A}$ has only components of the form taken by the real or imaginary part of $(\theta^{2i-1}\pm\sqrt{-1}\,\theta^{2i})\wedge(\eta+\sqrt{-1}\,dt)$.

(ii) Therefore as a $\mathfrak{g}_P$-valued 2-form the curvature form $F_{\mathbf{A}}$ for a $\{(n-1)!\}^{-1}$-generalized monopole has no (1,1)- components.

(iii) Further $F_{\mathbf{A}}$ for $-\{(n-1)!\}^{-1}$-generalized monopole is a primitive (1,1)-form, i.e., a (1,1)-form orthogonal to the $\tilde{\omega}$.

Proof: It suffices to check only the last part. Since from the first part $F_{\mathbf{A}}$ is written by the linear combination of $(\theta^{2i-1}-\sqrt{-1}\,\theta^{2i})\wedge(\eta+\sqrt{-1}\,dt)$, $F_{\mathbf{A}}$ is clearly orthogonal to $\tilde{\omega}$.

If the almost complex structure J on $M\times\mathbf{R}$ is integrable, namely, the contact structure η on M is normal, then from (iii) of Proposition 7 the SU(2) connection $\mathbf{A}$ which associates with a $-\{(n-1)!\}^{-1}$-generalized monopole (A,Φ) induces a holomorphic vector bundle over $M\times\mathbf{R}$ equipped with an Einstein Hermitian bundle metric.

V. FINAL REMARKS.

To conclude this note, we give several remarks.

Pedersen and Poon,[11] and Galicki and Poon[6] gave another generalization of 3-dimensional magnetic monopole over $\mathbf{R}^{3n}=\mathbf{R}^3\otimes\mathbf{R}^n$ by multitimes independent instantons on $\mathbf{R}^{4n}$. However, our generalization is valid over any odd dimensional contact manifold, even though over the Eucliden space $\mathbf{R}^{\ell}$, $\ell>3$, with the natural contact form η non-trivial solutions of our generalized Bogomolny equations are not yet obtained. An arbitrary compact semisimple Lie group can be taken as a gauge group G which in this note we specialized as SU(2). For an arbitrary compact semisimple Lie group G we impose the gauge invariant ansatz on Higgs fields. Actually we consider for this general case configurations satisfying the asymptotical decay conditions (☆2) and further require that the Higgs field at infinity Φ_∞ has the image sitting inside an adjoint action orbit in the Lie algebra so that Φ_∞ is regarded as a map from ∂M_∞ to a homogeneous space G/K (K is the isotropy subgroup for the orbit). See for this Itoh,[12] Horváthy, and Rawnsley.[4]

So in this most general situation we observe the following topological phenomenon entirely different from the 3-dim original monopoles. For $M=\mathbf{R}^{2n+1}, n \geq 2$ the boundary at infinity ∂M_∞ is diffeomorphic to S^{2n} and the Higgs field at infinity Φ_∞ defines a class in $\pi_{2n}(G/K)$ which happens to be trivial, e.g., $n=2$ and $G/K=\mathbf{C}P^k$, $k \geq 2$ so that this homotopy triviality might give a strict restriction on the topological invariant $p(\Phi_\infty)$.

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