Gradient estimate of the heat kernel on modified graphs *

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Abstract

We obtain a condition on the modification of graphs which guarantees the preservation of the Gaussian upper bound for the gradient of the heat kernel.

1 Introduction

Let us consider the simple random walk on an infinite graph X and denote the kernel of the associated heat semigroup by k_n. Our interest is to see whether the gradient of the heat kernel ∇k_n has the Gaussian upper bound:

$$\nabla k_n(x,y) \leq \frac{C}{\sqrt{n}V(x,\sqrt{n})} e^{-cd^2(x,y)/n} \quad \forall x,y \in X, \forall n \in \mathbb{N}^*, \tag{1}$$

where $V(x,r)$ is the volume of the ball centered at x with radius r for the combinatorial distance d and C, c are some positive constants.

In [7], Hebisch and Saloff-Coste proved that (1) is satisfied on discrete groups of polynomial volume growth. After that, the author generalized their result to the case of nilpotent covering graphs in [8] (see also [5], [6] and [9]). In the proof of all of them, the periodicity of the graph is used essentially. In this article, we prove that (1) is preserved under certain modification of the graph.

It should be noted that there is a connection between the estimate of (1) and the L^p-boundedness of the Riesz transform $\nabla\Delta^{-1/2}$ on non-compact Riemannian manifolds, where Δ is the Laplace-Beltrami operator. Let M be a non-compact complete Riemannian manifold with volume doubling property

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and p_t the kernel of $e^{-t\Delta}$. We assume that p_t satisfies the on-diagonal upper estimate

$$p_t(x,x) \le \frac{C}{V(x,\sqrt{t})},$$

for all $x \in M$, $t > 0$ and some constant $C > 0$. By Theorem 1.4 in [1], the estimate

$$|\nabla p_t(x,y)| \le \frac{C}{\sqrt{t}V(x,\sqrt{t})} e^{-cd(x,y)^2/t}$$

is sufficient to ensure the L^p-boundedness of the Riesz transform for all $1 < p < \infty$, namely there exists C_p such that for any compact supported smooth function $f \in C_0^\infty(M)$,

$$\|\nabla \Delta^{-1/2} f\|_p \le C_p \|f\|_p, \quad 1 < p < \infty.$$

This fact is true also for graphs by Russ ([14]).

There are a lot of results about boundedness of the Riesz transform. See [1] and the literature therein.

1.1 Notation and result

First we fix a graph X_B which is the basis of our modification. Let $X_B = (V_B, E^B)$ be an oriented, locally finite connected infinite graph. Here V_B is the set of vertices and E^B is the set of oriented edges. For $e \in E^B$, we denote by $o(e)$, $t(e)$ and $\overline{e}$ the origin of e, the end of e and the inverse of e, respectively. We always assume that the set of edges of a graph includes their inverse edges. For $x \in V_B$, let $E_x^B = \{e \in E^B \mid o(e) = x\}$ and $m_B(x) := \deg_B x := \# E_x^B$ the weight on x. For $x, y \in V_B$, $n \in \mathbb{N}$, the transition probability $p_n^B(x,y)$ for the simple random walk on X_B is given by

$$\begin{aligned} p_0^B(x,y) =& \chi_y(x), \\ p_n^B(x,y) =& \sum_{(e_1,e_2,\dots,e_n) \in C_{x,n}} p^B(e_1)p^B(e_2)\cdots p^B(e_n)\chi_y(t(e_n)) \quad n \in \mathbb{N}^*, \end{aligned}$$

where χ_y is the characteristic function for $\{y\} \subset V_B$, $C_{x,n}$ is the set of paths from x of length n and $p^B(e) = (\deg_B o(e))^{-1}$.

The transition operator P_B associated to the simple random walk on X_B is an operator acting on a function f on V_B by

$$P_B f(x) = \sum_{e \in E_x^B} p^B(e) f(t(e)).$$

The n-th iteration $u(n,x) = P_B^n f(x)$ gives the solution of the heat equation

$$\begin{aligned} (\partial_1 + \Delta) u(n,x) =& 0 \\ u(0,x) =& f(x), \end{aligned}$$

where $\partial_1 u(n,x) = u(n+1,x) - u(n,x)$ and

$$\Delta u(n,x) = \sum_{e \in E_x^B} p^B(e)(u(n,o(e)) - u(n,t(e))).$$

Then the kernel $k_n^B(x,y)$ of P_B^n w.r.t. the measure m_B is written as $k_n^B(x,y) = p_n^B(x,y) m_B^{-1}(y)$.

For $x, y \in V_B$, let $d_B(x,y)$ be the combinatorial distance from x to y and $V_B(x,r) := \sum_{d_B(x,z) \leq r} m_B(z)$ the measure of $B_B(x,r) = \{y \in V_B \,|\, d_B(x,y) \leq r\}$. We assume that X_B satisfies the *volume doubling property*, namely there exists a constant $C > 0$ such that for any $x \in V_B$ and $r > 0$,

$$V_B(x,2r) \leq C V_B(x,r).$$

It follows that m_B is uniformly finite. Because, for any $y \sim x$, the neighbors in V_B,

$$m_B(x) \leq \sum_{d_B(y,z) \leq 1} m_B(z) = V_B(y,1) \leq C V_B(y,1/2) = C m_B(y).$$

Then we have

$$C m_B(x) = C V_B(x,1/2) \geq V_B(x,1) = \sum_{d_B(x,y) \leq 1} m_B(y) \geq \sum_{d_B(x,y) \leq 1} \frac{1}{C} m_B(x) = \frac{1}{C} m_B^2(x),$$

namely

$$m_B(x) \leq C^2 \quad \forall x \in V_B.$$

We also assume that there exist positive constants C and c such that

$$k_n^B(x,y) \leq \frac{C}{V_B(x,\sqrt{n})} e^{-c d_B(x,y)^2/n}, \tag{2}$$

$$\nabla^B k_n^B(x,y) \leq \frac{C}{\sqrt{n} V_B(x,\sqrt{n})} e^{-c d_B(x,y)^2/n} \tag{3}$$

for all $x, y \in V_B$ and $n \in \mathbb{N}^*$, where $\nabla^B k_n^B(x,y)$ is the gradient of k_n^B for the first variable which is defined by

$$\nabla^B k_n^B(x,y) = \left(\sum_{d_B(x,\omega) \leq 1} \left| k_n^B(\omega,y) - k_n^B(x,y) \right|^2 p_1^B(x,\omega) \right)^{1/2}. \tag{4}$$

In this paper, we consider the stability of (3) under the modification of $X_B = (V_B, E^B)$ given by

$$X = (V,E) = (V_B \backslash V_B', (E^B \cup \mathbb{A}^+) \backslash \mathbb{A}^-).$$

Here $\mathbb{A}^+$ is the set of edges of a graph whose set of endpoints A_+ is a subset in V_B and $\mathbb{A}^-$ is the set of edges of a subgraph in X_B with the following conditions. If $(V_B, (E^B \cup \mathbb{A}^+)\backslash \mathbb{A}^-)$ is connected, this is the modified graph X. Then $V_B' = \phi$. Otherwise, we choose a connected component as the modified graph X. In this case, we retake $\mathbb{A}^-$ by adding all edges of other components and denote by V_B' the set of vertices of them. Then the chosen connected component can be written as $(V_B \backslash V_B', (E^B \cup \mathbb{A}^+)\backslash \mathbb{A}^-)$.

We identify $\mathbb{A}^+$ and $\mathbb{A}^-$ with their associated graphs, respectively.

In other words, X is constructed by the following three operations from X_B:

1. Add edges $\mathbb{A}^+$.
2. Remove (retaken) edges $\mathbb{A}^-$.
3. Remove isolated vertices V_B'.

For $x, y \in V$, let $d(x, y)$ be the combinatorial distance between x and y on X.

We assume that X is uniformly finite, namely the weight $m(x) = \deg x$ on $x \in V$ is uniformly finite. Moreover, we assume that the natural inclusion $I : (X, d) \to (X_B, d_B)$ is a *quasi-isometry*, namely

(i) for a sufficiently large $\epsilon > 0$, the ϵ-neighborhood of $I(V)$ coincides with V_B,

(ii) there exist constants $Q_1 \geq 1$ and $Q_2 \geq 0$ such that

$$Q_1^{-1} d(x, y) - Q_2 \leq d_B(I(x), I(y)) \leq Q_1 d(x, y) + Q_2 \tag{5}$$

for all $x, y \in V$

(see [11]).

For $x, y \in V$, let $p_n(x, y)$ be the transition probability for the simple random walk on X and P^n the associated semigroup acting on the function on V. The kernel $k_n(x, y)$ of P_n w.r.t. the weight $m(y) = \deg y$ is written as $k_n(x, y) = p_n(x, y) m^{-1}(y)$. The gradient $\nabla k_n(x, y)$ for the first variable is also defined by the same manner of $\nabla^B k^B$ in (4). It is known that a quasi-isometric modification preserves the Gaussian upper bound for the heat kernel (2) (see [3]). Hence there exist positive constants C and c such that

$$k_n(x, y) \leq \frac{C}{V(x, \sqrt{n})} e^{-cd(x,y)^2/n} \tag{6}$$

for $x, y \in V$ and $n \in \mathbb{N}^*$, where $V(x, r) := \sum_{d(x,z) \leq r} m(z)$. We remark that X also has the volume doubling property.

Let $A_- = o(\mathbb{A}^-) \cap V \subset V \subset V_B$ be the set of endpoints of $\mathbb{A}^-$ in V. For $x \in V$, we denote by $V_A(x, r)$ the volume of $A = A_+ \cup A_-$ in X defined by

$$V_A(x, r) = \sum_{\substack{d(x, a_+) \leq r, \\ a_+ \in A_+}} m(a_+) + \sum_{\substack{d(x, a_-) \leq r, \\ a_- \in A_-}} m(a_-).$$

Then the main result of this paper is the following:

Theorem 1 *If there exists a positive constant M such that*

$$\frac{iV_A(x,\sqrt{i})}{V(x,\sqrt{i})} < M, \quad \sum_{\ell=1}^{\infty} \frac{V_A(x,\sqrt{\ell})}{V(x,\sqrt{\ell})} < M \tag{7}$$

for all $x \in V$ and $i \in \mathbb{N}^*$, there exist positive constants C and c such that

$$\nabla k_n(x,y) \le \frac{C}{\sqrt{n}V(x,\sqrt{n})} e^{-cd(x,y)^2/n} \tag{8}$$

for $x, y \in V$ and $n \in \mathbb{N}^*$.

In this paper, we do not allow the modification of adding vertices. This is because we need the injection of X into X_B to obtain the expression of $p_n(x,y)$ with $p_n^B(x,y)$ as in Lemma 2.1. However, our hypothesis of the modification and the assumption (7) of the Theorem may be strong in some cases. Indeed, there is a result that finite modifications of $\mathbb{Z}^D$ lattice graph preserve (8) in [4]. Moreover, as in Example 1.1, our result does not show the preservation of (8) under our modifications of $\mathbb{Z}^D$ if $D \le 2$.

Example 1.1 *Let $X_B = (V_B, E^B)$ be the D-dimensional square lattice graph, namely*

$$V_B = \mathbb{Z}^D, \quad E^B = \{(x,y) \in \mathbb{Z}^D \times \mathbb{Z}^D \mid |x-y| \le 1\},$$

where $|\cdot|$ is the Euclidean norm. For $\alpha < D$, let us put

$$\mathbb{A}^- = E_{\mathbb{Z}^\alpha} = \cup_{x\in\mathbb{Z}^\alpha} E_x, \quad V_B' = \mathbb{Z}^\alpha,$$

where

$$\mathbb{Z}^\alpha = \{(x_1, x_2, \ldots, x_\alpha, 0, \ldots, 0) \in \mathbb{Z}^D \mid x_i \in \mathbb{Z},\, 1 \le i \le \alpha\}.$$

It is easy to see that if $\alpha \le D-2$, the modified graph

$$X = (V, E) = \left(V_B \backslash V_B', E^B \backslash \mathbb{A}^-\right)$$

is connected. Since $B_B(V_B \backslash V_B', 1) = X_B$, the condition (i) of the quasi-isometry is satisfied. Since there are no added edges in X, $d_B(x,y) \le d(x,y)$ for any $x, y \in V$. For $x, y \in V$, let $c = (x, c_1, c_2, \ldots, c_{k-1}, y)$ be a geodesic from x to y in X_B. If $c \cap \mathbb{Z}^\alpha = \phi$, $d(x,y) = d_B(x,y)$. If $c \cap \mathbb{Z}^\alpha \ne \phi$, it has one connected component. Then let us consider the parallel transformation $c \cap \mathbb{Z}^\alpha + e_D \subset V_B \backslash V_B'$ of $c \cap \mathbb{Z}^\alpha$, where $e_D = (0, 0, \ldots, 0, 1)$. Let $\{c_{i_1}, c_{i_2}\} = \partial(c \cap \mathbb{Z}^\alpha)$ ($i_1 < i_2$), the boundary of $c \cap \mathbb{Z}^\alpha$. Since

$$d(c_{i_1-1}, c_{i_1} + e_D) \le 2, \quad d(c_{i_2} + e_D, c_{i_2+1}) \le 2,$$

$$\begin{aligned} d(x,y) \le & d(x, c_{i_1-1}) + d(c_{i_1-1}, c_{i_1} + e_D) + d(c_{i_1} + e_D, c_{i_2} + e_D) \\ & + d(c_{i_2} + e_D, c_{i_2+1}) + d(c_{i_2+1}, y) \\ \le & d_B(x, c_{i_1-1}) + 2 + d_B(c_{i_1}, c_{i_2}) + 2 + d_B(c_{i_2+1}, y) \\ \le & d_B(x,y) + 2. \end{aligned}$$

Then we have

$$d(x,y) - 2 \le d_B(x,y) \le d(x,y).$$

Hence X is quasi-isometric to X_B if $\alpha \le D-2$.

It is proved that $\nabla^B k_n^B$ has the Gaussian estimate in [7]. The above theorem asserts that if $\alpha < D-2$, then the Gaussian estimate (8) for ∇k_n holds also on X.

Example 1.2 *More generally, let us denote $X_B = (V_B, E^B)$ the Cayley graph of a finitely generated torsion free nilpotent group Γ with a symmetric set S of generators including the identity e of Γ. Namely, X_B is the oriented graph defined by*

$$V_B = \Gamma, \quad (x,y) \in E^B \text{ if } y^{-1}x \in S.$$

From [7], the Gaussian upper bounds (2) and (3) hold on X_B.

As a modification of X_B, let us take a subgroup H of Γ, namely denote by $\mathbb{A}^- = E_H = \cup_{h\in H} E_h$ and $V'_B = H$. Let D be the volume growth of Γ, namely D is the number such that the volume of the ball $V_B(x,r)$ of centered at $x \in V_B$ with radius r in X_B satisfies

$$C^{-1} r^D \le V_B(x,r) \le C r^D, \quad \forall x \in \Gamma, r \in \mathbb{N}^*.$$

Similarly, let α be the number such that $B(x,r) \cap H$ satisfies

$$V_A(x,r) = \sum_{z \in B(x,r) \cap H} m(z) \le C r^\alpha, \quad \forall x \in V_B, r \in \mathbb{N}^*.$$

In [10], it is proved that, if $\alpha \le D-2$, the modified graph

$$X = (V_B \backslash V'_B, E^B \backslash \mathbb{A}^-)$$

is quasi-isometric to X_B. The previous theorem asserts that if $\alpha < D-2$, then the Gaussian estimate (8) for ∇k_n holds also on X.

2 Proof of the Theorem

2.1 Difference between p_n^B and p_n

For $a_+ \in A_+ \subset V$ and $a_- \in A_- \subset V$, let

$$\deg_+ a_+ := \# \mathbb{A}^+_{a_+} := \#\{e_+ \in \mathbb{A}^+ \mid o(e_+) = a_+\},$$
$$\deg_- a_- := \# \mathbb{A}^-_{a_-} := \#\{e_- \in \mathbb{A}^- \mid o(e_-) = a_-\}.$$

Let us denote P_+ and P_- the transition operator on $\mathbb{A}^+$ and $\mathbb{A}^-$, respectively. Then we have the following:

Lemma 2.1 *For $x, y \in V$ and $n \geq 1$,*

$$p_n(x,y) = p_n^B(x,y) - \sum_{\ell=1}^{n} \sum_{a_+ \in A_+} p_{n-\ell}(x,a_+) \frac{\deg_+ a_+}{\deg a_+} \left(p_\ell^B(a_+,y) - P_+ p_{\ell-1}^B(a_+,y) \right)$$
$$+ \sum_{\ell=1}^{n} \sum_{a_- \in A_-} p_{n-\ell}(x,a_-) \frac{\deg_- a_-}{\deg a_-} \left(p_\ell^B(a_-,y) - P_- p_{\ell-1}^B(a_-,y) \right). \tag{9}$$

Proof. For a real valued function f on V_B, let $I^* f$ be the pull-back to the function on V by the inclusion $I : V \to V_B$. First, we show the following:

$$P(I^* f)(x) = I^*(P_B f)(x) - \sum_{a_+ \in A_+} \chi_{a_+}(x) \frac{\deg_+ a_+}{\deg a_+} \left(I^*(P_B f)(a_+) - P_+ \left(f|_{\mathbb{A}^+} \right)(a_+) \right)$$
$$+ \sum_{a_- \in A_-} \chi_{a_-}(x) \frac{\deg_- a_-}{\deg a_-} \left(I^*(P_B f)(a_-) - P_- \left(f|_{\mathbb{A}^-} \right)(a_-) \right). \tag{10}$$

When $x \notin A_+ \cup A_-$, clearly $P(I^* f)(x) = I^*(P_B f)(x)$. In the case of $x = a_+ \in A_+ \backslash A_-$,

$$\begin{aligned}
P(I^* f)(a_+) =& \frac{1}{\deg a_+} \sum_{e \in E_{a_+}} I^* f(t(e)) = \frac{1}{\deg a_+} \sum_{e \in E_{a_+}} f(I(t(e))) \\
=& \frac{1}{\deg a_+} \left(\sum_{e_B \in E^B_{I(a_+)}} f(t(e_B)) + \sum_{e_+ \in \mathbb{A}^+_{I(a_+)}} f(t(e_+)) \right) \\
=& \frac{\deg_B a_+}{\deg a_+} \cdot \frac{1}{\deg_B a_+} \sum_{e_B \in E^B_{I(a_+)}} f(t(e_B)) + \frac{1}{\deg a_+} \sum_{e_+ \in \mathbb{A}^+_{I(a_+)}} f(t(e_+)) \\
=& \frac{\deg a_+ - \deg_+ a_+}{\deg a_+} P_B f(I(a_+)) + \frac{1}{\deg a_+} \sum_{e_+ \in \mathbb{A}^+_{I(a_+)}} f(t(e_+)) \\
=& P_B f(I(a_+)) - \frac{\deg_+ a_+}{\deg a_+} \left(P_B f(I(a_+)) - \frac{1}{\deg_+ a_+} \sum_{e_+ \in \mathbb{A}^+_{I(a_+)}} f(t(e_+)) \right) \\
=& I^*(P_B f)(a_+) - \frac{\deg_+ a_+}{\deg a_+} \left(I^*(P_B f)(a_+) - I^* P_+ f(a_+) \right).
\end{aligned}$$

When $x = a_- \in A_- \backslash A_+$, similarly we have

$$\begin{aligned}
P(I^* f)(a_-) =& \frac{1}{\deg a_-} \left(\sum_{e_B \in E^B_{I(a_-)}} f(t(e_B)) - \sum_{e_- \in \mathbb{A}^-_{I(a_-)}} f(t(e_-)) \right) \\
=& \frac{\deg a_- + \deg_- a_-}{\deg a_-} P_B f(I(a_-)) - \frac{1}{\deg a_-} \sum_{e_- \in \mathbb{A}^-_{I(a_-)}} f(t(e_-)) \\
=& I^*(P_B f)(a_-) + \frac{\deg_- a_-}{\deg a_-} \left(I^*(P_B f)(a_-) - I^* P_- f(a_-) \right).
\end{aligned}$$

Finally, if $x = a \in A_+ \cap A_-$, we have

$$\begin{aligned}
P(I^* f)(a) =& \frac{1}{\deg a} \left(\sum_{e_B \in E^B_{I(a)}} f(I(t(e_B))) + \sum_{e_+ \in \mathbb{A}^+_{I(a)}} f(t(e_+)) - \sum_{e_- \in \mathbb{A}^-_{I(a)}} f(t(e_-)) \right) \\
=& \frac{\deg a - \deg_+ a + \deg_- a}{\deg a} P_B f(I(a)) + \frac{1}{\deg a} \sum_{e_+ \in \mathbb{A}^+_{I(a)}} f(t(e_+)) \\
&- \frac{1}{\deg a} \sum_{e_- \in \mathbb{A}^-_{I(a)}} f(t(e_-)) \\
=& I^*(P_B f)(a) - \frac{\deg_+ a}{\deg a} \left(I^*(P_B f)(a) - I^* P_+ f(a) \right) \\
&+ \frac{\deg_- a}{\deg a} \left(I^*(P_B f)(a) - I^* P_- f(a) \right).
\end{aligned}$$

Hence (10) is obtained.

From the definition of $p_n(x,y)$, we have

$$
\begin{aligned}
p_n(x,y) &= \sum_{(e_1,e_2,\dots,e_{n-1})\in C_{x,n-1}} p(e_1)p(e_2)\cdots p(e_{n-1}) \sum_{e\in E_t(e_{n-1})} p(e)p_0^B(t(e_{n-1}),y) \\
&= \sum_{(e_1,e_2,\dots e_{n-1})\in C_{x,n-1}} p(e_1)p(e_2)\cdots p(e_{n-1}) \\
&\quad \times \Big(P_B p_0^B(t(e_{n-1}),y) - \sum_{a_+\in A_+} \chi_{a_+}(t(e_{n-1})) \frac{\deg_+ a_+}{\deg a_+} \left(P_B p_0^B(a_+,y) - P_+ p_0^B(a_+,y)\right) \\
&\qquad + \sum_{a_-\in A_-} \chi_{a_-}(t(e_{n-1})) \frac{\deg_- a_-}{\deg a_-} \left(P_B p_0^B(a_-,y) - P_- p_0^B(a_-,y)\right) \Big) \\
&= \sum_{(e_1,e_2,\dots e_{n-1})\in C_{x,n-1}} p(e_1)p(e_2)\cdots p(e_{n-1}) p_1^B(t(e_{n-1}),y) \\
&\quad - \sum_{a_+\in A_+} p_{n-1}(x,a_+) \frac{\deg_+ a_+}{\deg a_+} \left(p_1^B(a_+,y) - P_+ p_0^B(a_+,y)\right) \\
&\quad + \sum_{a_-\in A_-} p_{n-1}(x,a_-) \frac{\deg_- a_-}{\deg a_-} \left(p_1^B(a_-,y) - P_- p_0^B(a_-,y)\right).
\end{aligned}
$$

By using (10) inductively, we conclude

$$
\begin{aligned}
p_n(x,y) = p_n^B(x,y) &- \sum_{\ell=1}^{n} \sum_{a_+\in A_+} p_{n-\ell}(x,a_+) \frac{\deg_+ a_+}{\deg a_+} \left(p_\ell^B(a_+,y) - P_+ p_{\ell-1}^B(a_+,y)\right) \\
&+ \sum_{\ell=1}^{n} \sum_{a_-\in A_-} p_{n-\ell}(x,a_-) \frac{\deg_- a_-}{\deg a_-} \left(p_\ell^B(a_-,y) - P_- p_{\ell-1}^B(a_-,y)\right).
\end{aligned}
$$

□

2.2 Estimate by induction

We prove the Theorem by an induction on time. Since $\nabla k_1(x,y)$ is bounded w.r.t. x, y and the support of $\nabla k_1(x,y)$ as a function of $y \in V$ is included in $B(x,2)$, it is trivial to see that there exist positive constants T_1 and t_1 such that

$$\nabla k_1(x,y) \le \frac{T_1}{V(x,1)} e^{-t_1 d^2(x,y)}.$$

Next, for $\nu > 1$, let us assume that there exist positive constants T_ν and t_ν such that

$$\nabla k_i(x,y) \le \frac{T_\nu}{\sqrt{i}V(x,\sqrt{i})} e^{-t_\nu d^2(x,y)/i}$$

for all $x, y \in V$ and $1 \le i \le \nu - 1$. Let us separate $\nu = m + n$ so that $m = n$ or $m = n + 1$ depending on whether ν is even or odd. By the Cauchy-Schwarz inequality, we have

$$e^{2t_\nu d^2(x,y)/\nu}\nabla k_\nu(x,y) \le \|e^{4t_\nu d^2(x,\cdot)/n}\nabla k_n(x,\cdot)\|_{L^2}\|e^{4t_\nu d^2(\cdot,y)/m}k_m(\cdot,y)\|_{L^2}.$$

Since k_m has the Gaussian upper bound (6), if t_ν is sufficiently small,

$$\|e^{4t_\nu d^2(\cdot,y)/m}k_m(\cdot,y)\|_{L^2} \le \frac{C'(t_\nu)}{V^{1/2}(y,\sqrt{m})} \le \frac{C(t_\nu)}{V^{1/2}(y,\sqrt{\nu})}.$$

We aim to show that

$$\|e^{4t_\nu d^2(x,\cdot)/n}\nabla k_n(x,\cdot)\|_{L^2} \le \frac{C(t_\nu)(T_\nu + c(t_\nu))^{1/2}}{\sqrt{n}V^{1/2}(x,\sqrt{n})}. \tag{11}$$

For $n = 1$, this is trivial. Thus we always assume that $n > 1$.

By using Lemma 2.1, we have

$$\begin{aligned}
&\|e^{4t_\nu d^2(x,\cdot)/n}\nabla k_n(x,\cdot)\|_{L^2}^2 = \sum_{z\in V} e^{8t_\nu d^2(x,z)/n} \sum_{d(x,\omega)\le 1} |k_n(\omega,z)-k_n(x,z)|^2\, p_1(x,\omega)m(z) \\
=&\sum_{z\in V} e^{8t_\nu d^2(x,z)/n} \sum_{d(x,\omega)\le 1} |p_n(\omega,z)-p_n(x,z)|^2\, p_1(x,\omega)m(z)^{-1} \\
=&\sum_{z\in V} e^{8t_\nu d^2(x,z)/n} \sum_{d(x,\omega)\le 1} \Big| p_n^B(\omega,z)-p_n^B(x,z) \\
&-\sum_{\ell=1}^{n}\sum_{i=+,-} i \sum_{a_i\in A_i}(p_{n-\ell}(\omega,a_i)-p_{n-\ell}(x,a_i))\frac{\deg_i a_i}{\deg a_i}(p_\ell^B(a_i,z)-P_i p_{\ell-1}^B(a_i,z))\Big|^2 \\
&\times p_1(x,\omega)m(z)^{-1} \\
\le& 2\sum_{z\in V} e^{8t_\nu d^2(x,z)/n} \sum_{d(x,\omega)\le 1} \left|p_n^B(\omega,z)-p_n^B(x,z)\right|^2 p_1(x,\omega)m(z)^{-1} \\
&+2\sum_{z\in V} e^{8t_\nu d^2(x,z)/n} \sum_{d(x,\omega)\le 1} \Big|\sum_{\ell=1}^{n}\sum_{i=+,-} i \sum_{a_i\in A_i}(p_{n-\ell}(\omega,a_i)-p_{n-\ell}(x,a_i)) \\
&\frac{\deg_i a_i}{\deg a_i}(p_\ell^B(a_i,z)-P_i p_{\ell-1}^B(a_i,z))\Big|^2 p_1(x,\omega)m(z)^{-1} \\
=& I_1(n,x)+I_2(n,x).
\end{aligned}$$

We estimate I_1 and I_2 separately.

2.3 Estimate of I_1

Let us recall the property of X_B and X. Since m_B and m are uniformly finite and the inclusion $I : V \to V_B$ is a quasi-isometry between X and X_B, there exists $C > 0$ depending only on $M_B = \max_{x\in V_B} m_B(x)$, Q_1 and Q_2 in (5) such that

$$I_1(n,x) \le C\sum_{z\in V_B} e^{8t_\nu Q_1^2 d_B^2(x,z)/n} \sum_{d_B(x,\omega)\le Q_1+Q_2} \left|k_n^B(\omega,z)-k_n^B(x,z)\right|^2 m_B(z).$$

For $d_B(x,\omega) \le Q_1 + Q_2$, it is easy to see that

$$\left|k_n^B(\omega,z) - k_n^B(x,z)\right|^2 \le (Q_1+Q_2)M_B^2 \sum_{d_B(x,\omega') \le Q_1+Q_2} \left|\nabla^B k_n^B(\omega',z)\right|^2. \quad (12)$$

Hence $I_1(n,x)$ is estimated by

$$C \sum_{z \in V_B} e^{8t_\nu Q_1^2 d_B^2(x,z)/n} \sum_{d_B(x,\omega) \le Q_1+Q_2} \left|\nabla^B k_n^B(\omega,z)\right|^2 m_B(z).$$

Since $\nabla^B k_n^B$ has the Gaussian upper bound (3),

$$\begin{aligned}
&\sum_{z \in V_B} e^{8t_\nu Q_1^2 d_B^2(x,z)/n} \sum_{d_B(x,\omega) \le Q_1+Q_2} \left|\nabla^B k_n^B(\omega,z)\right|^2 m_B(z) \\
&\le C \sum_{z \in V_B} e^{8t_\nu Q_1^2 d_B^2(x,z)/n} \sum_{d_B(x,\omega) \le Q_1+Q_2} \frac{C^2}{nV(\omega,\sqrt{n})^2} e^{-2cd_B^2(\omega,z)/n} m_B(z) \\
&\le \frac{C'}{nV(x,\sqrt{n})^2} \sum_{z \in V_B} e^{-(2c-8t_\nu Q_1^2)d_B^2(x,z)/n} m_B(z).
\end{aligned}$$

Consequently, if $t_\nu < c/4Q_1^2$, there exists $C(t_\nu)$ independent of T_ν such that

$$I_1(n,x) \le \frac{C(t_\nu)}{nV(x,\sqrt{n})}$$

which gives a desired bound (11).

2.4 Estimate of I_2

In order to make the induction work, we estimate I_2 by the following:

$$\begin{aligned}
&2 \sum_{z \in V} \sum_{i,j=+,-} \sum_{\ell_1,\ell_2=1}^{n} \sum_{a_{1i} \in A_i} \sum_{a_{2j} \in A_j} e^{8t_\nu d^2(x,z)/n} \\
&\sum_{d(x,\omega) \le 1} |k_{n-\ell_1}(\omega,a_{1i}) + k_{n-\ell_1}(x,a_{1i})| \deg_i a_{1i} \left|k_{\ell_1}^B(a_{1i},z) - P_i k_{\ell_1-1}^B(a_{1i},z)\right| \\
&\times |k_{n-\ell_2}(\omega,a_{2j}) - k_{n-\ell_2}(x,a_{2j})| \deg_j a_{2j} \left|k_{\ell_2}^B(a_{2j},z) - P_j k_{\ell_2-1}^B(a_{2j},z)\right| \\
&\times m_B(z)^2 m(x)^{-1} m(z)^{-1}.
\end{aligned}$$

By the Cauchy-Schwarz inequality for ω, I_2 is less than

$$
\begin{aligned}
&2\sum_{z\in V}\sum_{i,j=+,-}\sum_{\ell_1,\ell_2=1}^{n}\sum_{a_{1i}\in A_i}\sum_{a_{2j}\in A_j} e^{8t_\nu d^2(x,z)/n}\\
&\times\left(\sum_{d_(x,\omega)\le 1}\left|k_{n-\ell_1}(\omega,a_{1i})+k_{n-\ell_1}(x,a_{1i})\right|^2 m(x)^{-1}\right)^{1/2}\deg_i a_{1i}\\
&\times\left|k^B_{\ell_1}(a_{1i},z)-P_i k^B_{\ell_1-1}(a_{1i},z)\right|\left(\sum_{d(x,\omega)\le 1}\left|k_{n-\ell_2}(\omega,a_{2j})-k_{n-\ell_2}(x,a_{2j})\right|^2 p_1(x,\omega)\right)^{1/2}\\
&\times\deg_j a_{2j}\left|k^B_{\ell_2}(a_{2j},z)-P_j k^B_{\ell_2-1}(a_{2j},z)\right| m_B(z)^2 m(z)^{-1}\\
\le&2\sum_{z\in V}e^{8t_\nu^2 d(x,z)/n}\sum_{\ell_1=1}^{n}\sum_{i=+,-}\sum_{a_{1i}\in A_i}2\sup_{d(x,\omega)\le 1}k_{n-\ell_1}(\omega,a_{1i})\deg_i a_{1i}\\
&\times\left|k^B_{\ell_1}(a_{1i},z)-P_i k^B_{\ell_1-1}(a_{1i},z)\right|\sum_{\ell_2=1}^{n}\sum_{j=+,-}\sum_{a_{2j}\in A_j}\nabla k_{n-\ell_2}(x,a_{2j})\deg_j a_{2j}\\
&\times\left|k^B_{\ell_2}(a_{2j},z)-P_j k^B_{\ell_2-1}(a_{2j},z)\right| m_B(z)^2 m(z)^{-1}.
\end{aligned}
$$

Using the same arguments in (12),

$$
\begin{aligned}
&\left|k^B_\ell(a_i,z)-P_i k^B_{\ell-1}(a_i,z)\right|\\
&=\left|\left(k^B_\ell(a_i,z)-k^B_{\ell-1}(a_i,z)\right)+\left(k^B_{\ell-1}(a_i,z)-P_i k^B_{\ell-1}(a_i,z)\right)\right|\\
&\le\frac{1}{\deg a_i}\sum_{e_B\in E^B_{a_i}}\left|k^B_{\ell-1}(t(e_B),z)-k^B_{\ell-1}(a_i,z)\right|+\frac{1}{\deg_i a_i}\sum_{e_i\in\mathbb{B}^i_{a_i}}\left|k^B_{\ell-1}(a_i,z)-k^B_{\ell-1}(t(e_i),z)\right|\\
&\le\nabla^B k^B_{\ell-1}(a_i,z)+M_B\#B_B(a_i,Q_1+Q_2)\sum_{d_B(a_i,\omega)\le Q_1+Q_2}\nabla^B k^B_{\ell-1}(\omega,z)\\
&\le 2M_B\#B_B(a_i,Q_1+Q_2)\sum_{d_B(a_i,\omega)\le Q_1+Q_2}\nabla^B k^B_{\ell-1}(\omega,z). \qquad (13)
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
I_2(n,x)\le&C'\sum_{z\in V}e^{8t_\nu d^2(x,z)/n}\\
&\sum_{\ell_1=1}^{n}\sum_{i=+,-}\sum_{a_{1i}\in A_i}\sup_{d(x,\omega)\le 1}k_{n-\ell_1}(\omega,a_{1i})\sum_{d_B(a_{1i},\omega_1)\le Q_1+Q_2}\nabla^B k^B_{\ell_1-1}(\omega_1,z)\\
&\sum_{\ell_2=1}^{n}\sum_{j=+,-}\sum_{a_{2j}\in A_j}\nabla k_{n-\ell_2}(x,a_{2j})\sum_{d_B(a_{2j},\omega_2)\le Q_1+Q_2}\nabla^B k^B_{\ell_2-1}(\omega_2,z)m_B(z)^2 m(z)^{-1}.
\end{aligned}
$$

Lemma 2.2

$$\sum_{\ell_2=1}^{n}\sum_{j=+,-}\sum_{a_{2j}\in A_j}\nabla k_{n-\ell_2}(x,a_{2j})\sum_{d_B(a_{2j},\omega_2)\le Q_1+Q_2}\nabla^B k^B_{\ell_2-1}(\omega_2,z)m_B(z)^2m(z)^{-1}$$
$$\le\frac{C(t_\nu)T_\nu+C}{\sqrt{n}V(x,\sqrt{n})}+\frac{C(t_\nu)T_\nu}{\sqrt{n}V(z,\sqrt{n})}. \tag{14}$$

Proof. From the induction hypothesis for ∇k_n and the Gaussian upper bound for $\nabla^B k^B_n$,

$$\sum_{\ell_2=1}^{n}\sum_{j=+,-}\sum_{a_{2j}\in A_j}\nabla k_{n-\ell_2}(x,a_{2j})\sum_{d_B(a_{2j},\omega_2)\le Q_1+Q_2}\nabla^B k^B_{\ell_2-1}(\omega_2,z)m_B(z)^2m(z)^{-1}$$
$$\le\sum_{i=+,-}\sum_{a_i\in A_i}\Big\{\frac{T_\nu}{\sqrt{n-1}V(x,\sqrt{n-1})}e^{-t_\nu d^2(x,a_i)/(n-1)}\chi_{B_B(a_i,Q_1+Q_2+1)}(z)$$
$$\sum_{\ell=2}^{n-1}\frac{T_\nu}{\sqrt{n-\ell}V(x,\sqrt{n-\ell})}e^{-t_\nu d^2(x,a_i)/(n-\ell)}\frac{C}{\sqrt{\ell-1}V_B(z,\sqrt{\ell-1})}e^{-cd_B^2(a_i,z)/(\ell-1)}$$
$$+\chi_{B(x,1)}(a_i)\frac{C}{\sqrt{n-1}V_B(a_i,\sqrt{n-1})}e^{-cd_B^2(a_i,z)/(n-1)}\Big\}$$
$$\le\frac{C'T_\nu}{\sqrt{n-1}V(x,\sqrt{n-1})}$$
$$+\sum_{\ell=2}^{n/2}\sum_{i=+,-}\sum_{a_i\in A_i}\frac{C'T_\nu}{\sqrt{n}V(x,\sqrt{n})}\frac{C}{\sqrt{\ell-1}V_B(z,\sqrt{\ell-1})}e^{-cd_B^2(a_i,z)/(\ell-1)}$$
$$+\sum_{\ell\ge n/2}^{n-1}\sum_{i=+,-}\sum_{a_i\in A_i}\frac{T_\nu}{\sqrt{n-\ell}V(x,\sqrt{n-\ell})}e^{-t_\nu d^2(x,a_i)/(n-\ell)}\frac{C'}{\sqrt{n}V_B(z,\sqrt{n})}$$
$$+\frac{C'}{\sqrt{n-1}V_B(x,\sqrt{n-1})}e^{-c'd_B^2(x,z)/(n-1)}.$$

Since

$$\begin{aligned}\sum_{a_i\in A_i}e^{-t_\nu d^2(x,a_i)/\ell}&=\sum_{k=0}^{\infty}\sum_{k\le d^2(x,a_i)/\ell<k+1}e^{-t_\nu d^2(x,a_i)/\ell}\\&\le\sum_{k=0}^{\infty}\#\{a\in A\,|\,t_\nu d^2(x,a_i)/\ell\le k+1\}e^{-k}\\&\le\sum_{k=0}^{\infty}V_A\left(x,\sqrt{t_\nu^{-1}(k+1)\ell}\right)e^{-k}\end{aligned}$$

and $V_B(x,r)$ is comparable with $V(x,r)$, (14) is less than

$$
\begin{aligned}
&\frac{C'T_\nu + C}{\sqrt{n-1}V(x,\sqrt{n-1})} \\
&+ \frac{C'T_\nu}{\sqrt{n}V(x,\sqrt{n})} \sum_{\ell=2}^{n/2}\sum_{k=0}^{\infty} \frac{CV_A(z,\sqrt{c^{-1}(k+1)(\ell-1)})}{\sqrt{\ell-1}V_B(z,\sqrt{\ell-1})} e^{-k} \\
&+ \sum_{\ell\geq n/2}^{n-1}\sum_{k=0}^{\infty} \frac{T_\nu V_A(x,\sqrt{t_\nu^{-1}(k+1)(n-\ell)})}{\sqrt{n-\ell}V(x,\sqrt{n-\ell})} e^{-k} \frac{C'}{\sqrt{n}V_B(z,\sqrt{n})}.
\end{aligned}
$$

By the volume doubling property on X,

$$
\begin{aligned}
&\sum_{\ell=2}^{n/2}\sum_{k=0}^{\infty} \frac{V_A(z,\sqrt{c^{-1}(k+1)(\ell-1)})}{\sqrt{\ell-1}V_B(z,\sqrt{\ell-1})} e^{-k} \\
&= \sum_{\ell=2}^{n/2}\sum_{k=0}^{\infty} \frac{V_A(z,\sqrt{c^{-1}(k+1)(\ell-1)})}{V(z,\sqrt{c^{-1}(k+1)(\ell-1)})} \cdot \frac{V(z,\sqrt{c^{-1}(k+1)(\ell-1)})}{\sqrt{\ell-1}V_B(z,\sqrt{\ell-1})} e^{-k} \\
&\leq \sum_{k=0}^{\infty} \left(\sum_{\ell=2}^{n/2} \frac{V_A(z,\sqrt{c^{-1}(k+1)(\ell-1)})}{\sqrt{\ell-1}V(z,\sqrt{c^{-1}(k+1)(\ell-1)})} \right) C(c^{-1}(k+1))^{\delta} e^{-k}
\end{aligned}
$$

for some $\delta > 0$. From the assumption (7) for V_A in Theorem 1, this is uniformly bounded for $z \in V$. Then the lemma follows. □

By this lemma and the Gaussian upper bounds for $\nabla^B k_n^B$ and k_n, I_2 is estimated by

$$
\begin{aligned}
&C' \sum_{z\in V} e^{8t_\nu d^2(x,z)/n} \Big\{ \sum_{i=+,-}\sum_{a_{1i}\in A_i} \frac{C}{V(x,\sqrt{n-1})} e^{-cd^2(x,a_{1i})/(n-1)} \chi_{B_B(a_{1i},Q_1+Q_2+1)}(z) \\
&+ \sum_{\ell_1=2}^{n-1}\sum_{i=+,-}\sum_{a_{1i}\in A_i} \frac{C}{V(x,\sqrt{n-\ell_1})} e^{-cd^2(x,a_{1i})/(n-\ell_1)} \\
&\quad \times \frac{C}{\sqrt{\ell_1-1}V_B(a_{1i},\sqrt{\ell_1-1})} e^{-cd_B^2(a_{1i},z)/(\ell_1-1)} \\
&+ \sum_{i=+,-}\sum_{a_{1i}\in A_i} \chi_{B(x,1)}(a_{1i}) \frac{C}{\sqrt{n-1}V_B(a_{1i},\sqrt{n-1})} e^{-cd_B^2(a_{1i},z)/(n-1)} \Big\} \\
&\times \left(\frac{C(t_\nu)T_\nu + C}{\sqrt{n}V(x,\sqrt{n})} + \frac{C(t_\nu)T_\nu}{\sqrt{n}V(z,\sqrt{n})} \right).
\end{aligned}
$$

Since

$$
e^{8t_\nu d^2(x,z)/n} \leq e^{16t_\nu d^2(x,a_{1i})/(n-\ell_1)} e^{32Q_1^2Q_2^2t_\nu} e^{32Q_1^2t_\nu d_B^2(a_{1i},z)/(\ell_1-1)}
$$

for $1 < \ell_1 < n$,

$$
\begin{aligned}
I_2(n,x) \le & C'' \sum_{z\in V} \Big\{ \sum_{i=+,-} \sum_{a_{1i}\in A_i} \frac{C}{V(x,\sqrt{n-1})} e^{-(c-16t_\nu)d^2(x,a_{1i})/(n-1)} \chi_{B_B(a_{1i},Q_1+Q_2+1)}(z) \\
& + \sum_{\ell_1=2}^{n-1} \sum_{i=+,-} \sum_{a_{1i}\in A_i} \frac{C}{V(x,\sqrt{n-\ell_1})} e^{-(c-16t_\nu)d^2(x,a_{1i})/(n-\ell_1)} \\
& \times \frac{C}{\sqrt{\ell_1-1} V_B(a_{1i},\sqrt{\ell_1-1})} e^{-(c-32Q_1^2 t_\nu)d_B^2(a_{1i},z)/(\ell_1-1)} \\
& + \sum_{i=+,-} \sum_{a_{1i}\in A_i} \chi_{B(x,1)}(a_{1i}) \frac{C}{\sqrt{n-1} V_B(a_{1i},\sqrt{n-1})} e^{-(c-32Q_1^2 t_\nu)d_B^2(a_{1i},z)/(n-1)} \Big\} \\
& \times \left(\frac{C(t_\nu)T_\nu + C}{\sqrt{n} V(x,\sqrt{n})} + \frac{C(t_\nu)T_\nu}{\sqrt{n} V(z,\sqrt{n})} \right).
\end{aligned}
$$

By the volume doubling property for V,

$$
\begin{aligned}
\frac{1}{V(z,\sqrt{n})} e^{-cd^2(a,z)/n} =& \frac{1}{V(a,\sqrt{n})} \cdot \frac{V(a,\sqrt{n})}{V(z,\sqrt{n})} e^{-cd^2(a,z)/n} \\
\le & \frac{1}{V(a,\sqrt{n})} \cdot \frac{V(z,\sqrt{n}+d(a,z))}{V(z,\sqrt{n})} e^{-cd^2(a,z)/n} \\
\le & \frac{1}{V(a,\sqrt{n})} \left(1 + \frac{d(a,z)}{\sqrt{n}} \right)^{\delta} e^{-cd(a,z)/n} \\
\le & \frac{C}{V(a,\sqrt{n})} e^{-cd^2(a,z)/2n}.
\end{aligned}
$$

For sufficiently small $t_\nu > 0$, I_2 is estimated by

$$
\begin{aligned}
& \Big\{ \sum_{i=+,-} \sum_{a_{1i}\in A_i} \frac{C}{V(x,\sqrt{n-1})} e^{-(c-16t_\nu)d^2(x,a_{1i})/(n-1)} \\
& + C \sum_{\ell_1=2}^{n/2} \sum_{i=+,-} \sum_{a_{1i}\in A_i} \frac{C}{V(x,\sqrt{n})} e^{-(c-16t_\nu)d^2(x,a_{1i})/n} \frac{C}{\sqrt{\ell_1-1}} \\
& + C \sum_{\ell_1>n/2}^{n-1} \sum_{i=+,-} \sum_{a_{1i}\in A_i} \frac{C}{V(x,\sqrt{n-\ell_1})} e^{-(c-16t_\nu)d^2(x,a_{1i})/(n-\ell_1)} \frac{C'}{\sqrt{n}} \\
& + \frac{C}{\sqrt{n-1}} \Big\} \frac{C(t_\nu)T_\nu + C(t_\nu)}{\sqrt{n} V(x,\sqrt{n})}.
\end{aligned}
$$

Here we remark that

$$
\sum_{z\in V} \frac{1}{V(a,\sqrt{\ell})} e^{-cd^2(a,z)/\ell} \le C \quad \forall \ell \ge 1.
$$

By the same argument in Lemma 2.2 with the assumption (7) of the Theorem, we conclude that

$$\begin{aligned} I_2(n,x) \leq & C\Big\{\sum_{k=0}^{\infty}\frac{\sqrt{c(t_\nu)(k+1)(n-1)}V_A(x,\sqrt{c(t_\nu)(k+1)(n-1)})}{V(x,\sqrt{c(t_\nu)(k+1)(n-1)})}(c(t_\nu)(k+1))^{\delta}e^{-k} \\ & + \sum_{k=0}^{\infty}\frac{c(t_\nu)(k+1)(n-1)V_A(x,\sqrt{c(t_\nu)(k+1)(n-1)})}{V(x,\sqrt{c(t_\nu)(k+1)(n-1)})}(c(t_\nu)(k+1))^{\delta}e^{-k} \\ & + \sum_{k=0}^{\infty}\sum_{\ell_1>n/2}^{n-1}\frac{V_A(x,\sqrt{c(t_\nu)(k+1)(n-\ell_1)})}{V(x,\sqrt{c(t_\nu)(k+1)(n-\ell_1)})}(c(t_\nu)(k+1))^{\delta}e^{-k} + \frac{\sqrt{n}}{\sqrt{n-1}}\Big\} \\ & \cdot\frac{C(t_\nu)T_\nu + C'(t_\nu)}{nV(x,\sqrt{n})} \\ \leq & M\frac{C'(t_\nu)T_\nu + C''(t_\nu)}{nV(x,\sqrt{n})} \end{aligned}$$

which gives a desired bound for (11).

2.5 Conclusion of induction

From the above arguments, for small $t_\nu > 0$, we obtain

$$\nabla k_\nu(x,y) \leq \frac{(C(t_\nu)T_\nu + C'(t_\nu))^{1/2}}{\sqrt{\nu}V(x,\sqrt{\nu})^{1/2}}\frac{C(t_\nu)}{V(y,\sqrt{\nu})^{1/2}}e^{-2t_\nu d(x,y)^2/\nu}$$

under the hypothesis that

$$\nabla k_i(x,y) \leq \frac{T_\nu}{\sqrt{i}V(x,\sqrt{i})}e^{-t_\nu d(x,y)^2/i}$$

for $1 \leq i \leq \nu - 1$. Since X has the volume doubling property, by the same arguments as in the proof of Theorem 5.2 in [3], we have

$$\nabla k_\nu(x,y) \leq \frac{C''(t_\nu)(C(t_\nu)T_\nu + C'(t_\nu))^{1/2}}{\sqrt{\nu}V(x,\sqrt{\nu})}e^{-t_\nu d(x,y)^2/\nu}.$$

Finally, by taking T_ν large enough so that

$$C''(t_\nu)(C(t_\nu)T_\nu + C'(t_\nu))^{1/2} \leq T_\nu,$$

we obtain

$$\nabla k_\nu(x,y) \leq \frac{T_\nu}{\sqrt{\nu}V(x,\sqrt{\nu})}e^{-t_\nu d(x,y)^2/\nu}.$$

Then the proof of theorem is complete.

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